K-EQUIVALENCE IN BIRATIONAL GEOMETRY AND CHARACTERIZATIONS OF COMPLEX ELLIPTIC GENERA

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Abstract

We show that for smooth complex projective varieties the most general combinations of Chern numbers that are invariant under the K-equivalence relation consist of the complex elliptic genera.

1. Introduction and statements

The K-equivalence relation for $\mathbb{Q}$-Gorenstein projective varieties was introduced in [17] in order to abstract the notion of higher dimensional *composite of flops*. Two birational $\mathbb{Q}$-Gorenstein varieties X and X' are K-equivalent (denoted by $X =_K X'$) if there is a smooth birational model $\phi : Y \to X$ and $\phi' : Y \to X'$ such that $\phi^* K_X =_{\mathbb{Q}} \phi'^* K_{X'}$. This simple notion appears naturally in birational geometry and we are interested in characterizing those geometric/topological invariants that are invariant under it.

The basic strategy formulated in [17] is a *meta theorem*: in order to obtain numerical invariants under the K-equivalence relation, it suffices to have a suitable integration theory which admits a *nice* change of variable formula for birational morphisms and geometric/topological interpretations of the integration. For K-equivalent smooth complex projective varieties, the invariance of Betti and Hodge numbers has been verified in [17] [5] [3]. Based on this, one is tempted to conjecture that their ordinary cohomology groups (at least for the torsion free part) are canonically isomorphic under the cohomology correspondence induced from the graph of the given birational map. It is clear that this map respects the $\mathbb{Q}$-Hodge structures.

Not all interesting invariants are invariant under the K-equivalence relation. For example, while the cohomology groups for smooth threefolds are canonically isomorphic under a classical flop, their ring structures are in general different. We find two recent works that may lead to an explanation. One

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is the *quantum minimal model conjecture* raised by Ruan [15] (cf. §6). Another one is Totaro's work [16]: *complex elliptic genera = complex cobordism ring modulo classical flops*, or equivalently Chern numbers invariant under classical flops consist of precisely the complex elliptic genera.

We show in this paper a generalization of Totaro's result: *complex elliptic genera = complex cobordism ring modulo K-equivalence*. A surprising conclusion is that in the complex cobordism ring, the ideal generated by classical flops equals the seemingly much larger ideal generated by all K-equivalent pairs. To summarize our approach, we follow the meta theorem. The target invariants are genera (or Chern numbers), which by definition are certain integrals, so we only need a nice change of variable formula. This is treated in two steps:

Inspired by the work of Hirzebruch et. al. [8] on the characterization of the (real) elliptic genera (of Landweber and Stong) as the most general genera that are multiplicative on fiber bundles with fiber $\mathbb{P}^{2n-1}$ for all $n \in \mathbb{N}$, we characterize the complex elliptic genera (studied by Witten, Hirzebruch and subsequently by Krichever, Höhn and Totaro) in a similar flavor via blowing-ups along smooth centers.

Let X be a compact complex manifold or a proper smooth variety (over an algebraically closed field of arbitrary characteric). For a commutative ring R, an R-genus φ is defined by a power series $Q(x) \in R[\![x]\!]$ through Hirzebruch's multiplicative sequence K_Q (or K_φ). As usual we write $Q(x) = x/f(x)$.

Theorem A (residue theorem). *For any cycle D in X and for any blowing-up $\phi : Y \to X$ along smooth center Z with exceptional divisor E, one has for any power series $A(t) \in R[\![t]\!]$:*

$$\int_{\phi^* D} A(E) \, K_Q(c(T_Y)) \;=\; \int_D A(0) \, K_Q(c(T_X))$$
$$+ \int_{Z.D} \operatorname{Res}_{t=0} \Big(\frac{A(t)}{f(t) \prod_{i=1}^r f(n_i - t)} \Big) K_Q(c(T_Z)).$$

Here n_i's denote the formal Chern roots of the normal bundle $N_{Z/X}$ and the residue stands for the coefficient of the degree -1 term of a Laurent power series with coefficients in the cohomology ring or the Chow ring of X.

Theorem B (characterization of complex elliptic genera). *Consider the following sets of power series $f(x) = x + \cdots \in \mathbb{C}[\![x]\!]$ (or $\mathbb{C}$-genera ϕ's):*

$S_1 :$ *φ admits a first step change of variable formula. That is, for each $r \in \mathbb{N}$ there exists a power series $A(t, r)$ in t that serves as the Jacobian*

factor such that $A(0, r) = 1$ and

$$\int_X K_\varphi(c(T_X)) = \int_Y A(E, r) \, K_\varphi(c(T_Y))$$

for any blowing-up $\phi : Y \to X$ along the smooth center of codimension r with exceptional divisor E.

$S_2 :$ *$f(x)$ is a solution to the following functional equation:*

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}$$

for some power series $A(t)$ with $A(0) = 1$.

$S_3 :$ *φ is a specialization of the complex elliptic genera. That is, φ is parameterized by $(k, a, b, g_2) \in \mathbb{C}^4$ such that*

$$\frac{f'(x)}{f(x)} = -\frac{1}{2}\frac{\wp'(x) - b}{\wp(x) - a} + k.$$

Where $\wp(x)$ is the unique function with a pole of order 2 at zero such that $\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3$ with g_3 defined by $b^2 = 4a^3 - g_2 a - g_3$. Equivalently, φ is parametrized by $k \in \mathbb{C}$ and affine Weierstrass equations, which may define singular curves, with a marked point.

Then $S_1 \subset S_2 \subset S_3 \cong \mathbb{C}^4$. Moreover, S_2 contains precisely those f's with (a, b) not a 2 torsion point plus exceptional cases $e^{kx}\sinh(sx)/s$ with $s^2 = 6f_3 = f'''(0)$. In the former cases, $A(x)$ is uniquely determined by f and for the exceptional cases $A(x) = e^{-kx}(a_1 \sinh(sx)/s + \cosh(sx))$ where $a_1 = A'(0)$. S_1 contains all f's with (a, b) a non-torsion point. In particular, the generic point of the complex elliptic genera (the power series f with parameters) is a solution to S_1 and S_2.

Let $(a, b) = (\wp(z), \wp'(z))$, in terms of the Weierstrass σ function we will write down f and find a candidate for $A(t, r)$ in §4, namely

$$f(x) = e^{(k+\zeta(z))x}\frac{\sigma(x)\sigma(z)}{\sigma(x+z)}, \qquad A(t, r) = e^{-(r-1)(k+\zeta(z))t}\frac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}.$$

This shows that for blowing-ups along codimension r centers, there could be *no* change of variable formula when one specializes z to an r torsion point.

The second step is the *full* change of variable formula for birational morphisms. At this moment we can only prove this for morphisms that can be factorized into composite of blowing-ups and blowing-downs along smooth centers. In the algebraic case, we may complete the proof in the characteristic zero case using recent result of Wlodarsczyk et. al. [1] [18] on the *weak factorization theorem.*

Theorem C (change of variable formula). *Let φ be the complex elliptic genera. Then for any algebraic cycle D in X and birational morphism $\phi : Y \to X$ with $K_Y = \phi^* K_X + \sum e_i E_i$, we have*

$$\int_D K_\varphi(c(T_X)) = \int_{\phi^* D} \prod_i A(E_i, e_i + 1) \, K_\varphi(c(T_Y)).$$

Or equivalently, $\phi_* \prod_i A(E_i, e_i + 1) \, K_\varphi(c(T_Y)) = K_\varphi(c(T_X))$.

It is well-known that the Todd genus is the only complex genus that is absolutely birationally invariant. In this case, $Q(x) = x/(1 - e^{-x})$, $A(t, r) = 1 = A(x)$ and Theorem C is a special case of the Grothendieck-Riemann-Roch Formula $\phi_* \mathrm{td}(T_Y) = \mathrm{td}(T_X)$, which is valid without any restriction on the ground field. Our guiding principle is that one should regard the Todd genus as a *rational measure* and the complex elliptic genera as the *elliptic measure* — think of $K_\varphi(c(T_X))$ as $d\mu_X$! Theorem C together with Totaro's result [16] imply

Theorem D (invariance under K-equivalence). *The most general Chern numbers that are preserved under the K-equivalence relation among smooth proper complex varieties consist of the complex elliptic genera. Moreover, complex elliptic genera are precisely those genera that could be defined on log-terminal singular varieties through the change of variable formula, with the result to be independent of the chosen smooth model.*

Our original plan is to regard Theorem C as a Grothendieck-Riemann-Roch type problem and to prove Theorem D via Theorem B and C only. One reason for doing so is to get a characteristic free treatment of all these results in the algebraic case. We realize only part of it in the current work. Also it should be interesting to compare our approach with Totaro's, which uses the rigidity property of the complex elliptic genera.

Historically, physicists in the 1980's has predicted that the two parameter elliptic genera (with $k = 0$) can be defined on singular Calabi-Yau varieties (at least for orbifolds) and should agree with the value on the (non-unique) crepant resolutions, if there are any. This has motivated many earlier works on this subject. In this context, a recent preprint of Borisov and Libgober [4] contains results that are related to our present work. In particular, they also showed that elliptic genera (with $k = 0$) could be defined on varieties with at most log-terminal singularities.

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2. A residue theorem for genera

Let X be a compact complex manifold or a proper smooth algebraic variety over an algebraically closed field. Let $c(T_X) = \prod_i (1 + x_i)$ be the formal decomposition into Chern roots. For any commutative ring R, an R-genus $\varphi(X)$ is defined by a power series $Q(x) = 1 + \cdots \in R[\![x]\!]$ through Hirzebruch's recipe of multiplicative sequence

$$\varphi(X) = \int_X K_\varphi(c(T_X)) = \int_X \prod_i Q(x_i).$$

We will also write K_φ as K_Q if Q defines φ.

In order to represent $\varphi(X)$ in terms of data on Y under a blowing-up $\phi : Y \to X$ along Z, we will need a localization result. To begin with, let us recall and extend some results from [8], Ch.3. As usual, let $Q(x) = x/f(x)$ with $f(x) = x + \cdots$. Then $f(u+v) = F(f(u), f(v))$ for an unique power series $F(y_1, y_2) = \sum_{(r,s) \neq (0,0)} a_{rs} y_1^r y_2^s = y_1 + y_2 + \cdots$. For cycles $u_j \in H^2(M, \mathbb{Z})$ (or $Z_{n-1}(M)$), the virtual genus is defined by $\varphi(\prod u_j) := \int_M K_Q(c(T_M)) \prod f(u_j)$. When $\prod u_j$ is represented by a smooth subvariety V, $\varphi(\prod u_j) \equiv \varphi(V)$. Then one has [8], p.37:

$$\varphi(u + v) = \sum_{(r,s) \neq (0,0)} a_{rs} \varphi(u^r v^s).$$

Moreover, for $D \subset M$ an analytic/algebraic cycle, we may define $\varphi_D(\prod u_j)$ by $\int_D K_Q(c(T_M)) \prod f(u_j)$. Then the above is also valid for φ_D.

Now let $g := f^{-1}$, the inverse power series (so $g'(y) = \sum_{n \geq 0} \varphi(\mathbb{P}^n) y^n$). Let $H_{ij} \subset \mathbb{P}^i \times \mathbb{P}^j$ be the degree $(1,1)$ hypersurfaces, then by [8], p.39:

$$F(y_1, y_2) g'(y_1) g'(y_2) = \sum_{(i,j) \neq (0,0)} \varphi(H_{ij}) y_1^i y_2^j.$$

Proposition 2.1. *For any complex genus φ and two blowing-ups $\phi : Y \to X$ and $\phi' : Y' \to X'$ along isomorphic smooth center $Z \cong Z'$, if $N_{Z/X} \cong N_{Z'/X'}$ and there are cycles $D \subset X$, $D' \subset X'$ such that $Z.D = Z'.D'$, then $\varphi_D(X) - \varphi_{\phi^* D}(Y) = \varphi_{D'}(X') - \varphi_{\phi'^* D'}(Y')$.*

Proof. It is clear that one may reduce the proof to the case that $X' = \mathbb{P}_Z(N \oplus \mathbf{1})$ and $Y' = \mathrm{Bl}_Z \mathbb{P}_Z(N \oplus \mathbf{1})$, where N denotes the normal bundle of Z and $\mathbf{1}$ is the trivial line bundle. To get relations among these spaces, we apply *deformations to the normal cone* to the inclusion $i : Z \to X$ [6]. That is, we form the blowing-up $\Phi : M \to X \times \mathbb{P}^1$ along $Z \times \{\infty\}$. Then $X \cong M_0 \sim M_\infty = Y \cup \mathbb{P}_Z(N \oplus \mathbf{1}) =: u + v$, with $uv = \mathbb{P}_Z(N)$ and $(u + v)u = 0 = (u + v)v$. If we plug in $y_1 = f(u)$ and $y_2 = f(v)$ into the previous formula, then since $\varphi((u + v)u) = 0 = \varphi((u + v)v)$ and $u^i = (-1)^{i-1} uv^{i-1}$, $v^j = -uv^{j-1}$, we see that only the constant term of g' contributes and then

$$
\begin{aligned}
\varphi(X) &= \varphi(u + v) = \int_M f(u + v) \, K_Q(c(T_M)) \\
&= \int_M f(u + v) g'(f(u)) g'(f(v)) \, K_Q(c(T_M)) \\
&= \int_M F(f(u), f(v)) g'(f(u)) g'(f(v)) \, K_Q(c(T_M)) \\
&= \sum_{(i,j) \neq (0,0)} \varphi(H_{ij}) \varphi(u^i v^j) \\
&= \varphi(Y) + \varphi(\mathbb{P}_Z(N \oplus \mathbf{1})) + \sum_{i+j \geq 2} (-1)^{i-1} \varphi(H_{ij}) \varphi(E^{i+j-1}).
\end{aligned}
$$

Notice that this is almost a change of variable formula except the appearance of the term $\varphi(\mathbb{P}_Z(N \oplus \mathbf{1}))$. If we apply this formula to the inclusion $i' : Z \to N \to X' = \mathbb{P}_Z(N \oplus \mathbf{1})$ with $Y' = \mathrm{Bl}_Z \mathbb{P}_Z(N \oplus \mathbf{1})$, then the last term is exactly $-\varphi(Y')$. Hence in particular

$$
\varphi(X) - \varphi(Y) = \varphi(\mathbb{P}_Z(N \oplus \mathbf{1})) - \varphi(\mathrm{Bl}_Z \mathbb{P}_Z(N \oplus \mathbf{1})).
$$

If instead of the total space M we use the cycle $\Phi^*(D \times \mathbb{P}^1)$, by restricting the above computation to this cycle, we get the result. $\qquad \square$

Remark 2.2. Proposition 2.1 should be a standard fact at least when $D = X$ and $D' = X'$, and it is easily visualized in the case of complex manifolds. The above proof works in both analytic and algebraic cases.

With this proposition, it suffices to prove Theorem A for any other pair $Y' \to X'$. First of all, if $A(0) = 0$, then the whole thing localizes to $E \cong \mathbb{P}_Z(N) \to Z$ and becomes a fiber integration formula, so it does not depend on which pair $Y \to X$ we use. If $A(0) \neq 0$, we may normalize it so that $A(0) = 1$ and we may rewrite the power series $A(t)$ as $J(f(t))$, then the

equality

$$\int_{X'} K_\varphi(c(T_{X'})) = \int_{Y'} J(f(E)) K_\varphi(c(T_{Y'})) - \int_Z \mathrm{Res}$$

and Proposition 2.1 implies that $\varphi(X) - \varphi(Y) = \varphi(X') - \varphi(Y')$ equals

$$\int_{Y'} (J(f(E)) - 1) K_\varphi(c(T_{Y'})) - \int_Z \mathrm{Res} = \varphi(J(E) - 1) - \int_Z \mathrm{Res}.$$

All terms in the last expression of virtual genus involves E^i with $i \geq 1$, hence they localize to E and the whole expression equals the same expression $\varphi(J(E) - 1) - \int_Z \mathrm{Res}$ but with ambient space Y. By reversing the computations, we get the first step change of variable formula for the pair $Y \to X$. The same argument applies to the general case involving D etc. too.

So we may assume that $X = \mathbb{P}_Z(N \oplus \mathbf{1})$ with $i : Z \to X$ the embedding as the zero section $Z \to N \to \mathbb{P}_Z(N \oplus \mathbf{1})$. Then $N = i^*\mathcal{Q}$ where $\mathcal{Q}$ is the universal quotient bundle in the tautological sequence

$$0 \to \mathcal{S} \to p^*(N \oplus \mathbf{1}) \to \mathcal{Q} \to 0.$$

Here $p : \mathbb{P}_Z(N \oplus \mathbf{1}) \to Z$ is the projection map. In such a case, there are explicit formula relating $c(T_X)$ and $c(T_Y)$ ([6]; p.301, (a)):

Lemma 2.3. *If in a blowing-up $\phi : Y \to X$ along smooth center Z, the normal bundle $N = i^*\mathcal{Q}$ for some vector bundle $\mathcal{Q}$ on X, then*

$$c(T_Y) = \phi^* c(T_X) \phi^* c(\mathcal{Q})^{-1} (1 + E) c(\phi^*\mathcal{Q} \otimes \mathcal{O}(-E)),$$

where E is the exceptional divisor.

Let q_i's be the formal Chern roots of $\mathcal{Q}$ and let

$$R(t) = \sum_{k \geq 0} R_k(\phi^*\mathcal{Q}) \, t^k = Q(t) \prod_{i=1}^r Q(\phi^*q_i - t) A(t),$$

a power series with coefficients in the cohomology ring $H^*(Y, \mathbb{Q})$ (or in the Chow ring). Then

$$\int_Y A(E) \, K_Q(c(T_Y))$$

$$= \int_Y \left[Q(E) \prod_{i=1}^r Q(\phi^*q_i - E) A(E) \right] \phi^* \left(K_Q(c(T_X)) K_Q(c(\mathcal{Q}))^{-1} \right)$$

$$= \int_Y R(E) \, \phi^* \left(K_Q(c(T_X)) K_Q(c(\mathcal{Q}))^{-1} \right).$$

It is clear that the constant term $R(0) = \phi^* K_Q(c(\mathcal{Q}))$ gives rise to the main term:

$$\int_Y A(0) \, \phi^* K_Q(c(T_X)) = \int_X A(0) \, K_Q(c(T_X)).$$

Let $j : E \to Y$ be the inclusion map and $\bar{\phi} : E = \mathbb{P}_Z(N) \to Z$ be the restriction of ϕ. One has $j^* \phi^* \mathcal{Q} = \bar{\phi}^* i^* \mathcal{Q} = \bar{\phi}^* N$ and also

$$0 \to \bar{\phi}^* T_Z \to \bar{\phi}^* i^* T_X \equiv j^* \phi^* T_X \to \bar{\phi}^* N \to 0,$$

hence that $j^* \phi^* (K_Q(c(T_X)) \, K_Q(c(\mathcal{Q}))^{-1}) = \bar{\phi}^* K_Q(c(T_Z))$. If 1 is the fundamental class of E then $j_*(1) = E$ and $e = E|_E = j^* E$ is the class of $\mathcal{O}_{\mathbb{P}(N)}(-1)$. Since $\alpha \cdot E = \alpha \cdot j_*(1) = j^* \alpha$, all non-constant terms localize to E. Let $R_1(t) = \sum_{k \geq 1} R_k(\bar{\phi}^* N, r) \, t^{k-1}$. Then the remaining terms give rise to

$$\mathcal{R} = \int_E R_1(e) \, \bar{\phi}^* K_Q(c(T_Z)) = \int_Z \bar{\phi}_* R_1(e) \, K_Q(c(T_Z)).$$

To perform the fiber integration, we make use of Segre classes as in [16], Lemma 5.1:

Lemma 2.4. ([6], p.47) *Let* $s(N) = \sum s_k(N)$ *with* $s(N)c(N) = 1$, *then*
  (1) $\bar{\phi}_* e^k = 0$ *for* $0 \leq k \leq r-2$ *and*
  (2) $\bar{\phi}_* e^{(r-1)+k} = (-1)^{(r-1)+k} s_k(N)$ *for* $k \geq 0$.

Apply this Lemma we get

$$\mathcal{R} = -\int_Z \Big[\sum_{k \geq 0} (-1)^{r+k} R_{r+k} \, s_k(N) \Big] K_Q(c(T_Z)).$$

Now $R_{r+k} = R_{r+k}(N)$ is the coefficient of degree $r+k$ term in

$$R(t) = Q(t) \prod_{i=1}^r Q(n_i - t) \, A(t).$$

And as power series, $s_t(N)c_t(N) = (\sum s_k(N) \, t^k) \prod(1 + n_i t) = 1$, which when replacing t by $-1/t$, one gets

$$\Big(\sum_{k \geq 0} (-1)^k s_k(N) \, \frac{1}{t^k} \Big) \prod \Big(1 - \frac{n_i}{t} \Big) = 1.$$

This implies that $\bar{\phi}_* R_1(e) = -[\cdots]$ is $(-1)^{r+1}$ times the coefficient of degree r term of $R(t) S_{-1/t}(N)$, that is, degree -1 term of the Laurent power series in t:

$$Q(t) \prod_{i=1}^r Q(n_i - t) \, A(t) \frac{1}{t^{r+1}} \frac{1}{\prod_{i=1}^r (1 - \frac{n_i}{t})} =: \frac{A(t)}{f(t) \prod_{i=1}^r f(n_i - t)}.$$

Here we agree to denote that

$$\frac{1}{n_i - t} := \frac{-1}{t} \frac{1}{1 - n_i/t} := -\frac{1}{t} - \frac{n_i}{t^2} - \frac{n_i^2}{t^3} - \cdots \text{(finite terms)}.$$

So the pole still occurs at $t = 0$.

Notice that in all the above computations, we may replace the integrals on X, Y, Z and E by D, $\phi^* D$, $i^* D = Z.D$ and $E.\phi^* D = \bar{\phi}^*(i^* D) = \bar{\phi}^*(Z.D)$. we have thus proved Theorem A. $\qquad \square$

3. Functional equations versus differential equations

In this and next sections we give the proof of Theorem B. We will show that $S_1 \subset S_2 \subset S_3$ in this section.

For $S_1 \subset S_2$, let $r \geq 2$ be a fixed integer. By Theorem A, the defining condition of S_1 is equivalent to that, for any proper smooth variety Z (of arbitrary dimension) and any rank r vector bundle $N \to Z$, the coefficient of the degree -1 term (residue at $t = 0$) of $\dfrac{A(t, r)}{f(t) \prod_{i=1}^{r} f(n_i - t)}$ is zero. By our definition of $\dfrac{1}{n_i - t}$, in order to compute the residue at $t = 0$, we may treat n_i as distinct complex parameters and then compute the total residue at $t = 0$ and $t = n_1, \ldots, n_r$. By the Cauchy residue theorem, the total residue is given by

$$\frac{1}{\prod_{i=1}^{r} f(n_i)} - \sum_{j=1}^{r} \frac{A(n_j, r)}{f(n_j) \prod_{i \neq j} f(n_i - n_j)}.$$

So the defining property of S_1 is equivalent to the functional equations of formal power series in x_i's:

$$\frac{1}{\prod_{i=1}^{r} f(x_i)} = \sum_{j=1}^{r} \frac{A(x_j, r)}{f(x_j) \prod_{i \neq j} f(x_i - x_j)}.$$

When $r = 2$, let $A(t) = A(t, 2)$. The vanishing of the residue for arbitrary rank two bundle $N \to Z$ is then equivalent to the following power series identity:

$$\frac{1}{f(x) f(y)} = \frac{A(x)}{f(x) f(y - x)} + \frac{A(y)}{f(y) f(x - y)}.$$

This is just the functional equation defining S_2.

For $S_2 \subset S_3$, let $f(x) = \sum_{i \geq 1} f_i x^i$ with $f_1 = 1$ and $A(x) = \sum_{i \geq 0} a_i x^i$ with $a_0 = 1$. The functional equation is equivalent to FE:

$$f(x - y) f(y - x) = A(x) f(y) f(x - y) + A(y) f(x) f(y - x).$$

To solve all such f and A, our guiding principle is that an identity of this type should be closely related to the addition law of Weierstrass $\wp$ functions. Since elliptic functions of such type always satisfy certain ordinary differential equations, we will try to transform our functional equation into certain ODE's so that we can explicitly solve them. Before doing so, we notice that if $(f(x), A(x))$ is a solution, then $(e^{kx} f(x), e^{-kx} A(x))$ will also be a solution. Hence without loss of generality we may assume that $f_2 = 0$.

First of all, by differentiating FE in y and setting $y = 0$, one gets FE':

$$A(x) f(x) + a_1 f(-x) f(x) + f(-x) f'(x) = 0,$$

That is, $A(x)$ is completely determined by f and a_1. Now we plug in $A(x)$ and $A(y)$ into FE and get FE*:

$$f(x)f(y)f(x-y)f(y-x)$$
$$+\big(a_1 f(x)f(-x) + f'(x)f(-x)\big)f(x-y)f(y)^2$$
$$+\big(a_1 f(y)f(-y) + f'(y)f(-y)\big)f(y-x)f(x)^2 = 0.$$

Now we differentiate FE* in y twice and set $y = x$ to get DE-1:

$$1 + a_1\big(f(x)f'(-x) + f'(x)f(-x)\big) + f'(x)f'(-x)$$
$$+ \left(2\frac{f(-x)f'(x)^2}{f(x)} - f(-x)f''(x)\right) = 0.$$

If we differentiate FE* in y three times and set $y = 0$, we get DE-2:

$$6f_3 f(x)f(-x) + 2a_1\big(f(x)f'(-x) + f'(x)f(-x)\big) + 2f'(x)f'(-x)$$
$$+ \left(2\frac{f(-x)f'(x)^2}{f(x)} - f(-x)f''(x)\right) = 0.$$

To motivate the following calculations, notice that $\wp(x)$ is even with principal part $1/x^2$ at $x = 0$ and with no constant terms. Since

$$\frac{-1}{f(x)f(-x)} = \frac{1}{x^2}(1 - f_3 x^2 + \cdots)(1 - f_3 x^2 + \cdots) = \frac{1}{x^2} - 2f_3 + \cdots,$$

an ambitious guess will be that $P(x) := \dfrac{-1}{f(x)f(-x)} + 2f_3$ is simply $\wp(x)$! Since $\wp$ satisfies $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, by taking differentiation one gets that $\wp'' - 6\wp^2$ is a constant $-g_2/2$. Thus we would like to compute $P'' - 6P^2$. To simplify the presentation, let $g(x) = 1/f(x)$ and so $P(x) = -g(x)g(-x) + 2f_3$. Then DE-1 and DE-2 takes the form DE-1* and DE-2*:

$$g(x)^2 g(-x)^2 - a_1(g'(x)g(-x) + g'(-x)g(x))$$
$$+ g'(x)g'(-x) + g''(x)g(-x) = 0$$
$$6f_3 g(x)g(-x) - 2a_1(g'(x)g(-x) + g'(-x)g(x))$$
$$+ 2g'(x)g'(-x) + g''(x)g(-x) = 0.$$

Since the only term that is not symmetric with respect to $x \to -x$ is the last term $g''(x)g(-x)$, we must have

$$g''(x)g(-x) = g''(-x)g(x).$$

But then $(g'(x)g(-x) + g'(-x)g(x))' = 0$ because it equals

$$g''(x)g(-x) - g'(x)g'(-x) - g''(-x)g(x) + g'(-x)g'(x) = 0.$$

That is, $g'(x)g(-x) + g'(-x)g(x)$ is a constant. By expanding out the power series one sees that this constant is $6f_4$. With these understood, then

$$
\begin{aligned}
P''(x) - 6P(x)^2 &= (-g'(x)g(-x) + g(x)g'(-x))' - 6(-g(x)g(-x) + 2f_3)^2 \\
&= -g''(x)g(-x) + 2g'(x)g'(-x) - g(x)g''(-x) \\
&\quad -6g^2(x)g^2(-x) + 24f_3g(x)g(-x) - 24f_3^2.
\end{aligned}
$$

This is exactly $-6(\text{DE-1*}) + 4(\text{DE-2*}) + 12a_1f_4 - 24f_3^2$. So

$$
P''(x) - 6P(x)^2 = 12a_1f_4 - 24f_3^2,
$$

which integrates into the Weierstrass equation with $g_2 = -24a_1f_4 + 48f_3^2$. That is, there exists periods lattice Λ such that $P(x) = \wp(x)$. (When the cubic curve is singular, Λ is of rank one and $\wp$ is a trigonometric function.)

In order to determine $f(x)$, recall that

$$
g'(x)g(-x) + g'(-x)g(x) = 6f_4.
$$

Also from the derivative of the equation $-g(x)g(-x) = \wp(x) - 2f_3$ one gets

$$
-g'(x)g(-x) + g'(-x)g(x) = \wp'(x).
$$

This gives that $g'(x)g(-x) = (6f_4 - \wp'(x))/2$. Hence

$$
\frac{f'(x)}{f(x)} = -\frac{g'(x)}{g(x)} = -\frac{1}{2}\frac{\wp'(x) - 6f_4}{\wp(x) - 2f_3}.
$$

Choose z such that $\wp(z) = 2f_3$; then $-g(z)g(-z) = \wp(z) - 2f_3 = 0$. The choice of z is up to sign, we choose the one such that $g(-z) = 0$. Then $(6f_4 - \wp'(z))/2 = g'(z)g(-z) = 0$, that is, $6f_4 = \wp'(z)$. So

$$
\frac{f'(x)}{f(x)} = -\frac{1}{2}\frac{\wp'(x) - \wp'(z)}{\wp(x) - \wp(z)}.
$$

Let $a := 2f_3 = \wp(z)$, $b := 3f_4 = \wp'(z)$ and g_2 be the corresponding coefficient in the Weierstrass equation. Together with the fact that the extra factor e^{kt} contributes simply an additive constant k to $(\log f(x))'$. We see that $S_2 \subset S_3$.

Remark 3.1. (coordinates of S_3)

(1) Since $S_3 = \operatorname{Spec} \mathbb{C}[k, a, b, g_2]$, (k, a, b, g_2) is the algebraic coordinates of S_3. In this paper we have ignored the integral structure of S_3 completely. Readers interested in it may consult [16] for more details.

(2) When we represent $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $(a, b) = (\wp(z), \wp'(z))$, we obtain the analytic parameter system $(k, \omega_1, \omega_2, z)$. This will be useful in the proofs of Theorem C and D.

(3) The obvious scaling $Q(sx)$ for $s \in \mathbb{C}^\times$ all correspond to proportional genera (Chern numbers). They correspond to recalling of the lattice, so the complex elliptic genera is also usually regarded as depending on three parameters $\tau = \omega_2/\omega_1$, z and k only.

4. Complex elliptic genera under blowing-up

In this section we will complete the proof of Theorem B by showing that under the analytic parameter system, S_2 contains precisely those points with z not a 2 torsion point and S_1 contains all points with z a non-torsion point.

There are several equivalent definitions of complex elliptic genera in the literature. It depends on the choices of elliptic-like functions and the parameter systems. It turns out that our definition is very close to Krichever's [13]. Namely the complex genus φ defined by the four parameter $(k, \omega_1, \omega_2,$ and $z)$ power series

$$f(x) = e^{kx} \, e^{-\zeta(z)x} \, \frac{\sigma_{\omega_1,\omega_2}(x)\sigma_{\omega_1,\omega_2}(-z)}{\sigma_{\omega_1,\omega_2}(x-z)}.$$

To see this, one may write out the function $f(x)$ we get in last section in terms of the Weierstrass σ function. Recall that $\zeta(x) = -\int^x \wp = 1/x + \cdots$ and $\sigma(x) = e^{\int^x \zeta} = x + \cdots$, both are odd functions. Then we have the well-known formula (see eg. [2])

$$-\frac{1}{2}\frac{\wp'(x) - \wp'(z)}{\wp(x) - \wp(z)} = \zeta(x) + \zeta(z) - \zeta(x + z).$$

So $\log f(x) = \log \sigma(x) - \log \sigma(x + z) + \zeta(z)x + \lambda$. Where λ is easily seen to be $\log \sigma(z)$ by comparing coefficients. Since the solution is always up to a normalization factor e^{kx}, the general solution is thus given by

$$f(x) = e^{kx}e^{\zeta(z)x} \, \frac{\sigma(x)\sigma(z)}{\sigma(x+z)}.$$

This agrees with Krichever's definition when we replace z by $-z$. We will use our definition throughout this paper.

Recall that $\sigma(z + \omega_i) = -e^{\eta_i(z+\omega_i/2)}\sigma(z)$ with $\eta_i = \zeta(\omega_i/2)$. If $\lambda \in \mathbb{Z}$ then for $\vartheta(z) := \sigma(\lambda z + a)$, this quasi-periodicity of σ implies that

$$\vartheta(z + \omega_i) = (-1)^\lambda e^{\eta_i(\lambda^2 z + \lambda^2 \omega_i/2 + \lambda a)}\vartheta(z).$$

Hence the following well-known fact

Lemma 4.1. *The function* $\prod_{j=1}^r \frac{\sigma(\lambda_j z - a_j)}{\sigma(\mu_j z - b_j)}$ *is elliptic; that is, doubly periodic if* $\sum \lambda_j^2 = \sum \mu_j^2$, $\sum \lambda_j a_j = \sum \mu_j b_j$ *and* $\sum \lambda_j \equiv \sum \mu_j \pmod 2$.

This Lemma also holds if a_j and b_j takes values in the nilpotent elements of some commutative algebra; for example, even cohomologies or the Chow rings. In that case, the pole of $1/\sigma(\lambda z - a)$ is still at $z = 0$ according to our definition (see the end of §2).

Now we are ready to prove that S_1 (resp. S_2) contains all f's in S_3 with z a non-torsion (resp. non 2-torsion) point. In fact we will show that for φ the complex elliptic genera as defined above, the residue term in Theorem A for a blowing-up along codimension r center is zero for z not an r torsion point. Notice that when the Weierstrass equations define singular cubic curves, the periods lattice degenerates to rank one and $\wp(x)$, $\zeta(x)$ and $\sigma(x)$ are all trigonometric functions with the same defining properties as in the non-singular case.

Direct substitution shows that the residue is given by

$$\operatorname{Res}_{t=0}\left(e^{-kc_1(N)}e^{(r-1)(k+\zeta(z))t}\frac{\sigma(t+z)}{\sigma(t)\sigma(z)}\prod_{i=1}^{r}\frac{\sigma(n_i-t+z)}{\sigma(n_i-t)\sigma(z)}\,A(t,r)\right).$$

In order for this to be zero, by Lemma 4.1, if we choose (notice that $rz \notin \Lambda$)

$$A(t,r) = e^{-(r-1)(k+\zeta(z))t}\frac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)},$$

then since $-rz + \sum_{i=1}^{r}(n_i + z) - z = \sum_{i=1}^{r} n_i - z$, we conclude that the above power series is an elliptic function (with value in the Chow ring) and with $t = 0$ the only pole (notice that the factor $\sigma(t+z)$ is canceled out). The contour integration over a parallelogram domain now shows that the coefficient of the degree -1 term (the residue) must be identically zero. When the lattice degenerates to rank one, we may use the contour integral along the boundary of a thin tube and then take limits to conclude the same result. Hence the proof.

It remains to consider the case that z is a 2 torsion point. Suppose that $f(x)$ is a solution to the functional equation. Let $z = \omega/2$ for some period ω and let $e = \wp(\omega/2)$. As before we may also first assume that $k = 0$. In this case $f(x) = 1/\sqrt{\wp(x) - e}$ is an odd function (this is the real elliptic genera considered in [8]). Since $f(-x) = -f(x)$, the formula for $A(x)$ in §3 reduces to $A(x) = a_1 f(x) + f'(x)$. Plug in this into the functional equation and replace x by $-x$. After simplification we get

$$f(x+y) = f'(x)f(y) + f'(y)f(x).$$

By expanding out the power series and equating the coefficients term by term, one sees that the solution, if it exists, is uniquely determined by f_3. It is then easy to see that the general solution is given by $f(x) = \sinh(sx)/s$ with

$s^2 = f'''(0) = 6f_3 \in \mathbb{C}$. In this case the corresponding lattice is degenerate. The proof of Theorem B is completed. □

Remark 4.2. (uniqueness of $A(t,r)$)

(1) At least when the cubic curve is smooth, we expect that $A(t,r)$ is uniquely determined by f and its existence is equivalent to that z is not an r torsion point. The case $r = 3$ has been settled by J.-K. Yu using MATHEMATICA (personal communications in 2000). In general, the author do not know a proof of this for $r \geq 4$.

(2) Instead of being an universal Jacobian factor, in specific cases with $Y \to X$ fixed, if we allow $A(t,r)$ to have coefficients in cohomology classes, then the choices is no longer unique. In fact any power series $e^{(r-1)kt}B(r,t)$ with the same value as the chosen one at $t = 0$ and with $B(r,t)$ satisfying the same transformation property will do the job. We will see this non-uniqueness during the proof of Theorem C.

Remark 4.3. There is an alternative way to prove that $S_2 \subset S_3$ based on the knowledge that S_2 contains at least those f's of S_3 with z not a 2 torsion. The strategy is to compute the degree of freedom of S_2. Using the twisting $e^{kx}f(x)$, we may first normalize $f(x)$ such that $f_2 = 0$. Now we expand out the functional equation via power series in x and y. Given $d \geq 2$ and $1 \leq p \leq d - 1$, comparing the coefficient of $x^p y^{d-p}$ gives

$$\sum_{i=1}^{p-1}(-1)^i f_i f_{d-i} C_p^d = \sum_{i=1}^{p}\sum_{j=1}^{d-j} C_{p-i}^{d-i-j}\big(a_i f_j + (-1)^{d-i-j}a_j f_i\big)f_{d-i-j}$$
$$+ \sum_{j=1}^{d-p}(-1)^{d-j} f_j f_{d-j} C_p^{d-j} + \sum_{i=1}^{p}(-1)^i f_i f_{d-i} C_{p-i}^{d-i}.$$

For $d = 2, 3$, these are trivial identities. For $d = 4$, all three equations reduces to $a_2 - 3f_3 = 0$. So we may allow f_3 and a_1 to be arbitrary and then solve $a_2 = 3f_3$. With this, for $d = 5$, all the four equations are equivalent and we see that $a_3 = 2f_4 + a_1 f_3$ with f_4 arbitrary. For $d = 6$, we get

$$a_4 = 2a_1 f_4 + \frac{3}{2}f_3^2, \qquad f_5 = \frac{3}{10}f_3^2 + \frac{3}{5}a_1 f_4.$$

We want to solve $f(x)$ and $A(x)$ inductively. For $d \geq 6$, each time there appears two new coefficients a_{d-2} and f_{d-1} and with $d - 1$ relations. If we show that there are at least two independent relations among those $d - 1$'s then $f(x)$ and $A(x)$ are uniquely determined by f_3, f_4 and a_1, if they exist.

First of all, a_{d-2} occurs only for $p = 2$ or $p = d - 2$ (which by the symmetry of the functional equation correspond to the same relation) as the term

$1 \cdot a_{d-2} f_1 f_1 = a_{d-2}$. However, for all p, f_{d-1} appears in the relations with coefficients

$$C_p^d + (-1)^d C_p^d + (-1)^d C_p^{d-1} + C_{p-1}^{d-1},$$

which is always nonzero. Hence there are at least two independent relations and $f(x)$ is uniquely determined by a_1, f_3 and f_4. Moreover if $f_4 \neq 0$ then $f(x)$ is uniquely determined by f_3, f_4 and f_5. By writing out $f(x)$ which defines the complex elliptic genera with $k = 0$, we find that

$$f(x) = x + \frac{a}{2}x^3 + \frac{b}{6}x^4 + \Big(\frac{3a^2}{8} - \frac{g_2}{40}\Big)x^5 + \cdots.$$

This establishes a one to one correspondence between S_2 with $f_2 = 0$, $f_4 \neq 0$ and S_3 with $k = 0$, $b \neq 0$ — that is with z a non 2 torsion point.

In fact this research started from solving the functional equation inductively. We are confident with our approach since it has been verified in MAPLE V and MATHEMATICA up to degree 20; that $f(x)$ and $A(x)$ are uniquely solvable in f_2, f_3, f_4 and a_1.

5. The change of variable formula

Theorem 5.1. (transition formula) *Let φ be the complex elliptic genera. Let E_i', $i = 1, \ldots, p$ be p irreducible divisors in X and $e_i \in \mathbb{R} \backslash \{-1\}$. Consider a blowing-up $\phi : Y \to X$ along smooth center of codimension r with exceptional divisor E_0. Let $\phi^* E_i' = E_i + m_i E_0$ with E_i the proper transform of E_i'. Let D be an cycle in X. If $e_0 := \sum_{i=1}^{p} e_i m_i + (r-1) \neq -1$ then*

$$\int_D \prod_{i=1}^{p} A(E_i', e_i + 1) \, K_\varphi(c(T_X)) = \int_{\phi^* D} \prod_{i=0}^{p} A(E_i, e_i + 1) \, K_\varphi(c(T_Y)).$$

Proof. For the complex elliptic genera, Theorem A implies that for any power series $F(\mathbf{t}) = F(t_1, \ldots, t_p)$ and cycles D, $\mathbf{E}' = (E_1', \ldots, E_m')$ in X,

$$\int_D F(\mathbf{E}') \, K_Q(c(T_X)) = \int_{\phi^* D} F(\phi^* \mathbf{E}') \, A(E_0, r) \, K_Q(c(T_Y)).$$

(Since $\bar{\phi}_* R_1(e) = 0$.) With this, the left hand side in the theorem becomes

$$\int_{\phi^* D} \prod_{i=1}^{p} A(\phi^* E_i', e_i + 1) \, A(E_0, r) \, K_\varphi(c(T_Y)).$$

And the right hand side can be written as

$$\int_{\phi^* D} \prod_{i=1}^{p} A(\phi^* E_i' - m_i E_0, e_i + 1) \, A(E_0, e_0 + 1) \, K_\varphi(c(T_Y)).$$

Now we plug in $A(t,r) = e^{-(r-1)(k+\zeta(z))t} \dfrac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}$ and analyze the map ϕ. The dominant variable E_0 is again replaced by the variable t in the fiber integration calculation.

The extra *Jacobian factors* of both integrals have the same exponential factor

$$e^{-e_0(k+\zeta(z))t-\sum(-m_i)e_i(k+\zeta(z))t} = e^{-(e_0-\sum m_ie_i)(k+\zeta(z))t} = e^{-(r-1)(k+\zeta(z))t},$$

and also the same relevant transformation factor: for the first one, it is $e^{2\pi i(-rz+z)} = e^{-2\pi i(r-1)z}$; for the second integral, the exponent is $2\pi i$ times

$$\sum_{i=1}^p \big(m_i\phi^*E_i' + m_i(e_i+1)z\big) - \sum_{i=1}^p \big(m_i\phi^*E_i' + m_iz\big) - (e_0+1)z + z$$
$$= -(e_0 - \sum_{i=1}^p m_ie_i)z = -(r-1)z.$$

As in Remark 4.2, since both Jacobian factors become equal if we formally set $E_0 = 0$, this implies that both have the same effect in the fiber integration computation of $\bar\phi_*$ (which is zero), so both integrals are equal. $\square$

Now we prove Theorem C. Let $\phi' : X' \to X$ be a birational morphism with $K_{X'} = \phi'^*K_X + \sum_{i=1}^p e_iE_i'$. Consider a further blowing-up $\psi : Y \to X'$ along a smooth center of codimension r with $K_Y = \psi^*K_X + (r-1)E_0$. Let $\phi = \psi \circ \phi'$ and let $\psi^*E_i' = E_i + m_iE_0$. Then the canonical bundles satisfy the following relations

$$K_Y = \psi^*\Big(\phi'^*K_X + \sum_{i=1}^p e_iE_i'\Big) + (r-1)E_0$$
$$= \phi^*K_X + \sum_{i=1}^p e_iE_i + \Big(\sum_{i=1}^p e_im_i + (r-1)\Big)E_0.$$

By applying Theorem 5.1 to the blowing-up $\psi : Y \to X'$ we conclude that

$$\int_{\phi'^*D} \prod_{i=1}^p A(E_i', e_i+1)\, K_\varphi(c(T_{X'})) = \int_{\phi^*D} \prod_{i=0}^p A(E_i, e_i+1)\, K_\varphi(c(T_Y)).$$

In particular, this proves Theorem C in the case that $\phi : Y \to X$ is a composite of blowing-ups along smooth centers.

To prove Theorem C for a general birational morphism ϕ, we need to assume that $k = \mathbb{C}$ and make use of a recent result due to Wlodarsczyk and his co-workers [1] [18], namely the weak factorization theorem. It says that (in characteristic zero) any birational map $f : X \dashrightarrow X'$ can be factorized into the composite of $f_i : X_i \dashrightarrow X_{i+1}$, $i = 0, \ldots, q$ such that $X_0 = X$, $X_{q+1} = X'$ and each f_i is either a blowing-up or a blowing-down along the smooth center. We apply it to the morphism $\phi : Y \to X$.

Since the coefficient e_i in front of E_i is independent of the birational model we choose, as long as the divisor E_i has a nontrivial proper transform in that

model, they must transform correctly in all f_i. Theorem C then follows from the blowing-up case. $\qquad\square$

6. K-equivalence relation, proof of Theorem D and the statements of the main conjectures

Let X be an n dimensional complex normal $\mathbb{Q}$-Gorenstein variety. Recall that X has (at most) terminal (resp. canonical, resp. log-terminal) singularities if there is a (hence for any) resolution $\phi : Y \to X$ such that in the canonical bundle relation $K_Y =_{\mathbb{Q}} \phi^* K_X + \sum a_i E_i$, we have that $a_i > 0$ (resp. $a_i \geq 0$, resp. $a_i > -1$) for all i. Here, the E_i's vary among the prime components of all the exceptional divisors. For two $\mathbb{Q}$-Gorenstein varieties X and X', we say that X and X' are K-equivalent, written as $X =_K X'$, if there is a smooth variety Y and a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$, such that $\phi^* K_X =_{\mathbb{Q}} \phi'^* K_{X'}$. Notice that this property does not depend on the choices of Y.

To get a feeling on the objects involved, let us recall the following typical situations that lead to K-equivalence. By definition, any composite of flops induces K-equivalence. More generally, let $f : X \dashrightarrow X'$ be a birational map between two varieties with at most canonical singularities such that K_X (resp. $K_{X'}$) is nef along the exceptional locus $Z \subset X$ (resp. $Z' \subset X'$), then $X =_K X'$. In particular, this applies to birational minimal models [11] [17]. Also all cohomologically small resolutions of a singular variety, if they exist, are all K-equivalent [16].

The simplest type of flops are called *ordinary flops*. An ordinary $\mathbb{P}^r$-flop (or simply $\mathbb{P}^r$-flop) $f : X \dashrightarrow X'$ is a birational map such that the exceptional set $Z \subset X$ has a $\mathbb{P}^r$-bundle structure $\psi : Z \to S$ over some smooth variety S and the normal bundle $N_{Z/X}$ is isomorphic to $\mathcal{O}(-1)^{r+1}$ when restricting to any fiber of ψ. The map f and the space X' are then obtained by first blowing up X along Z to get Y, with exceptional divisor E a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S, then blowing down E along another fiber direction. Ordinary $\mathbb{P}^1$-flops are also called classical flops.

We now prove Theorem D. It is clear that if X and X' are K-equivalent proper smooth complex algebraic varieties, then by the change of variable formula (Theorem C), we know that they have the same complex elliptic genera at least for the parameter z not an r torsion point for $2 \leq r \leq \dim X$. But once we know that the complex elliptic genera coincide for generic z, they must coincide by continuity (or specialization). Conversely, if a complex genus φ is invariant under K-equivalence then it is invariant under classical

flops, hence by Totaro's theorem [16] it must belong to the complex elliptic genera.

Now let X be a complex $\mathbb{Q}$-Gorenstein variety with at most log-terminal singularities. Take any resolution of singularities $\phi : Y \to X$ with $K_Y =_\mathbb{Q} \phi^* K_X + \sum e_i E_i$. Since $e_i > -1$, one may simply define its complex elliptic genera to be

$$\int_Y \prod A(E_i, e_i + 1) \, K_\varphi(c(T_Y)),$$

where $A(t, r)$ is the same as before though now we plug in the variable r by rational numbers. Again this definition will cause difficulties for certain torsion values z, we avoid this problem by using the universal complex elliptic genera instead of its various specializations. In order to show that it is independent of the smooth model Y, suppose that $Y' \to X$ is another resolution, then by using the weak factorization theorem, Y and Y' are connected through blowing-ups and blowing-downs. Then Theorem 5.1 implies this independence because the coefficient e_i in front of E_i is independent of the birational model we choose.

Finally, it follows from Theorem B or Totaro's Theorem that there are no other genera which could be defined on singular varieties such that they are compatible with the change of variable formula. □

Just as in the case of birational minimal models, we expect that any K-equivalence can be decomposed into composite of some *nice flops*. However, this *rigid decomposition* seems to be very hard to achieve at this moment. Instead, we would like to state a series of conjectures on K-equivalent varieties, with the hope to reduce the necessity of a rigid decomposition result for most potential applications.

Main conjectures on K-equivalence relation. Fix a birational map $f : X \dashrightarrow X'$ between two proper smooth complex varieties and let $T := \phi'_* \circ \phi^*$ be the cohomology correspondence induced from a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$ which extends f and with smooth Y. T is determined by the closure of the graph $\bar{\Gamma}_f \subset X \times X'$ through the Künneth formula hence is independent of the choice of Y. Suppose that $X =_K X'$.

I (canonical isomorphism). T induces a canonical isomorphism on rational cohomologies, which respects the rational Hodge structures:

$$T : H^i(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X', \mathbb{Q}).$$

II (quantum cohomology/Kähler moduli). Under part I, T also induces an isomorphism on the (big) quantum cohomology rings in the sense of analytic continuations over the extended Kähler moduli spaces (compare with [15]).

III (birational complex moduli). X and X' have canonically isomorphic (at least local) complex moduli spaces. Moreover, suitably compactified polarized moduli spaces should again be K-equivalent.

IV (soft decomposition). X and X' admit symplectic deformations such that the K-equivalence relation deformed into copies of ordinary flops.

Most of these conjectures are known in dimension three based on classification theoretic results of flops in the minimal model theory [11] [15] [12]. Yet, the techniques involved are unlikely to work in higher dimensions. It seems that IV will play a key role toward the understanding of I, II and III.

Topological evidence for conjecture IV. Let Ω^U be the cobordism ring of stably almost complex manifolds. (Recall that a closed orientable manifold X is called stably almost complex if $T_X \oplus \xi$ admits a complex vector bundle structure for some trivial bundle ξ.) For any commutative ring R, an R-valued complex genus is a ring homomorphism $\varphi : \Omega^U \to R$. A theorem due to Milnor [14] says that the cobordism class is determined exactly by all the Chern numbers of the stable tangent bundle, or equivalently, determined by all its ($\mathbb{Q}$-valued) complex genera. So the above topological definition of genera is the same as the previous algebraic one. In terms of the cobordism theory, we may rephrase Theorem D in the following way:

Totaro proved that 'complex cobordism ring modulo classical flops' = 'complex elliptic genera'. Theorem D generalizes this to 'complex cobordism ring modulo K-equivalence' = 'complex elliptic genera'. That is, inside the complex cobordism ring, the ideal generated by $X - X'$ for X and X' which are related by classical flops are indeed the same as the seemingly much larger ideal generated by all $X - X'$ where $X =_K X'$. So Conjecture IV is true up to complex cobordism and, in this case, one needs only $\mathbb{P}^1$-flops instead of all ordinary flops.

Remark 6.1. In general, it is not enough to use only $\mathbb{P}^1$-flops for the decomposition problem: a $\mathbb{P}^r$-flop with $r \geq 2$ has codimension $r + 1 \geq 3$ exceptional locus which can not be the specialization of any $\mathbb{P}^1$-flops (which have codimension two exceptional loci) by dimension reason.

It is also important to remark that we do not include *Mukai flops* in IV since Huybrechts [9] has shown that such flops in hyperkähler manifolds can be deformed away in nearby complex structures. Recently Huybrechts [10] has shown that birational hyperkähler manifolds become isomorphic under some small deformations, thus our conjectures are solved in this case except that the cohomology correspondence constructed in [10] is induced from the limiting cycle of the nearby isomorphism graphs, which contains not only $\bar{\Gamma}_f$ but also certain *degenerate correspondences* in $X \times X'$.

7. Relations with equivalence of Hodge structures

The proof of the equivalence of Hodge numbers sketched in [17] uses the theory of motivic integration developed by Denef and Loeser [5]. In fact, for K-equivalent smooth complex projective varieties X and X' one has $[X] = [X']$ in a suitably completed localized Grothendieck ring of algebraic varieties $\widehat{\mathcal{M}}$. As is remarked in [5], the Hodge structure realization functor factors through this ring. Together with the fact that the category of pure $\mathbb{Q}$-Hodge structures is semi-simple, we conclude that X and X' have isomorphic $\mathbb{Q}$-Hodge structures on cohomologies. However, this does not provide any canonical morphism between them.

Hodge numbers and Hodge structures determine a substantial part of the complex elliptic genera and also give information to the complex moduli. For this, recall the formula in [16]:

$$\varphi(X) = \chi\Big(X, K_X^{\otimes(-k)} \otimes \prod_{m \geq 1} (\Lambda_{-y^{-1}q^m} T \otimes \Lambda_{-y^{-1}q^{m-1}} T^* \otimes S_{q^m} T \otimes S_{q^m} T^*)\Big).$$

Here we normalize the period lattice by $\omega_1 = 1$, $\omega_2 = \tau$, also $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$ and $T = T_X - n$ the rank zero virtual tangent bundle. The twisted χ_y-genus corresponds to the two parameter genera

$$\chi_y(X) := \chi\Big(X, K_X^{\otimes(-k)} \otimes \Lambda_y T_X^*\Big),$$

which is equivalent to knowing all $\chi(X, K_X^{\otimes(-k)} \otimes \Omega_X^p)$ for $p \geq 0$. If $n = \dim X \leq 11$, the twisted χ_y genus contains the same Chern numbers as the complex elliptic genera. So in this range, twisted χ_y genus contains precisely all Chern numbers that are invariant under the K-equivalence relation. If $n \leq 4$, the twisted χ_y genus contains all Chern numbers, so all Chern numbers are invariant under K-equivalence for dimensions up to 4.

It is clear that if K_X is trivial, that is, X is a Calabi-Yau manifold, then the twisted χ_y genus becomes Hirzebruch's χ_y genus $\sum_{p \geq 0} \chi(X, \Omega_X^p) y^p$. In particular, it is determined by the Hodge numbers. So the equivalence of elliptic genera (that is, $k = 0$) follows from the equivalence of Hodge numbers when $n \leq 11$. But when $n \geq 12$, the elliptic genera and Hodge numbers contain quite a different type of information.

For non Calabi-Yau manifolds, we can still use Hodge numbers to study twisted χ_y genus in some cases. First we show that:

Theorem 7.1. *Let X and X' be two K-equivalent smooth complex projective varieties with $D \subset X$ and $D' \subset X'$ be base point free divisors such that $\phi^* D = \phi'^* D'$ for some birational correspondence $(\phi, \phi') : X \leftarrow Y \rightarrow X'$. Then for all $\ell \in \mathbb{Z}$ and $p \geq 0$, $\chi(X, \mathcal{O}(\ell D) \otimes \Omega^p) = \chi(X', \mathcal{O}(\ell D') \otimes \Omega^p)$.*

Proof. We use induction on dimension $n = \dim X = \dim X'$. This is trivial if $n = 1$, so we may assume that the theorem is true up to dimension $n - 1 \geq 1$.

By Bertini's theorem, we may assume that D and D' are smooth, irreducible and correspond to each other under proper transform. Let $\tilde{D}$ be the proper transform of D and D' in Y with $\bar{\phi} := \phi|_{\tilde{D}}$ and $\bar{\phi}' := \phi'|_{\tilde{D}}$. Then $\bar{\phi}^* K_D = (\phi^*(K_X + D))|_{\tilde{D}} = (\phi'^*(K_{X'} + D'))|_{\tilde{D}} = \bar{\phi}'^* K_{D'}$. That is, D and D' are again K-equivalent.

We will prove by induction on $\ell \in \mathbb{N} \cup \{0\}$ that $\chi(X, \mathcal{O}(\ell D) \otimes \Omega^p) = \chi(X', \mathcal{O}(\ell D') \otimes \Omega^p)$, which is enough since they are polynomials in ℓ. For $\ell = 0$ this is true by equivalence of Hodge numbers. So let $\ell \geq 1$. From

$$0 \to \mathcal{O}((\ell - 1)D) \otimes \Omega^p \to \mathcal{O}(\ell D) \otimes \Omega^p \to \Omega^p|_D \to 0,$$

we get that

$$\chi(X, \mathcal{O}(\ell D) \otimes \Omega^p) = \chi(X, \mathcal{O}((\ell - 1)D) \otimes \Omega^p) + \chi(X, \mathcal{O}(\ell D) \otimes \Omega^p|_D).$$

By the induction hypothesis on ℓ, we only need to take care of the last term. From $0 \to T_D \to T_X|_D \to N_D \cong \mathcal{O}_D(D) \to 0$, we have that $0 \to \mathcal{O}_D(-D) \to \Omega^1|_D \to \Omega^1_D \to 0$, so $\Omega^p|_D = \Omega^p_D \oplus \mathcal{O}_D(-D) \otimes \Omega^{p-1}_D$. Hence that

$$\chi(X, \mathcal{O}(\ell D) \otimes \Omega^p|_D) = \chi(D, \mathcal{O}_D(\ell D) \otimes \Omega^p_D) + \chi(D, \mathcal{O}_D((\ell - 1)D) \otimes \Omega^{p-1}_D).$$

(For $p = 0$, it is understood that the third term is 0.) Since now $\dim D = \dim D' = n - 1$, $D =_K D'$ and $\bar{\phi}^*(D|_D) = \bar{\phi}'^*(D'|_{D'})$, the induction hypothesis on n then concludes that $\chi(X, \mathcal{O}(\ell D) \otimes \Omega^p|_D) = \chi(X', \mathcal{O}(\ell D') \otimes \Omega^p|_{D'})$. This completes the proof. $\square$

Corollary 7.2. *Let X and X' be two smooth complex projective varieties which are birational good minimal models, that is both X and X' have $K^{\otimes r}$ to be base point free for some $r \in \mathbb{N}$. Then for all $\ell \in \mathbb{Z}$ and $p \geq 0$,*

$$\chi(X, K_X^{\otimes \ell} \otimes \Omega^p) = \chi(X', K_{X'}^{\otimes \ell} \otimes \Omega^p).$$

Proof. Simply take $D = K_X^{\otimes r}$ and $D' = K_{X'}^{\otimes r}$ in the above theorem and notice that the equality holds for all $\ell \in r\mathbb{N}$ implies that it holds for all $\ell \in \mathbb{Z}$, since both terms are polynomials in ℓ. $\square$

Since birational minimal models are K-equivalent, Corollary 7.2 is just a special case of Theorem C. However, this alternative discussion has another aspect. Instead of using the Euler characteristic functor, if we write out the two corresponding long exact sequences for X and X' in the above proof, we may conclude inductively that under conjecture I,

$$H^q(X, K_X^{\otimes \ell} \otimes \Omega^p) \cong H^q(X', K_{X'}^{\otimes \ell} \otimes \Omega^p)$$

for all $\ell \in r\mathbb{N} \cup \{0\}$. It is likely that this will also hold for all $\ell \in \mathbb{Z}$. In that case we may take $\ell = -1$ and use Serre duality theorem for all $i \geq 0$ to get that

$$H^i(X, T_X) \cong H^i(X', T_{X'}).$$

(In the Calabi-Yau case, this is a direct consequence of the equivalence of Hodge structures.) We hope that this will be useful in attacking Conjecture III concerning the birational moduli spaces.

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