ASPECTS ON CALABI–YAU MODULI

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ABSTRACT. This is a personal update on some recent advances on the geometry of moduli spaces of Calabi–Yau manifolds, especially along the finite distance boundary with respect to the Weil–Petersson metric. Two main themes are metric completion and extremal transitions. Besides reviewing the known results, I will also raise some related questions.

0. INTRODUCTION

The Weil–Petersson metric g_{WP} on a complex manifold S for a polarized Kähler–Einstein family $(\mathfrak{X}, g) \to S$ is defined as, for $s \in S$,

$$g_{WP}(v, w) = \int_{\mathfrak{X}_s} (\rho(v), \rho(w))_{g_s}, \tag{0.1}$$

where $\rho : T_s S \to H^1(\mathfrak{X}_s, T_{\mathfrak{X}_s}) \cong \mathcal{H}^{0,1}_{\bar\partial}(T_{\mathfrak{X}_s})$ sends a vector v to the harmonic representative of its Kodaira–Spencer class in the Kähler–Einstein metric g_s. We always assume that $\pi : \mathfrak{X} \to S$ is effective in the sense that ρ is injective.

For a polarized family of Calabi–Yau n-folds with Ricci-flat metrics g_s given by Yau in the polarization class [48], the fact that the top holomorphic form $\Omega \in \Gamma(\mathfrak{X}_s, K_{\mathfrak{X}_s})$ is parallel with respect to g_s implies that (cf. [35])

$$g_{WP}(v, w) = \frac{\langle i(v)\Omega, i(w)\Omega\rangle}{\langle \Omega, \Omega\rangle}, \tag{0.2}$$

where $i(v)$ is the interior product by $\rho(v)$ and

$$\langle a, b\rangle = (-1)^q \sqrt{-1}^{n^2} (a, \bar b) \quad \text{for } a, b \in H^{n-q,q}$$

is the positive pairing (inner product). Hence (0.2) is an isometry between $T_s S$ and its image in $Hom(H^{n,0}, H^{n-1,1})$. Furthermore, the Bogomolov–Tian–Todorov theorem [35] asserts that the Kuranishi space of a Calabi–Yau manifold is smooth (unobstructed) of dimension $h^{n-1,1}$.

The two expressions (0.1) and (0.2) link the differential-geometric and algebrao-geometric aspects of Calabi–Yau moduli and suggest rich interactions between both techniques. However, it took a longer time than expected to make this realizable in specific problems. The purpose of this note is to review some of these developments related to my research.

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I worked on the Weil–Petersson geometry of Calabi–Yau moduli around 1994–1996 by observing the incompleteness phenomenon and its relation with degenerations acquiring canonical singularities [41]. Meanwhile I derived the formula for the Riemannian curvature tensor and analyzed its asymptotic behavior along degenerations (published much later in [44]). The original hope, partly suggested by Professor S.-T. Yau, was to obtain information on metrics and curvature to give a differential geometric proof of Viehweg's theorem on the quasi-projectivity of the moduli space of polarized Calabi–Yau manifolds (or with at most canonical singularities) [39]. The incompleteness of g_{WP} leads to serious obstructions.

A deeper thinking on it indicates the possibility to relate Viehweg's quasi-projective moduli of Calabi–Yau with at most canonical singularities with the metric completion of moduli of Calabi–Yau manifolds. In the algebraic side the needed machinery is the minimal model program (MMP in brief) in birational geometry, while in the analytic side the possible approach is via Hausdorff convergence. At that time both theories were less developed in high diemsnions and the project got stuck (cf. [42, §10]). Later on there were huge progresses made on both aspects during the last two decades.

After Shokurov's work in early 2000's on MMP in dimension 4, I wrote a note [43] explaining how the MMP leads to the equivalence between one parameter incomplete CY degenerations and the filling-in of a punctured CY family by a CY with at most canonical singularities. The MMP needed, in all dimensions, was later completed by Lai [13] and Fujino [9]. The metric completion is thus established in the one parameter case.

The idea of Hausdorff convergence in the study of degenerations of algebraic manifolds had also received significant progresses due to the works of Donaldson–Sun [5] and Rong–Zhang [26]. These were used by Tosatti [37] and Takayama [34] to deduce further geometric properties of the incomplete degenerations, e.g. its equivalence with *boundedness of diameters* in the family. I will review this in §3 and point out the further step needed to complete the study of the completion problem in its full generality.

Concerning with the curvature formula, it consists of a complex hyperbolic terms and a correction term related to the Yukawa coupling. In §2 I will review a recent construction by Sheng, Xu and Zuo [31] on the existence of maximal CY families with minimal length of Yukawa coupling, i.e. the correction terms vanish. The base of such a family is thus locally a ball quotient. It admits no maximal degenerate points nor essential finite distance degenerations. It might be possible to classify all of them.

The classification program of Calabi–Yau 3-folds in the form suggested by the Reid's fantasy [25] is build on extremal transitions which take place at incomplete degenerations. The conifold transitions are the simplest ones and a folklore conjecture expects that they form the building blocks of extremal transitions in some vague sense. The first evidence is the standard web of Green–Hubsch [10] which connects two CICY 3-folds (see §4 for the definition) by a sequence of conifold transitions. I will review in §4 an elementary proof due to S.-S. Wang [45].

For conifold transitions there is a pretty local exchange of quantum A models and B models obtained in my joint work with Y.-P. Lee and H.-W. Lin [16], which is explained in §5. Viewing conifold transitions as the semi-simple (or diagonalizable) case, our study might be useful for transitions through general canonical singularities. I end this expository survey by stating a global quantum transition via the notion of linked A and B models [16], and describing a concrete example due to T.-J. Lee and Lin [15].

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This expository article reflects only my personal experience on this subject. Due to the limitation of scope I apologize for the omission of many other important aspects and works on Calabi–Yau moduli.

1. DISTANCE

For a polarized family of Calabi–Yau n-folds $\pi : \mathfrak{X} \to S$, g_{WP} in (0.2) admits a Hodge theoretic description as the Ricci tensor of the Hodge line bundle $F^n = \pi_* K_{\mathfrak{X}/S}$ of holomorphic top forms. Namely

$$\omega_{WP} = c_1(F^n, \langle\,,\rangle) = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\langle\Omega,\Omega\rangle \tag{1.1}$$

where Ω is any local holomorphic section of F^n. g_{WP} is in general semi-positive. It is positive at T_sS if the associated VHS is effective at s in the sense that the *infinitesimal period map*

$$\sigma : T_sS \to Hom(H^{n,0}, H^{n-1,1}) \oplus Hom(H^{n-1,1}, H^{n-2,2}) \oplus \cdots \tag{1.2}$$

is injective in the first piece. This is the case if $S \subset \mathcal{M}$ is a sub-moduli.

Based on Schmid's nilpotent orbit theorem [27], I derived the following finite distance criterion when $\dim S = 1$. Denote the punctured disk by $\Delta^\times$.

Theorem 1.1. [41, Theorem 1.1] *Let $\mathcal{H} \to \Delta^\times$ be a polarized VHS of weight n with* rank $F^n = 1$. *Then $0 \in \Delta$ is at finite distance if and only if $NF^n_\infty = 0$.*

Here $F^\bullet_\infty$ is the limiting Hodge filtration and N is the nilpotent part of the monodromy operator. The extension to a higher dimensional base S, say the moduli, is still an unsolved question. Assume that we are in the typical situation that $S = (\Delta^\times)^r \times \Delta^m$, $0 \in \bar S = \Delta^n$, and $\bar S \setminus S = D_1 \cup \cdots \cup D_r$ a normal crossing divisor with nilpotent monodromy N_j along $D_j = (t_j)$.

Conjecture 1.2. *The point $0 \in \bar S$ is at finite g_{WP} distance if and only if $N_jF^n_\infty = 0$ for all j. Here $F^\bullet_\infty$ is the limiting Hodge filtration with respect to $N = \sum_{j=1}^r N_j$.*

One direction of Conjecture 1.2 is easy: If $N_jF^n_\infty = 0$ for all j, then the distance along the curve $t \mapsto (t, \ldots, t, 0, \ldots, 0)$ is finite since its monodromy is precisely $N = \sum_{j=1}^r N_j$. In fact, if the geodesic γ towards $0 \in \bar S$ lies in a holomorphic curve $C \subset S$ then the conjecture follows from the one parameter case: If C is parametrized by $t \mapsto (t^{d_j})_{j=1}^r$ in the first r coordinates with $d_j > 0$ then it has monodromy $N(\vec{d}) = \sum_{j=1}^r d_j N_j$. By [2, (3.3)], $N(\vec{d})$ has the same weight filtration as N. Hence

$$N(\vec{d})F_\infty^n = 0 \Longleftrightarrow NF_\infty^n = 0.$$

It remains to control the loci of γ. I pose it as a

Question 1.3. Is there a geodesic $\gamma \subset S$ towards $0 \in \bar{S}$ which lies in a holomorphic curve $C \subset S$? Is there a geodesic which fails this property?

By estimating the metric tensor carefully, Conjecture 1.2 for $r = 1$ can be deduced from Theorem 1.1. The case for $r = 2$ is already subtle which was only recently treated for families of Calabi–Yau 3-folds:

Theorem 1.4. [14] *Let $\pi : \mathfrak{X} \to S = (\Delta^\times)^2 \times \Delta^m$ be a polarized family of Calabi–Yau* 3*-folds. Then the distance measured by the dominant term of the Weil–Petersson potential is infinite if $N_j F_\infty^3 \neq 0$ for some $j \in \{1, 2\}$.*

I will explain the case $m = 0$ with $N_j F_\infty^3 \neq 0$, $j = 1, 2$. By the nilpotent orbit theorem, a section Ω of F^n near $0 \in \bar{S}$ takes the form

$$\Omega(t) = e^{z_1 N_1 + z_2 N_2} a(t_1, t_2) \in F_t^3$$

where $z_j = x_j + \sqrt{-1} y_j \in \mathbb{H}$ with $t_j = e^{2\pi\sqrt{-1}z_j} = e^{2\pi\sqrt{-1}x_j} e^{-2\pi y_j}$, and $a(t_1, t_2) = a_0 + \cdots$ is holomorphic with $a_0 \in F_\infty^n$.

The assumption $N_j a_0 \neq 0$, $j = 1, 2$ suggests that the potential $\langle \Omega, \Omega \rangle$ is essentially controlled by the *dominant term*

$$\langle e^{2\pi\sqrt{-1}\sum z_j N_j} a_0, e^{2\pi\sqrt{-1}\sum z_j N_j} a_0 \rangle = \langle e^{-4\pi \sum y_j N_j} a_0, a_0 \rangle =: p(y).$$

The metric constraint implies that $p(y)$ is a positive polynomial in $y \in (\mathbb{R}^+)^2$ such that $M(p) := -D^2 \log p(y)$ is positive definite for $y \in (\mathbb{R}^+)^2$.

In the one parameter case the story ends here since if $d = \deg p(y)$ then

$$M(p) = -\frac{d^2}{dy^2} \log p(y) = \frac{p'^2 - pp''}{p^2} \sim \frac{d}{y^2},$$

which is asymptotic to the Poincaré metric towards $y = +\infty$ ($t = 0$).

For $r \geq 2$, the natural question to ask is if

$$-\sum_{i,j=1}^r \partial_i \partial_j \log p(y)\, dy_i \otimes dy_j \sim \sum_{j=1}^r \frac{d_j}{y_j^2}\, dy_j^2$$

holds (where d_j is the nilpotency of N_j). Since $1 \leq d_j \leq n = 3$, for $r = 2$ there are 6 choices of (d_1, d_2) with $d_1 \leq d_2$. All such polynomials $p(y)$ were classified in [14], and all the off-diagonal terms were shown to be dominated by the diagonal ones in a case by case study.

Let d be the total degree of $p(y)$, then $d \le d_1 + d_2$. Let $\Lambda(p)$ be the convex hull of the set of exponents $(m_1, m_2) \in (\mathbb{Z}_{\ge 0})^2$ in p. We may assume that p is supported only on its higher degree boundary $\partial^+\Lambda(p)$.

For example, let $d_1 = d_2 = 1$. If $d = 1$, then $p = Ay_1 + By_2$ with $A, B > 0$,

$$M(p) = \frac{1}{p^2}\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}.$$

Let γ be a curve towards $t = 0$. Then

$$|\gamma| = \int_\gamma ds \ge \int_\gamma \frac{Ady_1 + Bdy_2}{p} = \int_\gamma d\log p = +\infty.$$

If $d = 2$, then $p = Ay_1 + By_1y_2 + Cy_2$ with $A, B, C > 0$ and

$$M(p) = \frac{1}{p^2}\begin{bmatrix} (A + By_2)^2 & AC \\ AC & (C + By_1)^2 \end{bmatrix}.$$

For large y_1 and y_2 the off-diagonal terms can be ignored. Hence we still conclude $|\gamma| \ge (\sqrt{2}^{-1} - \epsilon)\int_\gamma d\log p = +\infty$ as before.

For the other (d_1, d_2)'s, there are more terms and the major efforts are paid to get rid off the contributions from the off-diagonal terms.

When $N_1F_\infty^3 \ne 0$ but $N_2F_\infty^3 = 0$, the problem is reduced to the case $r = 1$ and $m = 1$. In this case the Weil–Petersson distance (not just the one approximated by the dominant term) is also shown to be infinite along *angular slices*. We refer the details to [14].

Remark 1.5 (Hessian geometry). Differential geometry with *hessian metrics* given by $ds^2 = \sum h_{ij}\,dy_i \otimes dy_j$, $(h_{ij}) = -D^2\log p(y)$, $p(y)$ a polynomial, deserves a systematic study (cf. [38]). It serves as the *real form* of the quasi-Hodge metrics [43] along period maps.

There is a classical recipe in [3, §1] in detecting completeness of the metric

$$h_{ij} = \frac{p_ip_j}{p^2} - \frac{p_{ij}}{p}.$$

Under the assumption that $(p_{ij}) = D^2p$ is invertible with inverse (p^{ij}), and denote by $(dp)^2 = \sum p^{ij}p_ip_j$, the inverse matrix (h^{ij}) of (h_{ij}) is given by

$$h^{ij} = -p\left(p^{ij} - \frac{p^ip^j}{(dp)^2 - p}\right).$$

We compute directly that

$$0 < \|\nabla h\|^2 = \sum h^{ij}h_ih_j = -\frac{1}{p}\left((dp)^2 - \frac{((dp)^2)^2}{(dp)^2 - p}\right) = \frac{(dp)^2}{(dp)^2 - p}.$$

Since $p > 0$, we have $(dp)^2 > p$. Moreover, for any positive function f,

$$\|\nabla h\| \le f \iff p \le (1 - f^{-2})(dp)^2. \tag{1.3}$$

For any such f and any path γ of unit speed towards ∞, we have

$$|\gamma| = \int_\gamma \|\gamma'\|\, ds \geq \int_\gamma \frac{1}{f}\,|\nabla h\cdot\gamma'|\, ds \geq \int_\gamma \frac{dh}{f}.$$

In order to conclude that $|\gamma| = \infty$ it is sufficient to prove the validity of (1.3) for f being in the one of the following form

$$f = c, \quad f = c\,h, \quad f = c\,h\log h, \quad f = c\,h\log h\log(\log h), \quad \cdots$$

near ∞, where $c > 0$ is a constant. There are obvious analogous statements for the Kähler (Weil–Petersson) metric $g_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log Q$ with $Q = \langle\Omega,\Omega\rangle$. The task is to relate the analogue of (1.3) with Conjecture 1.2.

2. CURVATURE

Based on (0.2) and Griffiths' curvature formula for Hodge bundles, the Riemannian curvature tensor for g_{WP} was calculated in

Theorem 2.1. [44, Theorem 2.1] *Let $\mathcal{H} \to S$ be an effective polarized variations of Hodge structures of weight n with $h^{n,0} = 1$. In terms of any holomorphic section Ω of $\mathcal{H}^{n,0}$, the full curvature tensor of $g_{WP} = \sum g_{i\bar{j}}\, dt_i \otimes d\bar{t}_j$ on S is given by*

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\,g_{k\bar{\ell}} + g_{i\bar{\ell}}\,g_{k\bar{j}}) + \frac{\langle\sigma_i\sigma_k\,\Omega, \sigma_j\sigma_\ell\,\Omega\rangle}{\langle\Omega,\Omega\rangle}, \tag{2.1}$$

where $\sigma_i = \sigma(\partial/\partial t_i)$ is the infinitesimal period map in (1.2).

A similar formula was derived earlier by Schumacher [29] by analytic method based on (0.1). For family of Calabi–Yau 3-folds, (2.1) is equivalent to Strominger's formula (cf. [44, Theorem 3.4])

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\,g_{k\bar{\ell}} + g_{i\bar{\ell}}\,g_{k\bar{j}}) + \sum_{p,q} g^{p\bar{q}}\,F_{pik}\overline{F_{qj\ell}},$$

where F_{ijk} is the Bryant–Griffiths cubic form

$$F_{ijk} = \frac{\int_X \partial_i\partial_j\partial_k\,\Omega\wedge\Omega}{\int_X \Omega\wedge\bar{\Omega}}.$$

Remark 2.2. For $n = 3, 4$, the Weil–Petersson metric determines the full Hodge metric g_{Hodge} which is induced from the period domain. For $n = 3$,

$$g_{Hodge} = (h^{2,1} + 3)g_{WP} + \mathrm{Ric}(g_{WP}).$$

This follows easily from (2.1) since it implies that $R_{i\bar{j}} = -(h^{2,1}+1)g_{i\bar{j}} + h_{i\bar{j}}$ where $h_{i\bar{h}}$ is the metric on $Hom(H^{n-1,1}, H^{n-2,2})$ (cf. [44, Theorem 3.2], see also [19] for a different derivation). Similarly, for $n = 4$ we have

$$g_{Hodge} = (2h^{2,1} + 4)g_{WP} + 2\mathrm{Ric}(g_{WP}).$$

In the physics literature the tensor F_{ijk} is known as *the Yukawa coupling*. Thus σ is also called the Yukawa coupling when no confusion may arise.

Definition 2.3. The length of Yukawa coupling $\ell(\pi)$ for a VHS $\pi : \mathcal{H} \to S$ is defined to be the largest integer ℓ with $\sigma_{i_1} \cdots \sigma_{i_\ell} \not\equiv 0$ for some $i_1, \ldots i_\ell$.

In the study of mirror symmetry for a (effective maximal) Calabi–Yau family $\pi : \mathfrak{X} \to S$, one needs the existence of maximal degenerate point $p_0 \in \bar{S}$ to start with. This implies that the π has maximal Yukawa coupling length $\ell(\pi) = n$. In such a case the family as well as the VHS over S is rigid by a result of Viehweg and Zuo [40, Proposition 8.2].

On the other extreme, there are examples of maximal families of Calabi–Yau manifolds $\pi : \mathfrak{X} \to \mathcal{M}$ (the moduli) which have minimal Yukawa coupling length $\ell(\pi) = 1$. By (2.1), this is equivalent to

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}).$$

That is, $(\mathcal{M}, g_{WP})$ is locally isometric to the complex hyperbolic ball.

The following construction is due to Sheng, Xu and Zuo. Let $\mathfrak{M}_{n,m}$ be the space of m-hyperplane arrangements of P^n in general positions. That is, each $A \in \mathfrak{M}_{n,m}$ is a set $A = \{H_1, \ldots, H_m\}$ of hyperplanes $H_i \subset P^n$ such that any $n+1$ members from A have empty intersection. Let $n \geq 3$ be an odd integer and $f_n : \mathfrak{X}_n \to \mathfrak{M}_{n,n+3}$ be the family of $r = \frac{1}{2}(n+3)$-fold cyclic cover $\pi_A : X_A \to P^n$ branched along the corresponding divisor $H_A = \bigcup_{i=1}^{n+3} H_i \subset P^n$. By the canonical bundle formula for branch covers,

$$K_{X_A} = \pi_A^* K_{P^n} + (r-1) H_A \sim 0.$$

Theorem 2.4. [31, Theorem 1.1] *There is a crepant resolution $\tilde{f}_n : \tilde{\mathfrak{X}}_n \to \mathfrak{M}_{n,n+3}$ which is a maximal family of Calabi–Yau n-folds with $\ell(\tilde{f}_n) = 1$.*

The most natural VHS with $\ell = 1$ comes from VHS of weight one. In fact f_n also comes from family of curves. Let $\gamma : (P^1)^n \to \mathrm{Sym}^n P^1 = P^n$ be the natural S_n Galois cover. Then γ induces a natural isomorphism

$$\Gamma : \mathfrak{M}_{1,n+3} \cong \mathfrak{M}_{n,n+3}$$

by sending $p_i \in P^1$ to $H_i = \gamma(\{p_I\} \times (P^1)^{n-1})$ [31, Lemma 3.4, 3.5]. In this way, $f_1 : \mathcal{C} = \mathfrak{X}_1 \to \mathfrak{M}_{1,n+3}$ is a family of curves C's which are r cyclic covers of P^1 at $n+3$ general points. In particular C has genus $g(C) = \frac{1}{4}(n+1)^2$. A key step towards the proof of Theorem 2.4 is the diagram

$$\begin{tikzcd} \mathcal{C}^n/G \arrow[rr, "\sim"] \arrow[dr, "h_n"'] & & \mathfrak{X}_n \arrow[dl, "f_n"] \\ & \mathfrak{M}_{n,n+3} & \end{tikzcd}$$

where $G = N \rtimes S_n$ for some abelian group N with h_n a $\mathbb{Z}/r$ Galois cover. To finish the description of these examples, I list their Hodge numbers:

Proposition 2.5. [31, Lemma 3.1] *The Hodge numbers of $H^n(\widetilde{X}_A)$ are*

$$h^{n-q,q} = \begin{cases} q+1 & \text{if } q \text{ is even,} \\ n+1-q & \text{if } q \text{ is odd.} \end{cases}$$

In particular, the family $\tilde{f}_n$ has dimension $h^1(T) = h^{n-1,1} = n = \dim \widetilde{X}_A$.

Now I briefly describe an on-going project with Sheng and Xu on characterizing maximal Calabi–Yau families π wth $\ell(\pi) = 1$. From Proposition 2.5, the dimension of complex deformations equals the complex dimension of the Calabi–Yau manifolds. We would like to know if this is the optimal case under the assumption that $\ell(\pi) = 1$.

The idea is to mimic the Bochner principle for Riemannian manifolds (M, g) with $\mathrm{Ric}(g) \geq 0$. In that case every harmonic one form is parallel whose dual is a parallel vector field. Thus we must have $h^1(M, \mathbb{R}) \leq \dim M$, with equality holds if and only if that the universal cover $(\tilde{M}, \tilde{g})$ is isometric to the Euclidean space $\mathbb{R}^{\dim M}$.

In the Calabi–Yau case $X = \mathfrak{X}_s$, $\pi : \mathfrak{X} \to S$ with $\ell(\pi) = 1$, every harmonic form $\sigma_i \Omega \in H^{n-1,1}(X)$ corresponds to a harmonic element

$$v = \sum v^{\beta}_{\bar{\alpha}} \frac{\partial}{\partial z^{\beta}} \otimes d\bar{z}^{\alpha} \in \mathcal{H}^{0,1}_{\bar{\partial}}(T_X),$$

and we ask if it a parallel section of $(T^*)^{0,1} \otimes T \cong Hom(\bar{T}, T)$. The equations $\sigma_j(\sigma_i \Omega) = 0$ for all j are then used to analyze this question.

Question 2.6. Classify maximal Calabi–Yau families π with $\ell(\pi) = 1$.

3. SINGULARITIES

For a semi-stable degeneration $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bigcup_i X_i$, the Clemens–Schmid exact sequence for MHS implies that

Theorem 3.1. [41, Theorem 2.1] *$NF^n_\infty = 0$ if and only if there is a component X_0 in $\mathfrak{X}_0$ with $h^{n,0} \neq 0$ ($\Longleftrightarrow$ there is exactly one component with $h^{n,0} = 1$ by the semi-continuity of geometric genus).*

Together with Theorem 1.1, this implies that a Calabi–Yau degeneration $\pi : \mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bar{X}$ irreducible and with at most canonical singularities is at finite g_{WP} distance [41, Corollary 2.3]. The converse follows from the minimal model theory in dimension $n + 1$:

Theorem 3.2. [43, Proposition 1.2] *Let $\pi : \mathfrak{X} \to \Delta$ is a finite distance degeneration of Calabi–Yau manifolds. Assuming MMP, then up to a finite base change and birational modifications on the central fiber, $\mathfrak{X}_0$ has only canonical singularities.*

Idea of proof. The idea is simple. Let $\pi' : \mathfrak{X}' \to \Delta$ be the semi-stable reduction of the given finite distance degeneration π. It is clear that π' is also at finite distance. By Theorem 3.1, we may write $\mathfrak{X}'_0 = \bigcup_{i=0}^N X'_i$ with X'_0 being the unique component with a canonical section $\Omega \in \Gamma(X'_0, K_{X'_0})$. Let $\pi'' : \mathfrak{X}'' \to \Delta$ be the relative minimal model of π' constructed from divisorial contractions and flips. Then the distinguished component X'_0 is never contracted during this process, for otherwise it will be covered by extremal rational curves and has Kodaira dimension $-\infty$.

The minimality of π'' easily implies that $K_{\mathfrak{X}''} \sim 0$. Also if $\mathfrak{X}''_0 = \sum_{i=0}^N X_i$ has more than one component then $-K_{X_{i,\mathrm{red}}}$ is a non-trivial effective divisor on $X_{i,\mathrm{red}}$ for each i [43, Lemma 5.2]. In particular, the component X_0 corresponding to X'_0 can not have a canonical section, hence leads to a contradiction. Finally we conclude $\mathfrak{X}''_0 = X_0$ which has at most canonical singularities. For the actual proof there are issues on normality on X_0 which needs to be taken care, which will not be repeated here. □

When Δ is replaced by a quasi-projective curve C, the MMP needed can be replaced by a weaker version, namely the semi-stable MMP for Calabi–Yau varieties, which was recently settled in [13, 9].

Instead of the algebraic method by MMP, the idea of using Hausdorff convergence to relate canonical singularity degenerations with Calabi–Yau family with *uniformly bounded diameters* was also raised in [42, §10] and [44, §7]. The starting point is that the finite distance condition $NF^n_\infty = 0$ is indeed equivalent to the continuity, or uniform boundedness, of the potential function $\langle \Omega(t), \Omega(t) \rangle$ in $t \in \Delta$ as $t \to 0$. Let ω_t be the Ricci-flat Kähler form in the polarization class L. By integrating the Monge–Ampère equation

$$\sqrt{-1}^{n^2}\, \Omega(t) \wedge \overline{\Omega(t)} = f(t)\, \omega_t^n,$$

we have $f(t) = \langle \Omega(t), \Omega(t) \rangle / (L^n) \leq M$ for some $M > 0$. Thus the finite distance is also equivalent to the point-wise estimate on $(\mathfrak{X}_t, g_t)$ for $t \neq 0$:

$$\omega_t^n \geq M^{-1}\, \Omega(t) \wedge \overline{\Omega(t)}.$$

Concerning with its equivalence to the uniform boundedness of diameters of $(\mathfrak{X}_t, g_t)$ in $t \in \Delta^\times$, denoted by $\mathrm{diam}_{g_t} \mathfrak{X}_t$, one direction was achieved by Rong and Zhang (stated in our notations):

Theorem 3.3. [26, Theorem 2.1] *Let $\mathfrak{X} \to \Delta$ be a degeneration of Calabi–Yau n-folds with $K_{\mathfrak{X}/\Delta} \sim 0$ and with ample line bundle (polarization) $\mathcal{L} \to \mathfrak{X}$. Let g_t, $t \in \Delta^\times$, be the Ricci-flat metric in the polarization class $L = c_1(\mathcal{L}|_{\mathfrak{X}_t})$. Then*

$$\mathrm{diam}_{g_t} \mathfrak{X}_t \leq 2 + c\, \langle \Omega(t), \Omega(t) \rangle,$$

where c is a constant independent of t.

When $\mathfrak{X} \to \Delta$ is a degeneration such that $\mathfrak{X}_0$ has only canonical singularities, we have $K_{\mathfrak{X}/\Delta} \sim 0$ and $\langle \Omega(t), \Omega(t) \rangle$ is continuous as $t \to 0$. Hence the diameter is uniformly bounded as expected.

Remark 3.4. In fact Theorem 3.3 serves as the a priori estimate in [26] to show the *smooth convergence* of $(\mathfrak{X}_t, g_t)$ to a *singular Ricci-flat metric* g_0 on $\mathfrak{X}_0$ which is smooth outside the singular loci. It follows from [6] that g_0 coincides with the singular metric constructed from the *complex Monge–Ampère equation with degenerate right side* in [48].

For the reverse implication, namely from the boundedness of diameters to degeneration with only canonical singularities, the machinery became mature only after the appearance of the fundamental work of Donaldson and Sun [5] on Gromov–Hausdorff limits of Kähler–Einstein manifolds. Their theory works for polarized families with *fixed total volume V*, *bounded Ricci curvature* $|\mathrm{Ric}| \le 1$, and the *volume non-collapsing property*

$$\mathrm{Vol}\, B_r \ge c\, r^{2n} \tag{3.1}$$

for any metric r ball with $r \le \mathrm{diam}\, X$. (Here $n = \dim_{\mathbb{C}} X$.) The class of such Kähler (projective) manifolds is denoted by $\mathcal{K}(n, c, V)$.

Theorem 3.5. [5, Theorem 1.2] *Given n, V and c, there exist $k, N \in \mathbb{N}$ such that any X in $\mathcal{K}(n, c, V)$ can be embedded in P^N by $\Gamma(X, L^{\otimes k})$.*

Moreover, let X_j be a sequence in $\mathcal{K}(n, c, V)$ with Gromov–Hausdorff limit X_∞. Then X_∞ is homeomorphic to a normal projective variety $W \subset P^N$. By passing to a subsequence and taking suitable projective transformations, we have $X_j \subset P^N$ converges to W in P^N.

The algebraic properties on the limit variety are discussed extensively in [5, §4.3, §4.4]. Especially the normality and the log-terminality are proved in [5, Lemma 4.12] and [5, Proposition 4.15] respectively. Thus to apply Theorem 3.5 in our setting, it is important to analyze (3.1) carefully. This has been done by Tosatti [37] and Takayama [34] recently. Below I briefly review their arguments and refer the details to their papers.

Tosatti [37, Theorem 1.1, $(e) \Leftrightarrow (f)$] made the important observation that (f) the volume non-collapsing (3.1) in a polarized family is equivalent to (e) the uniform boundedness of diameters. In fact $(e) \Rightarrow (f)$ follows from the Bishop volume comparison theorem [28, Theorem 1.3]:

$$\frac{\mathrm{Vol}_{g_t} B_r}{c_n\, r^{2n}} \ge \frac{\int_{\mathfrak{X}_t} \omega_t^n}{c_n\, (\mathrm{diam}_{g_t} \mathfrak{X}_t)^{2n}},$$

i.e. the ratio decreases in r, where $c_n = \pi^n / n!$ is the volume of unit ball in $\mathbb{C}^n$, and $(f) \Rightarrow (e)$ can be proved by covering a minimal geodesic with a bounded number of balls since the total volume V is fixed.

Based on Theorem 3.5 and the above equivalence, Takayama [34, Theorem 1.4] completed the reverse implication. In fact, for a given family $\mathfrak{X} \to \Delta$ with uniformly bounded diameters, (3.1) hold for all $\mathfrak{X}_t$, $t \ne 0$. By choosing the embedding of $\mathfrak{X}_t$ into P^N by $L_t^{\otimes k}$ for fixed k, N, he showed that the limit variety W from Theorem 3.5 indeed has at most canonical singularities and $K_W = 0$, hence it is at finite Weil–Petersson distance on the moduli [41, Corollary 2.3]. By a delicate comparison of period maps via Hilbert schemes, he then deduced that $\mathfrak{X} \to \Delta$ is a finite distance degeneration, hence admits a model (up to finite base change) such that $\mathfrak{X}_0$ has only canonical singularities. For technical reasons, Takayama stated and proved his result for families over a quasi-projective curve C instead of a disk Δ. It would be interesting to see if this requirement is really necessary.

Now we are in a position to discuss the metric completion problem mentioned in the introduction.

Convention. From now on, a Calabi–Yau variety is a projective $\mathbb{Q}$-Gorenstein variety with $K \sim 0$ and $h^1(\mathcal{O}) = 0$, unless stated otherwise. And a Calabi–Yau n-fold is always assumed to be smooth.

Let $\mathcal{M}$ be an irreducible component of the moduli space of polarized Calabi–Yau n-folds, which is a quasi-projective orbifold by Viehweg's theorem [39] and the BTT theorem mentioned before. We equip $\mathcal{M}$ with the Weil–Petersson metric g_{WP} and turn $(\mathcal{M}, d_{WP})$ into a metric space. We ask if there is a partial compactification $\mathcal{M}^c \supset \mathcal{M}$ such that the extended distance function d^c_{WP} defines a complete metric space on $\mathcal{M}^c$. The immediate candidate is Viehweg's quasi-projective moduli $\mathcal{M}_h$ of polarized Calabi–Yau varieties (X, L)'s with at most canonical singularities and with given Hilbert polynomial h. That is,

$$h(k) = h^0(X, L^{\otimes k}) \quad \text{for } k \gg 0.$$

Notice that we implicitly assume that the symbol $\mathcal{M}_h$ represents the irreducible component containing $\mathcal{M}$. Of course, by Theorem 3.2, $\mathcal{M}_h$ is considered as part of the proposed completion $\mathcal{M}^c$. But it might not be the whole $\mathcal{M}^c$. For example, starting from a (polarized) finite distance degeneration, in order to obtain a model $\mathfrak{X} \to \Delta$ such that $\mathfrak{X}_0$ has at most canonical singularities it is often necessary to change the polarization during the MMP. Thus the best one can expect seems to be the following

Conjecture 3.6. *Given an irreducible component $\mathcal{M}$ of the moduli of polarized (smooth) Calabi–Yau n-folds, there exist a finite number of $\mathcal{M}_{h_j}$'s such that*

$$\mathcal{M}^c = \bigcup\nolimits_j \mathcal{M}_{h_j} \supset \mathcal{M} \tag{3.2}$$

is complete with respect to the induced Weil–Petersson metric. Here $\mathcal{M}_{h_j}$ is an irreducible component of Viehweg's quasi-projective moduli of polarized Calabi–Yau with canonical singularities and with Hilbert polynomial h_j,

In order for (3.2) to make sense, we need a natural embedding

$$\iota_h : \mathcal{M} \hookrightarrow \mathcal{M}_h$$

whenever "$\mathcal{M}_h$ contains a point in $\mathcal{M}$". For $n = 3$, this is guaranteed by Wilson's theorem on the deformation invariance of Kähler cone [46].

Special attention needs to be paid on the jumping loci (X, L). This happens precisely when X contains a divisor E with $p : E \to C$ being a conic bundle over an elliptic curve C [46, Proposition 3.1]. Let $\psi = \Phi_{L^{\otimes k}} : X \to \bar{X}$ be the morphism contracting E to $C \subset \bar{X}$ so that $L = \psi^* \bar{L}$ for an ample line bundle $\bar{L}$ on $\bar{X}$. We still conclude that the pair $(\bar{X}, \bar{L})$ belongs to $\mathcal{M}_h$ since $\bar{X}$ has only canonical singularities.

The corresponding statement in higher dimensions are expected to hold, though a general proof does not seem to be available at this moment. Assuming it, then Conjecture 3.6 contains two major claims. The first one is purely algebraic and the second one is essentially differential-geometric:

(1) The existence of a finite cover of $\bigcup_h \mathcal{M}_h$ by $\mathcal{M}_{h_j}$'s.
(2) The completeness of the induced Weil–Petersson metric.

We emphasize that in (1) the singularities involved are not all the three dimensional canonical singularities. Otherwise there is no boundedness property to be expected. At this point, I should point our that based on the work of Donaldson and Sun [5], a version of metric completion using infinite covers by $\mathcal{M}_h$'s was studied in a recent preprint by Zhang [49].

For (2), the major problem is that so far Theorem 3.2 was proved only for one dimensional families. For families $\mathfrak{X} \to S$ with $\dim S \geq 2$, the problem is essentially reduced to its Hodge theoretic criterion, i.e. Conjecture 1.2.

An alternative approach is to use Hausdorff convergence. We pose

Question 3.7. Given a polarized Calabi–Yau degeneration $\mathfrak{X} \to S$ such that $p \in S$ is at finite Weil–Petersson distance. Do we then have the uniform boundedness of the diameter of $\mathfrak{X}_t$ for t close to p?

This is true for $\dim S = 1$ as I have explained above. But the proof relies on the Hodge theoretic criterion Theorem 1.1. So in order to avoid using Conjecture 1.2 we need to find a direct proof of it. Notice that this is unknown even for $\dim S = 1$.

According to (0.1), the finiteness of Weil–Petersson distance can be regarded as certain L^2 bound when integrating over a path γ (e.g. a geodesic) in S, and the expected uniform bound on diameters can be regarded as a C^0 (or sup norm) bound along γ. It would be interesting to see if certain iteration method as in [48] may be applied to study Question 3.7.

4. Transitions

Two (non-singular) Calabi–Yau 3-folds X, Y are connected by an extremal transition, denoted by $X \nearrow Y$, if there is a projective degeneration $\pi : \mathfrak{X} \to \Delta$ with $\mathfrak{X}_t = X$ for some $t \neq 0$, $\mathfrak{X}_0 = \bar{X}$ a singular Calabi–Yau with at most canonical singularity, and a projective contraction/resolution $\psi : Y \to \bar{X}$. Such a transition is called a terminal transition if $\bar{X}$ has only terminal singularities, and a conifold transition if $\bar{X}$ has only ordinary double points (ODP) as its singularities. An ODP has has local analytic equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

The famous Reid's fantesy in 1987 [25] suggested the possibility to connect all simply connected Calabi–Yau 3-folds by a sequence of (possibly non-Kähler) conifold transitions. We work with the Kähler (hence projective) assumption first, and leave the non-projective case to Remark 4.6.

The early evidence to this is due to Green and Hubsch saying that any two CICY 3-folds (nef complete intersection in product of projective spaces) are indeed connected by conifold transitions [10]. Notice that "nef" means the defining equations are required to be only *semi-ample* instead of *ample*. This is now well-known as the *standard web*. In such a situation the contraction ψ is always a determinantal contraction (whose definition is recalled in Definition 4.3 below). More complicate examples arise from toric geometry naturally. For example, extremal transitions for hypersurfaces in weighted projective spaces or toric varieties were studied by Morrison [22] in connection with mirror symmetry.

Transitions of non-explicit Calabi–Yau 3-folds had also been studied extensively through combinations of birational geometry and deformation theory, notably the deformation invariance of Kähler cone due to Wilson [46] and smoothing results of Namikawa–Steenbrink [23] and Gross [11] for Calabi–Yau 3-folds with terminal or canonical singularities. In contrast to the naive belief that conifold transitions are "generic" among extremal transitions, Namikawa constructed examples of terminal transitions which can not be deformed to conifold transitions:

Example 4.1. [24, Remark 2.8] Let $S \to P^1$ be a rational elliptic surface with 6 singular fibers of type II (i.e., cuspidal rational curves). Then $\bar{X} = S \times_{P^1} S$ is a Calabi–Yau 3-fold with 6 singular points of cA_2 type:

$$x^2 - y^3 = u^2 - v^3.$$

$\bar{X}$ admits smoothings to $X = S_1 \times_{P^1} S_2$ with $S_i \to P^1$ having disjoint discriminant loci, and a small resolution $\pi : Y \to \bar{X}$ exists by explicit constructions. The π-exceptional loci can not be deformed to a disjoint union of $(-1,-1)$-curves since a singular fiber of type II splits up into at most 2 singular fibers of type I, and a general fiber of small deformation of a singularity of $\bar{X}$ which preserves small resolutions has 3 ODPs.

To understand this phenomenon, it is natural to consider the possibility on factoring an extremal transition into composition of conifold transitions up to flat deformations. Such decompositions are shown to exist for Namikawa's examples by Wang [45, §6]. As a by-product, he found an elementary proof that generic determinantal contractions $\psi : Y \to \bar{X}$ in the standard web do have $\bar{X}$ being a conifold, hence offered a detailed proof to the Green–Hubsch result. To explain this, we first notice that there are simple topological constraints associated to a terminal transition $X \nearrow Y$.

Proposition 4.2. [45, Proposition 1.2] *Let $Y \to \bar{X}$ be a small resolution of a terminal 3-fold $\bar{X}$ and X a smoothing of $\bar{X}$. Then $e(Y) - e(X) \geq 2\,|\mathrm{Sing}(\bar{X})|$ with equality holds if and only if all the singularities of X are ODPs.*

Indeed, let C_i be the exceptional loci, namely an effective one cycle, over a singularity $p_i \in \bar{X}$. Since $\bar{X}$ is terminal Gorenstein, p_i must be an isolated hypersurface singularity. We have the well-known formula

$$e(Y) - e(X) = \sum_i \mu_{p_i} + \sum \left(e(C_i) - 1\right),$$

where μ_{p_i} is the Milnor number of p_i. By classification theory, p_i is a cDV (compound Du Val) singularity, hence the support of C_i is union of smooth rational curves which meet transversally and thus the number $e(C_i) - 1$ is equal to the number of irreducible components of C_i, which is denoted by

n_i. Then $\sum \mu_{p_i} + \sum n_i \geqslant 2\,|\mathrm{Sing}(X)|$ with the equality holds if and only if $n_i = \mu_{p_i} = 1$ for all i. That is, p_i is an ODP.

Definition 4.3 (Determinantal contractions/transitions). Let $Y \subset S \times P^n$ be the zero loci of sections $s_i \in \Gamma(S \times P^n, \mathcal{L}_i)$ where $\mathcal{L}_i \to S \times P^n$ are line bundles of the form $\mathcal{L}_i = L_i \boxtimes \mathcal{O}_{P^n}(1)$ with L_i being semi-ample on S.

Let $[x_0 : \cdots : x_n]$ be the homogeneous coordinates on P^n. We write

$$s_i = \sum\nolimits_{j=0}^{n} s_{ij}\, x_j, \qquad i = 0, \ldots, n,$$

where $s_{ij} \in \Gamma(S, L_i)$. We are interested in study the restriction of the projection map $\pi : S \times P^n \to S$ to Y. Define $\bar{X} = \pi(Y) \subset S$ and

$$\psi = \pi|_Y : Y \to \bar{X}.$$

For $p \in \bar{X} \subset S$, since $s_{ij}(p)$'s are fixed, $\psi^{-1}(p)$ is not unique if and only if

$$\Delta(p) := \det s_{ij}(p) = 0.$$

The contraction $\psi : Y \to \bar{X}$ is called a determinantal contraction, and the variety $\bar{X}$ is defined by equations $\Delta = 0$ on S. Notice that

$$\Delta \in \Gamma(S, \bigotimes\nolimits_{i=0}^{n} L_i).$$

If for general sections $\tau \in \bigotimes_{i=0}^{n} L_i$ the variety X_τ defined by $\tau = 0$ is smooth, then it gives rise to a transition $Y \searrow X$. If furthermore $\bar{X}$ has only ODPs as singulrities, then we get a conifold transition.

The important special case of CICY 3-folds transitions had been studied extensively in the literature. The smoothness of the generic member X_τ follows from certain Bertini type result. The statement that $\bar{X}$ is a conifold for generic $Y \to \bar{X}$ was proved via

Proposition 4.4. [45, Proposition 3.4] *Given a determinantal transition $Y \searrow X$ as in Definition 4.3 with $E = \bigoplus_{i=1}^{n+1} L_i$, the defect of Euler numbers is given by*

$$e(Y) - e(X) = 2\int_S \left(c_2(E)^2 - c_1(E)c_3(E)\right).$$

Indeed, in the case of CICY 3-fold transitions it was shown that the number of singularities on $\bar{X}$ is also calculated by the above integral when $Y \to \bar{X}$ is a *generic* determinantal contraction. Thus by Propositions 4.2 and 4.4 we conclude that $\bar{X}$ is a conifold. Since Green and Hubsch already showed that any two CICY 3-folds can be connected through a sequence of determinantal transitions, putting these together we have connected the standard web via *projective conifold transitions*.

Concerning with the program on factoring a terminal transition, the following is regarded as the first step (cf. [11, (5.1)] for the case $\rho(Y/\bar{X}) = 1$):

Theorem 4.5. [45, Theorem 1.3] *Let $\pi : Y \to \bar{X}$ be a small projective resolution of a Calabi–Yau 3-fold $\bar{X}$. If the natural closed immersion $Def(Y) \hookrightarrow Def(\bar{X})$ of Kuranishi spaces is an isomorphism then the singularities of $\bar{X}$ are ODPs. Moreover, the number of ODPs is equal to the relative Picard number $\rho(Y/\bar{X})$.*

Remark 4.6 (Non-projective transitions). Let Y be a projective (or cohomologically Kähler) Calabi–Yau 3-fold with disjoint $(-1,-1)$ curves $P^1 \cong C_i \subset Y$, $1 \le i \le r$ satisfying $\sum_{i=1}^r a_i[C_i] = 0$ with $a_i \ne 0$ for all i. Let $\psi : Y \to \bar{X}$ be the analytic map contracting all C_i's. Then works of Clemens, Friedman and Tian [8, 36] imply that the conifold $\bar{X}$ can be smoothed to a non-singular complex 3-fold X, which is non-projective if ψ is not.

This happens if $[C_i]$'s span $N_1(Y)$, and then $h^2(X) = 0$. If we assume Y (hence X) is simply connected, then by the classification theory of Wall such a non-projective Calabi–Yau 3-fold X is homotopic equivalent to $(S^3 \times S^3)^{\#k}$ for some $k \in \mathbb{N}$. Despite its topological simplicity, so far we are still short of tools for further geometric studies in this non-projective category.

The dual picture was studied by Smith, Thomas and Yau [32]. They developed the *symplectic conifold transitions* $X \nearrow Y$ by collapsing Lagrangian spheres S_i's with a good relation $\sum a_i[S_i] = 0$, $a_i \ne 0$, in $H_3(X)$, and then proving the existence of symplectic resolutions to get symplectic Y.

Question 4.7. The non-projective conifold transitions are more flexible and easier to construct. It was shown in [8, 20, 47] that for a generic CICY 3-folds $Y \subset P = \prod_{i=1}^k P^{n_i}$ defined by $m = \sum_{i=1}^k n_i - 3$ *ample divisors*, the collection of curves C_i's in Remark 4.6 can be constructed. Can one extend this to all CICY 3-folds (i.e. defined by semi-ample divisors only)?

Can one connect the web of nef CICY 3-folds in toric varieties to the standard web via (projective or non-projective) conifold transitions?

5. Invariance

In order for extremal transitions among Calabi–Yau 3-folds to admit fruitful applications, it is necessary to have an effective control on the variations of geometric data under the process. Putting in the folklore statement from string theory, one expects that two string theories based on topologically distinct Calabi–Yau 3-folds should be equally powerful to determine each other, at least when they are connected by extremal transitions. This is the notion on *invariance* which I would like to explain in this final section.

There could be several meanings of invariance we may pursue, depending on the category we are working on. However, I will skip the viewpoint from physics to avoid inappropriate interpretations (by me). From the differential geometric viewpoint, the Ricci-flat metrics do behave continuously in the Hausdorff topology under extremal transitions. This was proved by Rong and Zhang in [26] (see also Song [33] for the special case of conifold transitions). So far it is still a challenging problem to apply this continuity result to control the variations of geometric data.

I will thus focus directly on the geometric data which are interesting in current study on Calabi–Yau 3-folds. These are the $\mathcal{A}$ model, which is the Gromov–Witten theory on virtual counting of stable maps, and the $\mathcal{B}$ model, which is the Kodaira–Spencer theory on complex deformations. In the genus zero picture both admit the structure of integrable connections.

For $\mathcal{A}$ model it is the Dubrovin connection, or equivalently the quantum cohomology ring. For $\mathcal{B}$ model it is the VHS, or equivalently the Gauss–Manin connection with Griffiths' transversality on the Hodge filtration.

Now I will review a recent joint work with Y.-P. Lee and H.-W. Lin [16] concerning with the invariance of the coupled theory $(\mathcal{A}, \mathcal{B})$, linked in a suitable sense, under a conifold transition.

Let $X \nearrow Y$ be a *projective* conifold transition of Calabi–Yau 3-folds X, Y through a singular Calabi–Yau variety $\bar{X}$ with k ODPs $p_1, \ldots, p_k \in \bar{X}$. During the complex degeneration $\pi : \mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bar{X}$, there are k vanishing 3-spheres $S_1, \ldots, S_k$ with $N_{S_i/X} = T^*S^3$. And during the Kähler degeneration (small contraction) $\psi : Y \to \bar{X}$, there are k vanishing 2-spheres (exceptional curves) $C_1, \ldots, C_k$ with $N_{C_i/Y} = \mathcal{O}_{P^1}(-1)^{\oplus 2}$. Schematically these data are encoded in the following diagram:

$$\begin{tikzcd} & C_i \subset Y \arrow[d, "\psi"] \\ S_i \subset X \arrow[r, squiggly, "\pi"] & p_i \in \bar{X} \end{tikzcd}$$

Let $\mu := h^{2,1}(X) - h^{2,1}(Y) > 0$ be the loss of complex moduli and $\rho := h^{1,1}(Y) - h^{1,1}(X) > 0$ be the gain of Kähler moduli. From $\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2)$, we get the following well-known elementary relation

$$\mu + \rho = k.$$

This implies that the ψ-exceptional curve classes $[C_i] \in NE(Y/\bar{X})$ admit μ independent relations, and the π vanishing cycles $[S_i] \in V \hookrightarrow H_3(X) \to H^3(\bar{X})$ admit ρ independent relations. (The vanishing cycle space V has $\dim V = \mu$.) Let A, B be the corresponding relation matrices:

$$\begin{aligned} A &= (a_{ij}) \in M_{k\times\mu}, &\quad \textstyle\sum_{i=1}^k a_{ij}[C_i] = 0, \\ B &= (b_{ij}) \in M_{k\times\rho}, &\quad \textstyle\sum_{i=1}^k b_{ij}[S_i] = 0. \end{aligned}$$

We have the following relations on vanishing A and B cycles:

Theorem 5.1 (Basic exact sequence)**.** [16, Theorem 1.14] *The Hodge realization of $\mu + \rho = k$ is represented by an exact sequence*

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} V \to 0$$

of weight two Hodge structures.

Indeed $V \cong H^{1,1}_{\infty} H^3(X)$ in the limiting Hodge diamond for π:

$$\begin{tikzcd}
 & & H^{2,2}_{\infty} H^3 \arrow[dl, dash] \arrow[dd, "N", "\sim"'] & & \\
\mathbb{C} \cong H^{3,0}_{\infty} H^3 & H^{2,1}_{\infty} H^3 & & H^{1,2}_{\infty} H^3 \arrow[dl, dash] & H^{0,3}_{\infty} H^3 \\
 & & H^{1,1}_{\infty} H^3 & &
\end{tikzcd}$$

and the *invariant subsystem* is $Gr^W_3 H^3(X) \cong H^3(Y)$.

Based on Theorem 5.1, we may proceed to describe local quantum transitions. By the BTT unobstructedness theorem [35] and its extension to Calabi–Yau conifolds by Ran and Kawamata [12], the moduli spaces $\mathcal{M}_Y$ and $\mathcal{M}_{\bar{X}}$ are smooth of dimension $h^{2,1}(Y)$ and $h^{2,1}(X)$ respectively. Also the contraction $\psi : Y \to \bar{X}$ deforms in projective families. This then identifies $\mathcal{M}_Y$ as a codimenison μ boundary strata in $\mathcal{M}_{\bar{X}}$ and locally near $[\bar{X}] \in \mathcal{M}_{\bar{X}}$ we have $\mathcal{M}_{\bar{X}} \cong \Delta^\mu \times \mathcal{M}_Y$.

We represent $V = \mathbb{C}\langle \Gamma_1, \ldots, \Gamma_\mu \rangle$ in terms of a basis Γ_j's. It was shown in [16, Proposition 3.15] that the α-periods

$$r_j = \int_{\Gamma_j} \Omega, \qquad 1 \le j \le \mu \tag{5.1}$$

form the *degeneration coordinates* around $[\bar{X}] \in \mathcal{M}_{\bar{X}} \cong \Delta^\mu \times \mathcal{M}_Y$.

In order to describe the discriminant loci of $M_{\bar{X}}$ near $[\bar{X}]$, we recall Friedman's result on (partial) smoothing of ODP's:

Proposition 5.2. [7] *Let $w_i = a_{i1} r_1 + \cdots + a_{i\mu} r_\mu$, then the divisor $D_i := \{w_i = 0\} \subset \mathcal{M}_{\bar{X}}$ is the loci where the sphere S_i shrinks to an ODP p_i.*

It is clear that the discriminant loci $D_B = \bigcup_{i=1}^k D_i$ is not a normal crossing divisor. Rather it is a *central hyperplane arrangement*. Schmid's nilpotent orbit theorem admits a simple extension in such a situation:

Theorem 5.3. [16, Theorem 3.13] *Consider a degeneration of Hodge structures over $\Delta^\mu \times M$ with discriminant locus $\mathfrak{D}$ being a central hyperplane arrangement with axis M. Let $T^{(i)}$ be the monodromy around the hyperplane $Z(w_i)$ with quasi-unipotency m_i, $N^{(i)} := \log((T^{(i)})^{m_i})/m_i$, and suppose that the monodromy group Γ generated by $T^{(i)}$'s is* abelian.

Let $\mathbb{D}$ denote the period domain and $\check{\mathbb{D}}$ its compact dual. Then the period map $\phi : \Delta^\mu \times M \setminus \mathfrak{D} \to \mathbb{D}/\Gamma$ takes the following form

$$\phi(r, s) = \exp\left(\sum_{i=1}^k \frac{m_i \log w_i}{2\pi\sqrt{-1}} N^{(i)} \right) \psi(r, s), \tag{5.2}$$

where $\psi : \Delta^\mu \times M \to \check{\mathbb{D}}$ is holomorphic and horizontal.

In the current situation, the Picard–Lefschetz theory implies that $m_i = 1$ for all i, $N^{(i)}N^{(j)} = 0$ for all i, j, and the monodromy group is abelian. Hence Theorem 5.3 can be applied. In particular, the local section Ω near D_B can be selected as in the normal crossing case. Logically, in [16] this is done before proving the integrals in (5.1) form coordinates. The way I present it here is for ease of exposition.

Under a suitable choice of homology symplectic basis, the β-periods in the transversal directions are given by

$$u_p = \partial_p u = \int_{\beta_p} \Omega$$

for some function u. The Bryant–Griffiths–Yukawa couplings are then extended over the boundary D_B and satisfy

$$u_{pmn} := \partial^3_{pmn} u = O(1) + \sum_{i=1}^{k} \frac{1}{2\pi\sqrt{-1}} \frac{a_{ip}a_{im}a_{in}}{w_i} \tag{5.3}$$

for $1 \le p, m, n \le \mu$. It is regular if one of the indices is outside this range.

The collection $\{u_{pmn}\}$ is the essential part of the Gauss–Manin connection ∇^{GM} on $\mathcal{M}_X$ which has regular singular extension over D_B.

Similarly, let $u = \sum_{p=1}^{\rho} u^p T_p \in H^2(Y)/H^2(X)$, $D^i := \{\sum_{p=1}^{\rho} b_{ip} u^p = 0\}$, $i = 1, \ldots, k$. By the multiple cover formula of Gromov–Witten invariants, which will not be recalled here, we know that $QH(Y)$, or the Dubrovin connection on Y, is regular singular along $D^A = \bigcup D^i$.

Let $y = \sum_{i=1}^{k} y_i e_i \in \mathbb{C}^k$, with $e^1, \cdots, e^k$ being the dual basis on $(\mathbb{C}^k)^\vee$. The trivial logarithmic connection on $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee \longrightarrow \mathbb{C}^k$ is defined by

$$\nabla^k = d + \frac{1}{z} \sum_{i=1}^{k} \frac{dy_i}{y_i} \otimes (e^i \otimes e_i^*).$$

The statement $A^t B = 0$ in Theorem 5.1 leads to an orthogonal sum

$$\mathbb{C}^k = \text{image}\, A \overset{\perp}{\oplus} \text{image}\, B \cong V^* \oplus H^2(Y)/H^2(X). \tag{5.4}$$

Theorem 5.4 (Local invariance)**.** [16, Theorem 4.1] *Under* (5.4)*,*

(1) ∇^k *restricts to the logarithmic part of* ∇^{GM} *on* V^*.
(2) ∇^k *restricts to the logarithmic part of* ∇^{Dubrovin} *on* $H^2(Y)/H^2(X)$.

Theorem 5.4 provides evidence to

"excess $\mathcal{A}$ theory" + "excess $\mathcal{B}$ theory" = "trivial"

through the partial exchange of quantum information attached to *vanishing cycles* on both the $\mathcal{A}$ and $\mathcal{B}$ theories.

Remark 5.5 (Duality for vanishing cycles). The mirror of a CY 3-folds transition $X \nearrow Y$ through $\bar{X}$ is conjecturally to be another transition $Y^\vee \nearrow X^\vee$ through some $\bar{Y}$, where $X^\vee$ (resp. $Y^\vee$) is the mirror of X (resp. Y) [22, 1].

For conifold transitions, the basic exact sequence in Theorem 5.1 is compatible with mirror symmetry in the sense that its mirror corresponds to the dual sequence. Then the roles of vanishing $\mathcal{A}$ cycles and vanishing $\mathcal{B}$ cycles, as well as the matrices A and B, switch under duality.

Since the vanishing $\mathcal{A}$ cycles are $(-1,-1)$ curves, which are supersymmetric cycles, mirror symmetry then suggests that the vanishing $\mathcal{B}$ cycles should also be supersymmetric. That is, the vanishing sphere S_i should be representable by a special Lagrangian sphere, and hence is rigid by a theorem of McLean [21]. So far we only know that S_i can be chosen to be Lagrangian by a result of Seidel and Donaldson [30].

Remark 5.6 (Extremal transitions). Suppose that in the extremal transition of CY 3-folds $X \nearrow Y$ the variety $\bar{X}$ has more general canonical singularities, and assume that the contraction $\psi : Y \to \bar{X}$ can be deformed to $\psi_t : Y_t \to \bar{X}_t$ such that $\bar{X}_i$ is a conifold. This is equivalent to that the simultaneous contraction Ψ over $\mathcal{M}_Y$

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\Psi"] \arrow[dr] & & \overline{\mathcal{X}} \arrow[dl] \\ & \mathcal{M}_Y & \end{tikzcd}$$

is generically a contraction of $(-1,-1)$ curves to a conifold. In this case, there are natural extensions of the basic exact sequence (Theorem 5.1) and the local invariance (Theorem 5.4) to family versions. There is also an extension to extremal transitions which are compositions of conifold transitions up to flat deformations in the above sense.

For the full information on quantum $\mathcal{A}$, $\mathcal{B}$ theories, we proved the following result:

Theorem 5.7. [16, Theorem 0.3] *Let* $[X]$ *be a nearby point of* $[\bar{X}]$ *in* $\mathcal{M}_{\bar{X}}$,

(1) $\mathcal{A}(X)$ *is a sub-theory of* $\mathcal{A}(Y)$ *(e.g. quantum sub-ring in genus 0).*
(2) $\mathcal{B}(Y)$ *is a sub-theory of* $\mathcal{B}(X)$ *(invariant sub-VHS).*
(3) $\mathcal{A}(Y)$ *can be reconstructed from a "refined* $\mathcal{A}$ *theory" on*

$$X^\circ := X \setminus \bigcup_{i=1}^k S_i$$

"linked" by the vanishing spheres in $\mathcal{B}(X)$.

(4) $\mathcal{B}(X)$ *can be reconstructed from the variations of MHS on* $H^3(Y^\circ)$,

$$Y^\circ := Y \setminus \bigcup_{i=1}^k C_i,$$

"linked" by the exceptional curves in $\mathcal{A}(Y)$.

Again, due to the "$\mathcal{B}$ model nature" of this survey article, I will skip the aspect on Gromov–Witten theory completely.

To sketch the idea of proof to Theorem 5.7 (4), we go back to (5.2) and notice that the exponential term is completely determined by the relation matrix A of $[C_i]$'s. This might not be completely obvious at first sight for the monodromy matrix $N^{(i)}$ along D_i, which is indeed determined by A completely from the Yukawa coupling expression (5.3) (see [16, Corollary 3.18] for the precise formula of $N^{(i)}$).

To determine the VHS $\phi(r,s)$ from (5.2), we need to understand the holomorphic horizontal map $\psi(r,s)$. It turns out that $\psi(r,s)$ is determined by the "period map" of the VMHS on $H^3(Y^\circ)$. In this manner we regard this as a refined $\mathcal{B}$ model on Y linked by the exceptional curves C_i's.

Instead of explaining the technical part of the proof, I will include an explicit example for Theorem 5.7 (4) due to T.-J. Lee and H.-W. Lin to demonstrate the effective computability of the linked theory:

Example 5.8. [15] Consider a conifold transition $X \nearrow Y$ of (anti-canonical) Calabi–Yau hypersurfaces arising from toric degenerations (cf. [1]):

$$\begin{tikzcd} & Y \subset \hat{P} = \hat{P}(2,4)\ . \arrow[d, "\Psi"] \\ X \subset G = G(2,4) \arrow[r, rightsquigarrow] & P(2,4) \end{tikzcd}$$

The $\mathcal{B}$ model on X is governed by the tautological systems τ_G recently introduced by Lian, Song and Yau in [17, 18], while the $\mathcal{B}$ model on Y is govern by the so called extended GKZ system $\tau_{\hat{P}}$ as shown in [15]. In fact they proved that the extended GKZ can be regarded as a tautological system in this special case.

The goal is to determine τ_G from $\tau_{\hat{P}}$ and the collection of rational curves C_i's contracted by $\psi = \Psi|_Y$.

For τ_G, the symmetry operators come from $\mathrm{SL}(4,\mathbb{C})$, which has $16 - 1 = 15$ dimensions. It consists of 12 roots and 3 torus action.

For $\tau_{\hat{P}}$, the extended GKZ has symmetry $\mathrm{Aut}^0(\hat{P})$ generated by T^4 and 14 "roots". Here for a toric variety defined by a fan Σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$, the set of roots are defined by Cox [4] to be

$$R(\Sigma, N) = \{\alpha \in M \mid \exists p \in \Sigma_1, (\alpha, p) = -1,\ (\alpha, p') \geq 0 \quad \forall p' \neq p\}.$$

Now the 2 roots $\pm(1,1,1,1)$ are discarded since they do not preserve Ψ. The remaining 12 roots then correspond to the 12 roots in τ_G under suitable coordinate transformations. Thus $(\tau_{\hat{P}}, \bigcup C_i)$ determine τ_G.

Question 5.9. Generalize Theorem 5.3 to VHS associated to a smoothing family $\mathfrak{X} \to S$ of a Calabi–Yau 3-fold with at most terminal or canonical singularities. The simple splitting expression of nilpotent orbit theorem (5.2) no longer exists since the monodromy term and the deformation term for open space "linked with each other" in a deeper level. It might be related to the local fundamental group of the singualrity.

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