Complex Analysis II 4/7, 2015

Jacobi's Theta Functions $q = e^{\pi i \tau}$ (just like e_i, λ, level 2 str is used)

$\vartheta_3(z,\tau) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z}$ $\quad\vartheta_4(z,\tau) := \vartheta_3(z+\frac{1}{2},\tau)$

$\vartheta_1(z,\tau) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2}) z} = -i\, e^{\pi i z + \pi i \tau/4}\, \vartheta_4(z+\frac{\tau}{2},\tau)$

$\vartheta_2(z,\tau) := \vartheta_1(z+\frac{1}{2},\tau)$ $\quad\to$ simple zero at Λ_τ :

$$* \begin{cases} \vartheta_1(z+1) = -\vartheta_1(z) \\ \vartheta_1(z+\tau) = -q^{-1} e^{-2\pi i z}\, \vartheta_1(z) \end{cases}$$

Heat Eq'n: $4\pi i \frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial z^2}$

$\frac{1}{2\pi i}\int_{\partial P} \frac{\vartheta'}{\vartheta} = 1.$

	ϑ_1	ϑ_2	ϑ_3	ϑ_4
1	-1	-1	1	1
τ	$-N$	N	N	$-N$

$N = q^{-1} e^{2\pi i z}$

Product Expansion:

$$\vartheta_4(z) = \prod_{n=1}^{\infty}(1-q^{2n}) \cdot \prod_{n=1}^{\infty}(1-q^{2n-1}e^{2\pi i z})(1-q^{2n-1}e^{-2\pi i z}) =: G \cdot f(z)$$

Pf: f & ϑ_4 both satisfy $*$, same zeros, thus $\exists\, G$ constant in z.

Similarly $\vartheta_3(z) = G \cdot \prod (1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})$,

$\vartheta_1 = 2G q^{1/4} \sin\pi z \prod (1-q^{2n}e^{2\pi i z})(1-q^{2n}e^{-2\pi i z})$,

$\vartheta_2 = 2G q^{1/4} \cos\pi z \prod (1+q^{2n}e^{2\pi i z})(1+q^{2n}e^{-2\pi i z})$

Jacobi's Triple Product:

$\vartheta_1'(0) = \pi\, \vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$

Caution: In many books, including Whittaker-Watson, e^{2inz} is used instead of $e^{2\pi i n z}$

Thus we have the additional π factor

Pf: $\vartheta_2'(z) = \vartheta_2(z)\left[-\pi\tan\pi z + \sum \frac{2\pi i\, q^{2n} e^{2\pi i z}}{1+q^{2n}e^{2\pi i z}} + \frac{-2\pi i\, q^{2n} e^{-2\pi i z}}{1+q^{2n}e^{-2\pi i z}}\right] \Rightarrow \vartheta_2'(0) = 0$

$\Rightarrow \vartheta_2''(0) = \vartheta_2(0)\left[-\pi^2 + 8\pi^2 \sum \frac{q^{2n}}{(1+q^{2n})^2}\right]$

Also, $\vartheta_3''(0) = \vartheta_3(0)\, 8\pi^2 \sum \frac{q^{2n-1}}{(1+q^{2n-1})^2}$; $\vartheta_4''(0) = \vartheta_4(0)\left[-8\pi^2 \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right]$

Similarly, $\vartheta_1(z) = \sin\pi z \cdot \phi(z)$, $\phi'(0) = 0$, $\phi''(0) = \phi(0)\cdot 8\pi^2 \sum \frac{q^{2n}}{(1-q^{2n})^2}$

$\vartheta_1'(0) = \pi\phi(0)$, $\vartheta_1''(0) = 0$, $\vartheta_1'''(0) = -\pi^3\phi(0) - 3\pi\phi''(0)$

$$\Rightarrow \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = -\pi^2 - 24\pi^2 \sum \frac{q^{2n}}{(1-q^{2n})^2} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}$$

by clever arrangement of the RHS 3 series.

Heat eq'n $\Rightarrow$

$$\frac{\partial_\tau \vartheta_1'(0)}{\vartheta_1'(0)} = \frac{\partial_\tau \vartheta_2(0)}{\vartheta_2(0)} + \frac{\partial_\tau \vartheta_3(0)}{\vartheta_3(0)} + \frac{\partial_\tau \vartheta_4(0)}{\vartheta_4(0)}$$

$\Rightarrow \vartheta_1'(0) = C\, \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$. $\quad q \to 0 \Rightarrow C = \pi$.

$\vartheta_1'(0) \to \pi q^{1/4}$, $\vartheta_2(0) \to -i q^{1/4}$, $\vartheta_3(0) \to 1$, $\vartheta_4(0) \to i$ by the original def'n in series. □

Completion of the pf of prod. formula: $G = \prod_{n=1}^{\infty}(1-q^{2n})$.

ϑ-prod. formula $\Rightarrow$ $2\pi\, G\, q^{1/4} \prod (1-q^{2n})^2$

$= \pi \cdot 2G\, q^{1/4} \prod (1+q^{2n})^2 \cdot G \prod (1+q^{2n-1})^2 \cdot G \prod (1-q^{2n-1})^2$

Clever cancelation $\Rightarrow$ $\prod (1-q^{2n})^2 = G^2$ $\Rightarrow$ $G = \prod (1-q^{2n})$. $\square$

(for $q \to 0$ $\vartheta_3(0,\tau) \to 1$ hence $G(0) = 1$.)

Ex. For $\omega_1 = 1$, $\omega_2 = \tau$; $\eta_1 = -\frac{1}{3} \cdot \frac{\vartheta_1'''(0)}{\vartheta_1'(0)}$; $\sigma(z) = e^{\eta z^2/2} \cdot \frac{\vartheta_1(z)}{\vartheta_1'(0)}$

Q: So, why is the theta theory better than Weierstrass theory?

Modularity: Jacobi's imaginary transformation ($\vartheta_3 \equiv \Theta$ in Stein, p. 290)

(J) $\vartheta_3(z,\tau) = \sqrt{i/\tau}\; e^{-\pi i z^2/\tau}\, \vartheta_3(-z/\tau, -1/\tau)$ where $\sqrt{i/\tau} \in \mathbb{H}$.

This is proved by Poisson summation for $z \in \mathbb{R}$, $\tau = it$, then analy. conti.

4/9 Thm: For $\eta(\tau) := q^{1/12} \prod_{n=1}^{\infty}(1-q^{2n})$, $\eta(-1/\tau) = \sqrt{-i\tau}\; \eta(\tau)$.

Hence Δ "=" $\eta(\tau)^{24}$ is a $SL(2,\mathbb{Z})$ cusp form of wt 12.

Will prove it based on Jacobi's triple product formula
+ imaginary transformation

PS Since $\dim M_{12}^0 = 1$, we have $\eta(\tau)^{24} \,/\!/\, g_2^3(\tau) - 27 g_3^2(\tau)$

Usual convention $\Delta := g_2^3 - 27 g_3^2 = \frac{(2\pi)^{12}}{1728}\, \eta(\tau)^{24}$

Here $q^2 = e^{2\pi i \tau}$ = "the q" for $SL(2,\mathbb{Z})$

Pf (Stein, p. 292 ; cf. Serre : A course in Arithmetic)

$\vartheta_3(z,\tau) = (1 + q\, e^{-2\pi i z}) \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n+1}e^{-2\pi i z})$

$\quad = 0$ at $z_0 = \frac{1}{2} + \frac{\tau}{2}$ Set $H(\tau) = \prod (1-q^{2n})^3$

$\Rightarrow$ $\vartheta_3'(z_0,\tau) = 2\pi i\; H(\tau)$

Do this in (J) at z_0, get $2\pi i\, H(\tau) = \sqrt{i/\tau}\; e^{-\pi i (\frac{1}{2}+\frac{\tau}{2})^2 \frac{1}{\tau}} \frac{-1}{\tau} \cdot 2\pi i \cdot H(-1/\tau)$

i.e. $e^{\frac{\pi i \tau}{4}} H(\tau) = \left(\frac{i}{\tau}\right)^{3/2} e^{-\frac{\pi i}{4}\frac{1}{\tau}}\, H(-1/\tau)$.

For $\tau = it$, $t > 0$, $\eta(\tau) > 0$, hence the cubic root gives the thm. $\square$

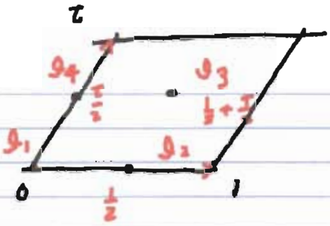
Ex For any $g_2, g_3 \in \mathbb{C}$ with $\Delta \neq 0$, $\exists!\, \Lambda$. (Hint : use j line)

Rmk: The modularity of ϑ_i can also be proved by comparing zeros (Liouville thm) and the triple product formula.

Easy formulae: $\vartheta_i^2(z+1) = \vartheta_i^2(z)$, $\vartheta_i^2(z+\tau) = q^{-2} e^{4\pi i z}\, \vartheta_i^2(z)$

choose a, b st. $\frac{a\vartheta_2^2(z) + b\,\vartheta_3^2(z)}{\vartheta_4^2(z)}$ has only simple pole $\Rightarrow$ a constant.

eg. $\vartheta_4^2(0)\, \vartheta_4^2(z) = \vartheta_3^2(0)\, \vartheta_3^2(z) - \vartheta_2^2(0)\, \vartheta_2^2(z)$. $z=0 \Rightarrow$ $\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4$



Pf Solving linear combinations

$$a\,\vartheta_2^2(z) + b\,\vartheta_3^2(z) = \vartheta_4^2(z)$$

- $z = \frac{1}{2} \Rightarrow b\,\vartheta_3^2(\tfrac{1}{2}) = b\,\vartheta_4^2(0) = \vartheta_4^2(\tfrac{1}{2}) = \vartheta_3^2(0)$
- $z = \frac{1}{2}+\frac{\tau}{2} \Rightarrow a\,\vartheta_2^2(\tfrac{1}{2}+\tfrac{\tau}{2}) = a\,\vartheta_1^2(\tfrac{\tau}{2})$

$$= -a\cdot e^{3\pi i\tau/2}\,\vartheta_4^2(\tau) = -a\,e^{-\pi i\tau/2}\,\vartheta_4^2(0) = \vartheta_4^2(\tfrac{1}{2}+\tfrac{\tau}{2}) = e^{-\pi i\tau/2}\,\vartheta_2^2(0) \qquad *$$

Addition formulae. Similarly, for

$$\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_3^2 = \vartheta_3^2(y)\,\vartheta_3^2(x) + \vartheta_1^2(y)\,\vartheta_1^2(x).$$

Fundamental Relations :

consider

$$2w' = -w+x+y+z \qquad [r] := \vartheta_r(w)\,\vartheta_r(x)\,\vartheta_r(y)\,\vartheta_r(z)$$
$$2x' = +w-x+y+z \qquad [r]' := \vartheta_r(w')\,\vartheta_r(x')\,\vartheta_r(y')\,\vartheta_r(z')$$
$$2y' = +w+x-y+z$$
$$2z' = +w+x+y-z \qquad N := q^{-1}e^{-2\pi i z}$$

Then,

	$[3]$	$[1]'$	$[2]'$	$[3]'$	$[4]'$
$+1$	$[3]$	$-[2]'$	$-[1]'$	$[4]'$	$[3]'$
$+\tau$	$N[3]$	$-N[4]'$	$N[3]'$	$N[2]'$	$-N[1]'$

the period transf. factors viewed as functions in z

eg $[1]' = \vartheta_1(\frac{1}{2}(-w+x+y+z))\,\vartheta_1(\frac{1}{2}(w-x+y+z))$

$\Rightarrow$ $+1$ or $+\tau$ really corr to $+1/2$, $+\tau/2$ in ϑ_1

Rmk : More general quartic identities holds for $A \in M_n(\mathbb{Z})$
$A^tA = m^2 I_n$ Here $m=2$, $n=4$ (Due to Riemann)

$[3]$ & $-[1]'+[2]'+[3]'+[4]'$ have the same factors 1 & N

Ratio has only one simple pole $\Rightarrow$ $A[3] = -[1]'+[2]'+[3]'+[4]'$.

Symmetry consideration $\Rightarrow$ A indep of w, x, y, z.

Set $w=x=y=z=0$ $\Rightarrow$ $A\,\vartheta_3^4 = \vartheta_2^4 + \vartheta_3^4 + \vartheta_4^4$, ★ $\Rightarrow$ $A = 2$.

Now set $w=x=y=z$, get
$w'=x'=y'=z'=z$ $\Rightarrow$ $\underline{\vartheta_1^4(z) + \vartheta_3^4(z)} = \underline{\vartheta_2^4(z) + \vartheta_4^4(z)}$.

the product form is rather interesting

Rmk : Other specializations give addition formulae.

Projective Embedding : Back to the "easy formulae"

$$\vartheta_4^2\,\vartheta_4^2(z) = \vartheta_3^2\,\vartheta_3^2(z) - \vartheta_2^2\,\vartheta_2^2(z)$$
$$\vartheta_4^2\,\vartheta_1^2(z) = \vartheta_2^2\,\vartheta_3^2(z) - \vartheta_3^2\,\vartheta_2^2(z)$$

define an algebraic curve $C \hookrightarrow \mathbb{CP}^3$

Theorem : $\varphi : \mathbb{C}/\Lambda_\tau \longrightarrow \mathbb{CP}^3$, $z \mapsto (\vartheta_1(2z) : \vartheta_2(2z) : \vartheta_3(2z) : \vartheta_4(2z))$
is an Isomorphism $\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} C$ as Riemann surfaces.

$2z$ to have same transf. law.

Sketch : The plane $a_1x_1 + \cdots + a_4x_4 = 0 \cap C$ at most 4 pts.
But $a_1\vartheta_1(2z) + \cdots + a_4\vartheta_4(2z) = 0$ does have 4 sol. as in $*$. $\square$

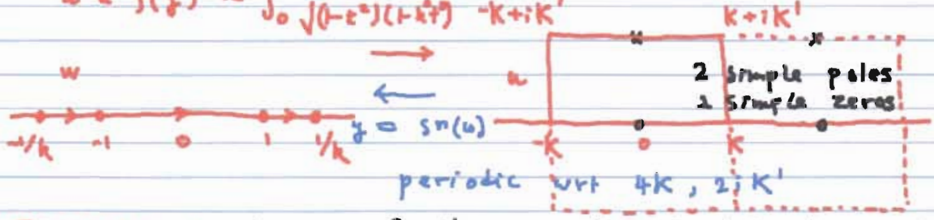
4/14

Jacobi Elliptic Functions recall for $0 \le k \le 1$: (modulus, moduli)

$u = f(y) := \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$

$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2)$ ← an old exercise

$K' = \int_1^{1/k}$



Question : A general theory of $sn(u,k)$ for all $k \in \mathbb{C}$?
Need explicit solution just like $y = \sin u$. (the case $k=0$)

Basic Ansatz : $\xi := \frac{\vartheta_1(z)}{\vartheta_4(z)}$ satisfies $\left(\frac{d\xi}{dz}\right)^2 = \pi^2(\vartheta_2^2 - \vartheta_3^2\xi^2)(\vartheta_3^2 - \vartheta_2^2\xi^2)$.

pf : $\xi' = \frac{1}{\vartheta_4^2(z)}\left(\vartheta_1'(z)\vartheta_4(z) - \vartheta_1(z)\vartheta_4'(z)\right)$ and $\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)}$ have the same periodicity factors $-1, 1$ wrt $1, \tau$

$\Rightarrow \phi(z) := \left(\vartheta_1'(z)\vartheta_4(z) - \vartheta_1(z)\vartheta_4'(z)\right)/\vartheta_2(z)\vartheta_3(z)$ is elliptic, poles $\frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}$

Key point : ϕ is periodic in $\frac{\tau}{2}$ indeed !

Q : this requires really explicit calculations, any theoretic reason ?

check : $\vartheta_1(z + \frac{\tau}{2}) = i\, q^{-1/4} e^{-\pi i z}\, \vartheta_4(z) = i M \vartheta_4(z)$ by def.

$\vartheta_4(z + \frac{\tau}{2}) = i M \vartheta_1(z)$ $\vartheta_2(z + \frac{\tau}{2}) = M \vartheta_3(z)$, $\vartheta_3(z + \frac{\tau}{2}) = M\vartheta_2(z)$.

$\Rightarrow$ the differentiation terms to $e^{-\pi i z}$ cancel out

$\Rightarrow$ only one simple pole at $\frac{1}{2}$ wrt $1, \frac{\tau}{2}$ $\Rightarrow$ constant function

$\vartheta_1'(0) = \pi \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$ $\Rightarrow$ $\phi(z) = \pi \vartheta_4^2$

$$\Rightarrow (\xi')^2 = \pi^2 \vartheta_4^2 \frac{\vartheta_2^2(z)}{\vartheta_4^2(z)} \frac{\vartheta_3^2(z)}{\vartheta_4^2(z)} = \pi^2 \frac{\vartheta_2^2\vartheta_4^2(z) - \vartheta_3^2\vartheta_1^2(z)}{\vartheta_4^2(z)} \frac{\vartheta_3^2\vartheta_4^2(z) - \vartheta_2^2\vartheta_1^2(z)}{\vartheta_4^2(z)} \quad \square$$

translate the proj imb eq'n by 1/2

$\Rightarrow$ For $y = \frac{\vartheta_3}{\vartheta_2}\xi$, $u = \pi\vartheta_3^2 z$, get $\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2)$
and $k^{1/2} = \vartheta_2/\vartheta_3$.

pf : $\left(\frac{dy}{du}\right)^2 = \frac{\vartheta_3^2}{\vartheta_2^2}\left(\frac{d\xi}{dz}\right)^2 \frac{1}{\pi^2\vartheta_3^4} = \left(1 - \frac{\vartheta_3^2}{\vartheta_2^2}\xi^2\right)\left(1 - \frac{\vartheta_2^2}{\vartheta_3^2}\xi^2\right)$. $\square$

That is, $y = sn(u) = \frac{\vartheta_3}{\vartheta_2} \cdot \frac{\vartheta_1}{\vartheta_4}\left(\frac{u}{\pi\vartheta_3^2}\right)$. do not confuse with constants !

quasi-periods $\pi\vartheta_3^2 =: 2K$, $\tau\pi\vartheta_3^2 =: 2iK'$; periods $4K, 2iK'$.

Now set $cn(u,k) = \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2}{\vartheta_4}\left(\frac{u}{\pi\vartheta_3^2}\right)$, $dn(u,k) = \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3}{\vartheta_4}\left(\frac{u}{\pi\vartheta_3^2}\right)$ (2 cos's)

Then :

1) $sn'(u) = cn(u)\, dn(u)$
2) $sn^2(u) + cn^2(u) = 1$
3) $k^2 sn^2(u) + dn^2(u) = 1$
4) $cn(0) = dn(0) = 1$

Additional law : Ex Show that, via 1) – 4)

$$sn(u+v) = \frac{sn\,u \cdot cn\,v\, dn\,v + sn\,v \cdot cn\,u\, dn\,u}{1 - k^2 sn^2 u\, sn^2 v} \quad \text{etc}$$

1) – 4) determine the theory.

Some Easy consequences as in trigonometric functions

$2\cdot sn(u)\cdot sn'(u) + 2\,cn(u)\,cn'(u) = 0 \Rightarrow cn'(u) = -sn(u)\,dn(u)$

Similarly, $dn'(u) = -k^2\,sn(u)\,cn(u)$

Using 2), 3) & $sn(u+v)$, get $\quad cn(u+v) = \dfrac{\overset{dn}{cn\,u}\cdot \overset{dn}{cn\,v} - \overset{k^2}{sn\,u\,sn\,v}\cdot \overset{cn}{dn\,u}\,\overset{cn}{dn\,v}}{1-k^2 sn^2u\, sn^2 v}$, $\overset{dn}{\text{and}}$

Complementary modulus k & k', and inversion problem

$\lambda(\tau) = k^2 := \dfrac{\vartheta_2^4(0,\tau)}{\vartheta_3^4(0,\tau)} \Rightarrow \tau$ exists unless $k^2 = 0, 1$ (stronger than WW p.481–484)*

then we set $k^2 + k'^2 = 1$, ie. $k'^2 = 1 - \dfrac{\vartheta_2^4}{\vartheta_3^4} = \dfrac{\vartheta_4^4(0,\tau)}{\vartheta_3^4(0,\tau)}$

ie we claim: $\dfrac{e_3 - e_2}{e_1 - e_2} = \dfrac{\vartheta_2^4}{\vartheta_3^4}$. In fact, $\wp(z,\tau) = A(\tau)\dfrac{\vartheta_4^2(z,\tau)}{\vartheta_1^2(z,\tau)} + B(\tau)$.

by the double pole property.

$$\Rightarrow \frac{e_3 - e_2}{e_1 - e_2} = \frac{A\,\dfrac{\vartheta_4^2(\frac{1}{2}+\frac{\tau}{2})}{\vartheta_1^2(\frac{1}{2}+\frac{\tau}{2})} + B - B}{A\,\dfrac{\vartheta_4^2(\frac{1}{2})}{\vartheta_1^2(\frac{1}{2})} + B - B} = \frac{\vartheta_2^4(0)}{\vartheta_3^4(0)} \Rightarrow k^2 \neq 0, 1\ \exists\, \tau \text{ st } \lambda(\tau) = k^2$$

as shown in Ahlfors.

By definition, $sn\,K = 1$, $cn\,K = 0$, $dn\,K = k'$

From $sn(u) = \dfrac{\vartheta_3}{\vartheta_2}\dfrac{\vartheta_1}{\vartheta_4}\left(\dfrac{u}{\pi\,\vartheta_3^2}\right)$, hence $\Rightarrow K = \dfrac{\pi}{2}\vartheta_3^2(0,\tau)$.

* Uniformization: Can also construct τ directly from k^2

Indeed, get K, K' by elliptic integrals. A simple check as in conformal mapping $\Rightarrow K, K'$ linearly indep /$\mathbb{R}$.

Then set $\tau = iK'/K$.

k, k' / τ, τ' Duality:

$K' := \displaystyle\int_1^{1/k} (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt =$ "K wrt. k'^2" ?

$k^2 = \dfrac{\vartheta_2^4(0,\tau)}{\vartheta_3^4(0,\tau)}$, $\quad k'^2 := 1 - k^2 = \dfrac{\vartheta_4^4(0,\tau)}{\vartheta_3^4(0,\tau)} \underset{\text{for } \tau' = -1/\tau}{\uparrow=} \dfrac{\vartheta_2^4(0,\tau')}{\vartheta_3^4(0,\tau')}$ Modularity (Ex.)

Then "K for k'^2" $= \dfrac{\pi}{2}\vartheta_3^2(0,\tau')$

$= -i\tau\,\dfrac{\pi}{2}\,\vartheta_3^2(0,\tau) = -i\tau K$ $\quad$ when $\tau = iK'/K$ this gives K'

The general case follows from analytic conti. (Q: Calculus pf ?)

Periodicity Properties:

K: $sn(u+K) = \dfrac{sn\,u\,cn\,K\,dn\,K + sn\,K\,cn\,u\,dn\,u}{1-k^2\,sn^2u\,sn^2K} = \dfrac{cn\,u\cdot dn\,u}{dn^2u} = cd\,u$ $\quad$ Glaisher's notations

$cn(u+K) = \dfrac{c_1c_2 - s_1s_2d_1d_2}{1-k^2s_1^2s_2^2} = \dfrac{-s_1d_1k'}{d_1^2} = -k'\,sd\,u$

$dn(u+K) = k'\,nd\,u$.

$\Rightarrow sn(u+2K) = \dfrac{cn(u+K)}{dn(u+K)} = \dfrac{-k'\,sd\,u}{k'\,nd\,u} = -sn\,u \Rightarrow 4K$ is a period.

also for cn, but $2K$ for dn.

$K+iK'$: All we need are $sn(K+iK') = k^{-1}$, $cn(K+iK') = -ik'/k$, $dn(\cdot) = 0$

Ex. Prove these and find the period formulae, also for $+iK'$.

Taylor Series: $sn\,u = u - \frac{1}{6}(1+k^2)u^3 + \cdots$, $sn(u+iK) = k^{-1}ns\,u = \dfrac{1}{ku} + \dfrac{1+k^2}{6k}u \cdots$

a simple pole at iK