Weierstrass Rep. of min surface in $\mathbb{R}^3$.

Ref: Osserman: A Survey of min. surface

M surface. oriented. $\hookrightarrow \mathbb{R}^n$

$\exists$ isothermal coor. $\Rightarrow$ M has R.S. structure

and $\varphi: M \to \mathbb{R}^n$ conformal

$\updownarrow$

$$ds^2 = \lambda(dx^2 + dy^2) = \frac{\lambda}{2} dz \otimes d\bar{z}$$

Laplace operator $\Delta = \frac{1}{\lambda}\partial_i(\lambda\cdot\lambda^{-1}\partial_i) = \frac{1}{\lambda}\sum\partial_i\partial_i$

$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j)$

$$= \frac{4}{\lambda}\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial\bar{z}}$$

$$\left(\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);\ \frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).\right)$$

so $\varphi: M \hookrightarrow \mathbb{R}^n$ minimal

$\Leftrightarrow \vec{H} \equiv \Delta\varphi = 0$ ie. $\frac{\partial}{\partial\bar{z}}\left(\frac{\partial\varphi}{\partial z}\right) = 0$

cf. Do Carmo p. 201

ie. the cpx functions $\frac{\partial\varphi^i}{\partial z} = \varphi^i_z$ $i = 1\dots n$ $\left(= \frac{1}{2}(\varphi^i_x - i\varphi^i_y)\right)$

are holomorphic.

Also $\sum_i {\varphi^i_z}^2 = \frac{1}{4}\left(\sum_i \varphi^i_x\varphi^i_x - 2i\sum_i \varphi^i_x\cdot\varphi^i_y - \sum_i \varphi^i_y\varphi^i_y\right) = 0$

(φ conformal)

$$|\varphi_z|^2 = \left(\frac{1}{2}\right)^2\cdot(|\varphi_x|^2 + |\varphi_y|^2) = \frac{\lambda}{2}$$

ie. φ_z is the cpx-coor. vector.

(Gauss Map introduce this at the very end)

If M has global cpx coor z, say $M = \mathbb{C}, D$ ..

then $\varphi_z: M \longrightarrow \mathbb{C}^n - 0$ is a holo. map.

with image inside $\sum z_i^2 = 0$.

In general, $\varphi_w = \varphi_z\cdot\frac{dz}{dw}$

hence the "pt" $[\varphi_z] \in \mathbb{CP}^{n-1}$ is always defined.

ie. get

$\Phi: M \longmapsto \mathbb{CP}^{n-1}$ hol. map.

$\searrow$ $Q = $ proj. v. $:= \{[z] \mid \sum z_i^2 = 0\} \hookrightarrow \mathbb{CP}^{n-1}$

This is the "Gauss Map".

in fact. $\varphi_z^1 dz, \cdots, \varphi_z^n dz$ is a system of hol. 1-forms. hence this is the ~~sub linear system~~

$P(M, K_M)$ in alg. geom.

Rmk: Φ is the cpx conjugate of the diff. geom. Gauss Map:

$$M \longrightarrow \underset{\text{oriented}}{Gr(2,n)} \cong SO(n)/SO(2)\times SO(n-2)$$

$\cong$ ← why?
Q

for $n=3$. ie. $\varphi: M \hookrightarrow \mathbb{R}^3$

we identify

$$Gr(2,3) \cong Gr(1,3) = S^2$$

Φ v.s. N

Summary: $\varphi: M \hookrightarrow \mathbb{R}^n$ minimal (under isothermal coor. or holo. coor.)

$\Longleftrightarrow$ the Gauss Map Φ is holomorphic

or in $n=3$, $N: M \to S^2$ is anti-conformal.

- Now it is VERY easy to construct minimal surfaces (especially when $M = \mathbb{C}^2$ or D) in $\mathbb{R}^4$:

(1). Find holo. 1-forms on M

$$\alpha = (\alpha^1, \cdots, \alpha^n) \text{ st. } \sum (\alpha^i)^2 = 0, \ \sum |\alpha^i|^2 > 0$$

(2). Let $\varphi := 2\cdot \mathrm{Re} \int_{z_0}^{z} \alpha : M \longrightarrow \mathbb{R}^n$

is then a minimal immersion.

Pf: Check that $\varphi_z = \frac{\partial}{\partial z} \int \alpha$ ※.

W - Rep in $\mathbb{R}^3$:

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$$

$$(\alpha_1 + i\alpha_2)(\alpha_1 - i\alpha_2) = -\alpha_3^2$$

$$g := \frac{\alpha_3}{\alpha_1 - i\alpha_2} \equiv \frac{-(\alpha_1 + i\alpha_2)}{\alpha_3}$$

g: global funct. meromorphic

$W := \alpha_1 - i\alpha_2$ hol. diff form

$$\exists \begin{cases} \alpha_1 + i\alpha_2 = -g\alpha_3 = -g^2 W \\ \alpha_1 - i\alpha_2 = W \end{cases}$$

get W: $\begin{cases} \alpha_1 = \frac{1}{2}(1-g^2) W \\ \alpha_2 = \frac{i}{2}(1+g^2) W \\ \alpha_3 = g W \end{cases}$

write $W = f\, dz$

$$\frac{\lambda}{2} = |\varphi_z|^2 = \frac{1}{4}|1-g^2|^2|f|^2 + \frac{1}{4}|1+g^2|^2|f|^2 + |g|^2|f|^2$$

$$= \frac{1}{4}|f|^2\left(2\left[1 + 2|g|^2 + |g|^4\right]\right)$$

$$= \frac{1}{2}|f|^2(1+|g|^2)^2$$

ie. $\lambda = |f|^2(1+|g|^2)^2$. This will be useful.

Prop: $g = P \circ N : M \to \mathbb{C} \cup \{\infty\}$ with

P = stereographic proj

N = Gauss Map : $M \to S^2$.

pf: $\frac{1}{2}(\varphi_x - i\varphi_y) = \varphi_z =: (a_1, a_2, a_3)$

$\alpha_i := a_i\, dz$, $a_i = u_i + i v_i$.

$$\varphi_x \times \varphi_y = 4 \operatorname{Re} a \times \operatorname{Re}(ia)$$

$$= -4(u_1, u_2, u_3) \times (v_1, v_2, v_3)$$

$$= 4 \cdot \operatorname{Im}(a_2\bar{a}_3, a_3\bar{a}_1, a_1\bar{a}_2)$$

$$= 4 \cdots = |f|^2(1+|g|^2)\left(2\operatorname{Re} g,\ 2\operatorname{Im} g,\ |g|^2-1\right)$$

eg. $$a_2\bar{a}_3 = \frac{i}{2}|f|^2(1+g^2)\bar{g}$$

$$\operatorname{Im} a_2\bar{a}_3 = \frac{|f|^2}{2i}\left[\frac{i}{2}(1+g^2)\bar{g} + \frac{i}{2}(1+\bar{g}^2)g\right]$$

$$= \frac{1}{4}|f|^2\left((g+\bar{g})(1+|g|^2)\right)$$

$$= \frac{1}{2}|f|^2(1+|g|^2)\operatorname{Re} g$$

similarly $\operatorname{Im} a_3\bar{a}_1 = \frac{1}{2}|f|^2(1+|g|^2)\operatorname{Im} g$

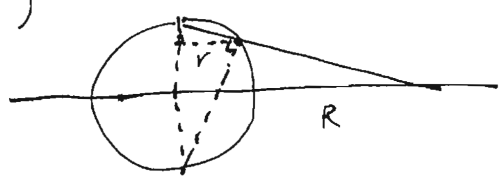
$$\operatorname{Im} a_1\bar{a}_2 = \operatorname{Im}\frac{-i}{4}(1-g^2)(1+\bar{g}^2)|f|^2$$

$$= -\frac{|f|^2}{4}\operatorname{Im}\left[i(1-|g|^4)+i(\bar{g}^2-g^2)\right]$$

$$= \frac{1}{4}|f|^2(1+|g|^2)(|g|^2-1).$$

$$\Rightarrow N = \frac{1}{\underset{\underset{|g|^2+1}{\shortparallel}}{\text{length}}}\Big(\underset{\underset{2x}{\shortparallel}}{2\operatorname{Re} g},\ \underset{\underset{2y}{\shortparallel}}{2\operatorname{Im} g},\ \underset{\underset{x^2+y^2-1}{\shortparallel}}{|g|^2-1}\Big)$$

$$(x^2+y^2-1)^2+4x^2+4y^2 = (x^2+y^2+1)^2$$



Since $$\frac{R}{r} = \frac{1}{1-\frac{|g|^2-1}{|g|^2+1}} = \frac{|g|^2+1}{2},$$

$$\Rightarrow p\circ N = (\operatorname{Re} g, \operatorname{Im} g) \equiv g \quad \square$$

- Summary for W-Rep in $\mathbb{R}^3$:

  Given g mero. f holo. (W) defines α.

  $\varphi := 2\operatorname{Re}\int\alpha : M \to \mathbb{R}^3$ min.

  $g = p\circ N$ projection of Gauss map.

  $\lambda = |f|^2(1+|g|^2)^2.$

Applications I: Isometric Deformations of Minimal Surfaces:

For given g, f, hence α, let $\theta \in [0, \pi]$

consider

$$\boxed{\varphi_\theta := 2\,\mathrm{Re}\left(e^{i\theta} \int \alpha \right)}$$

the image of φ_θ has the same λ. (1st fund. form)

hence all isometric.

$$\lambda = 2|(\varphi_\theta)_z|^2 = 2|e^{i\theta}\alpha|^2 \quad \text{indep. of } \theta.$$

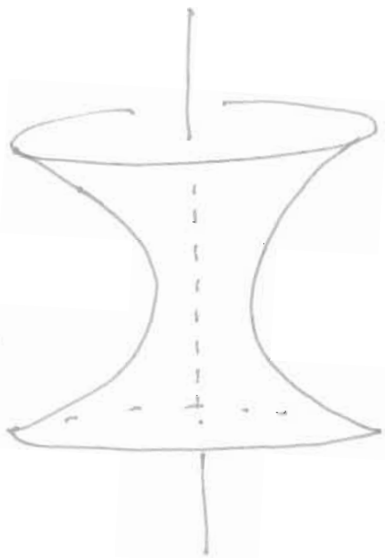
Examples I: Catenoid (the unique min. surface of revolution)

$M = \mathbb{C} - \{0\}$, $g(z) = z$; $f(z) = \frac{1}{z^2}$

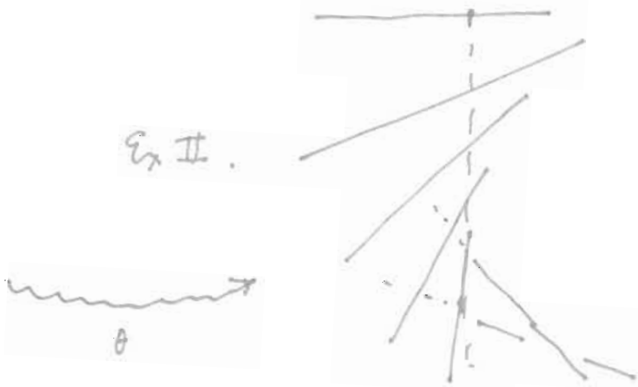
(so $f\,dz$ has no period !)

$y = a \cosh^{-1}\left(\frac{x}{a}\right)$

Ex I.



Ex II.



Example II: Helicoid ($\theta = \frac{\pi}{2}$, conjugate to I.)

since coor. functions are harmonic conjugate to $\theta = 0$

Ex III.

cf. Do Carmo p.205

Example III: Enneper's surface

$M = \mathbb{C}$, $g(z) = z$, $f(z) = 1$.

Applications II :

Osserman's Generalized Bernstein thm. (1959)

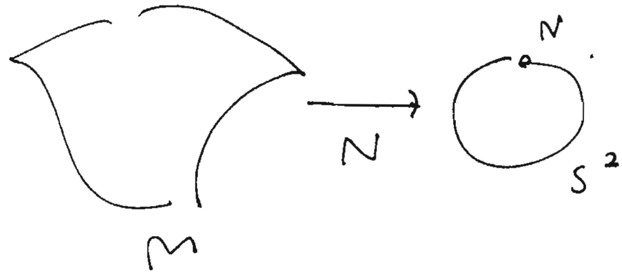
$\varphi : M \hookrightarrow \mathbb{R}^3$ complete min.

if Gauss map not dense then M = plane.

pf : Say $N \notin \operatorname{Im}\varphi$ of Gauss

$$\begin{tikzcd} \tilde{M} \arrow[d, "\pi"'] \arrow[dr, "\tilde{\varphi}"] & \\ M \arrow[r, "\varphi"'] & \mathbb{R}^3 \end{tikzcd}$$

univ. cover



$\tilde{\varphi}$ is still min. immersion.

Uniformization thm : $(\tilde{M}, \tilde{ds}^2)$ is conf. to $\begin{cases} S^2 \\ \mathbb{C} \\ D^2 \end{cases}$

S^2 impossible since $\tilde{\varphi}(\tilde{M}) = \varphi(M)$

if compact will have a pt. $K > 0 \Rightarrow H \neq 0$ $\times$

$\mathbb{C}$: compose Gauss map and stereographic projection

get a holo. function $g : \mathbb{C} \to \mathbb{C}$ (or anti-holo)

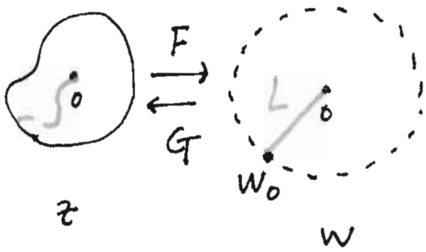
bounded $\Rightarrow \times$ to Liouville thm.

D : let $F : D^2 \to \mathbb{C}$

$w = F(z) = \int_0^z f(\zeta)\, d\zeta$

pick largest disk $|w| < R$

st $G = F^{-1}$ is defined



($R < \infty$ by Liouville thm on G)

let $w_0 \in \partial B_R$ be a non-ext'ble pt of G.

$L = \overline{0 w_0}$, Let $C = G(L)$.

Claim: C is a divergent path : otherwise for $w_n \to w_0$

$\exists\, z_n \to z_0 \in D^2$, but $F'(z_0) = f(z_0) \neq 0$

$F^{-1}(z_0)$ exists. (bec. g has no poles !)

$\times$. in Werest.-Rep. $\Sigma |\alpha|^2 \neq 0$.

Since M is a complete surface, will get $*$ if show $|C| < \infty$.

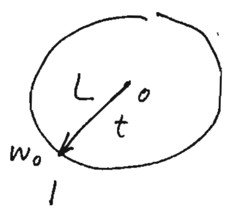
Now Gauss Map not dense $\Rightarrow$ $|g| \le A < \infty$

$$|C| = \int_C \sqrt{\frac{\lambda}{2}}\,|dz| = \frac{1}{\sqrt{2}} \int_C |f|\,(1+|g|^2)\,|dz|$$

$$\le \frac{1}{\sqrt{2}}(1+A^2) \int_C |f|\,|dz| = (\cdots) \int_L |dw| = (\cdots)\, R < \infty$$

$\Rightarrow f = F' = \frac{dw}{dz}$!

The pf is completed $\square$.



Generalizations:

(1). For $\varphi : M^2 \to \mathbb{R}^n$ complete min., same proof shows
If $\varphi(M)$ is not a plane, then the Gauss Map $\Phi : M \to \mathbb{C}P^{n-1}$ meets a dense set of hyperplanes.
(what does this say if φ degenerate ?)

(2). Minimal hypersurface graph in $\mathbb{R}^{n+1}$; on $\mathbb{R}^n$

i.e. $\partial_i \left(\frac{\partial_i f}{\sqrt{1+|\nabla f|^2}} \right) = 0$ on whole $\mathbb{R}^n$

J. Simons : For $n \le 7 \Rightarrow M$ is a hyp. plane

Bombierie : WRONG For $n \ge 8$!

(3). Xavier, Fujimoto thm : $M \hookrightarrow \mathbb{R}^3$ compl. min.

For Sherk Surf, N omits exactly 4 pts.

Xavier ('81) : Omit ≥ 7 pts $\Rightarrow$ plane

~~Fujimoto (1988) : Omit ≥ 4 pts $\Rightarrow$ plane.~~

optimal result !

↑

Final Reports.

In fact for $k \le 4$, the omited k pts can be prescribed on S^2. (See Osserman Thm 8.3).

Summary:

Assume the following two theorems:

Theorem 1. (Existence of Isothermal Coordinates)
For any geometric surface (M, ds^2), for each $p \in M$, there exists a neighborhood & coordinate system $(U, (x,y))$ such that $ds^2 = \lambda(dx^2 + dy^2)$.
(for minimal surfaces this is in 3.5 Ex. 13-(b))

Theorem 2. (Uniformization Theorem)
Any simply connected Riemann surface (ie. 1-dim complex manifold) is complex analytically equivalent to D, $\mathbb{C}$ or $\mathbb{C} \cup \{\infty\} = S^2$.

Then we want to prove Osserman's

Theorem (Generalized Bernstein Theorem)
Let $\varphi : M \to \mathbb{R}^3$ be a immersed complete orientable minimal surface. If the image of the Gauss map is not dense, then M is a plane.

Proof: (reduction to the case $M = D$):
Theorem 1 $\Rightarrow$ M has the structure of a Riemann surface.
Theorem 2 $\Rightarrow$ $\tilde{M}$, the universal cover of M is D, $\mathbb{C}$ or S^2.

$$\begin{tikzcd} \tilde{M} \arrow[d, "\pi"'] \arrow[dr, "\tilde{\varphi}"] & \\ M \arrow[r, "\varphi"'] & \mathbb{R}^3 \end{tikzcd}$$

$\tilde{M} \neq S^2$; since $H = 0 \Rightarrow K \leq 0$ but any compact surface in $\mathbb{R}^3$ has a point st $K > 0$ $*$.

$\tilde{M} \neq \mathbb{C}$; If $N \circ \varphi$ omits a open set of S^2, WLOG may assume this nbd is a nbd of north pole

$$\tilde{M} \xrightarrow[\tilde{\varphi}]{} \varphi(M) \xrightarrow[N]{} S^2 \xrightarrow{p} \mathbb{C}$$

if $\tilde{M} = \mathbb{C}$, then $p \circ N \circ \tilde{\varphi}$ is a bounded analytic function $\to\!\!\!\!\!*$ to the Liouville theorem.

So $\tilde{M} \simeq D$. Then use method of divergent path.

Algebraic Methods to construct Minimal Submanifolds

Almost cpx str.

M, $\dim M = 2n$, J tensor of type $(1,1)$ st.

$J_p : T_pM \to T_pM$ st. $J_p^2 := J_p \circ J_p = -\mathrm{id}_{T_pM}$

(reason : for $V \cong \mathbb{R}^{2n}$ if $V \cong \mathbb{C}^n$, then $i : V \to V$ has $i^2 = -1$. Conversely, if $J : V \to V$ st $J^2 = -\mathrm{id}_V$ then V has a $\mathbb{C}$-module (v.s) str. $(a+b_i)\cdot v := av + bJv$.)

g : Riem Metric. on (M, J)

g hermitian if $g(Jv, Jw) = g(v,w)$ $\forall v, w$

(Always exists, eg. $h(v,w) := g(v,w) + g(Jv,Jw)$)

let ∇ be the Levi-Civita conn. (wrt to her. g)

$\boxed{Q : \nabla J = 0 \ ?}$ (J parallel ?)

This is equiv. to $\nabla_v(JW) = J\nabla_v W$ $\forall v, w$

since $\nabla_v(JW) = (\nabla_v J)W + J\nabla_v W$.

Notice that this is NOT a trivial condition even if M is actually a complex mfd !

DEFINITION : (M, J, g) is called Kähler if $\nabla J = 0$.

Another point of view : for g, J

fundamental 2 form

$\omega(v,w) := g(Jv, w)$ (這和 Lawson 差一個負號 !)

$\omega \in \Lambda^2(M)$ since $\omega(w,v) = g(Jw, v) = g(JJw, Jv)$
$= -g(w, Jv) = -g(Jv, w)$
$= -\omega(v,w)$

Fund. Theorems :

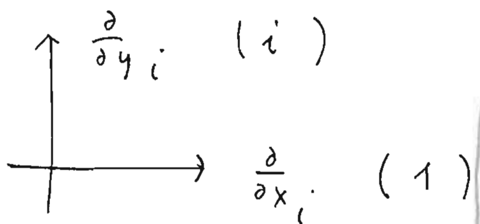
Theorem I. $\nabla J = 0 \iff (M, J)$ is a complex mfd and $d\omega = 0$.

Theorem II. Any cpx submfd (variety) in a Kähler mfd (eg. $\mathbb{C}^n$) is a stable min. sub. mfd (variety) which minimize area in its "homology class".

Examples:

(1) $\mathbb{C}^n$, $g = ds^2 = \sum_{i=1}^{2n} dx_i^2 = \sum_{i=1}^{n} dx_i^2 + \sum_{i=1}^{n} dy_i^2$

Almost cpx structure



since $i \cdot (1) = (i)$
$i \cdot (i) = -(1)$

so define

$$(*) \begin{cases} J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} & \forall i \\ J \frac{\partial}{\partial y_i} = - \frac{\partial}{\partial x_i} \end{cases}$$

Now $\omega(\partial x_i, \partial x_j) = g(J\partial x_i, \partial x_j) = g(\partial y_i, \partial x_j) = 0$

$\omega(\partial x_i, \partial y_j) = g(\partial y_i, \partial y_j) = \delta_{ij}$

hence $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$

clearly, $d\omega = 0$. Kähler.

For cpx submanifold $M \subset \mathbb{C}^n$, simply take

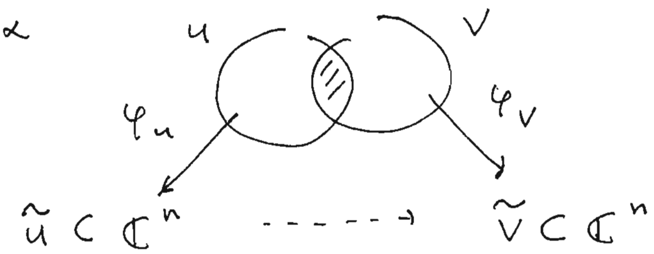
$M = \{ z \in \mathbb{C}^n \mid f_i(z) = 0 \quad i = 1 \ldots k \}$ the zero set of k holomorphic functions.

(If $f_i \in \mathbb{C}[z_1, \cdots, z_n]$, M is called affine variety)

Rmk: $\mathbb{C}^n$, M are not compact (why?)

For complex mfd M. i.e. $\exists$ holo coor. system

$M = \cup U_\alpha$



the corr. almost cpx str J is still given by $(*)$. Check it!

s.t. $\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \longrightarrow \varphi_V(U \cap V)$

is a bi-holomorphic mapping.

it is more convienent to use complex coor. system "even for Real purpose": $z_1, \cdots, z_n$,

with $\left. \begin{array}{l} z_i = x_i + \sqrt{-1}\, y_i \\ \bar{z}_i = x_i - \sqrt{-1}\, y_i \end{array} \right\}$ another set of "real coor"

The Precise Way : $\otimes \mathbb{C}$

Basis of $\Lambda^1(M) \otimes \mathbb{C}$: $dz_i := dx_i + \sqrt{-1}\, dy_i$

$d\bar{z}_i := dx_i - \sqrt{-1}\, dy_i$ $\quad JV = iV$

(dual) basis of $TM \otimes \mathbb{C}$: $\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$

(check it !) $J \circlearrowleft$

$\frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$ $\quad JV = -iV$

This SIMPLY MEANS that we extend everything by forcing $\mathbb{C}$-linearity :

eg. $\tilde{g}(v + iv', w) = g(v,w) + i g(v', w) \ldots$

then one can write, for any Riem metric g :

$$\tilde{g} = \tilde{g}_{\alpha\beta}\, dz^\alpha \otimes dz^\beta + \tilde{g}_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar{z}^\beta$$
$$+ \tilde{g}_{\bar\alpha\beta}\, d\bar{z}^\alpha \otimes dz^\beta + \tilde{g}_{\bar\alpha\bar\beta}\, d\bar{z}^\alpha \otimes d\bar{z}^\beta$$

Fact : g is hermitian $\Longleftrightarrow$ $\tilde{g}_{\alpha\beta} = 0 = \tilde{g}_{\bar\alpha\bar\beta}$

By def, $\tilde{g}_{\alpha\beta} = \tilde{g}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) = \tilde{g}\left(\frac{1}{2}(\partial_{x_\alpha} - i\partial_{y_\alpha}), \frac{1}{2}(\partial_{x_\beta} - i\partial_{y_\beta})\right)$

$$= \frac{1}{4}\Big(g(\partial_{x_\alpha}, \partial_{x_\beta}) + i\big[g(\partial_{x_\alpha}, \partial_{y_\beta}) + g(\partial_{y_\alpha}, \partial_{x_\beta})\big] - g(\partial_{y_\alpha}, \partial_{y_\beta})\Big) = 0$$

$g(\partial_{y_\alpha}, \partial_{x_\beta}) = g(J\partial_{x_\alpha}, -J\partial_{y_\beta})$

easier way: $\tilde{g}(\partial_\alpha, \partial_\beta) = -\tilde{g}(i\partial_\alpha, i\partial_\beta) = -\tilde{g}(J\partial_\alpha, J\partial_\beta)$
$= -\tilde{g}(\partial_\alpha, \partial_\beta) \Rightarrow = 0$.

Similarly for $\tilde{g}_{\bar\alpha\bar\beta} = 0$. and

$$\tilde{g}_{\alpha\bar\beta} = \frac{1}{4}\tilde{g}(\partial_{x_\alpha} - i\partial_{y_\alpha}, \partial_{x_\beta} + i\partial_{y_\beta})$$
$$= \frac{1}{4}\big[g(\partial_{x_\alpha}, \partial_{x_\beta}) + g(\partial_{y_\alpha}, \partial_{y_\beta})\big]$$
$$+ i\big[g(\partial_{x_\alpha}, \partial_{y_\beta}) - g(\partial_{y_\alpha}, \partial_{x_\beta})\big]$$

$g(\partial_{y_\alpha}, \partial_{x_\beta}) = + g(\partial_{x_\alpha}, \partial_{y_\beta})$

i.e. $\boxed{\tilde{g}_{\alpha\bar\beta} = \frac{1}{2}\left(g_{\alpha\beta} + i\, g_{\alpha, n+\beta}\right)}$

and $\tilde{g}_{\bar\alpha\beta} = \tilde{g}_{\beta\bar\alpha} = \overline{\tilde{g}_{\alpha\bar\beta}}$ bec. $g_{\alpha, n+\beta} = -g_{\beta, n+\alpha}$

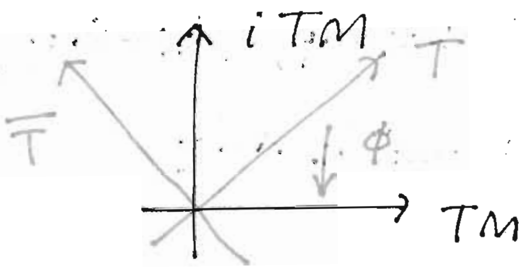
or equiv: $\tilde{g}\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = \frac{1}{2}\left[g\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) - i\, g\left(J\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right)\right]$.

CONCLUSION :

M, J	M
real $2n$ mfd	complex n-mfd
TM real tang.	(holomorphic) T cpx tangent bundle
basis $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}$	basis $\left\{ \frac{\partial}{\partial z^\alpha} \right\}$

$(\star)\quad TM \otimes \mathbb{C} = T \oplus \overline{T}$ ← anti-holomorphic tang

i-eigen space



Warning: Although

$T \cong TM$ $/\mathbb{R}$ via ϕ

even $/\mathbb{C}$ if $i \longmapsto J$

but we regard them as different via $(\star)$

hermitian metric g on TM

the $\mathbb{C}$-extension $\tilde{g}$ on $TM \otimes \mathbb{C}$, when restrict to T, gives the usual "hermitian metric"

via $h(v, w) := \tilde{g}(v, \overline{w})$, for $v, w \in T_p$

i.e. $h_{\alpha\beta} = \tilde{g}_{\alpha\bar{\beta}}$.

$$\boxed{g = \tilde{g}_{\alpha\bar{\beta}}\, dz^\alpha \otimes d\bar{z}^\beta + \tilde{g}_{\bar{\alpha}\beta}\, d\bar{z}^\alpha \otimes dz^\beta} \qquad \boxed{h = h_{\alpha\beta}\, dz^\alpha \otimes \overline{dz^\beta}}$$

abuse notation
still write
$g_{\alpha\bar{\beta}}$

$$\omega(\partial_\alpha, \partial_\beta) = \tilde{g}(J\partial_\alpha, \partial_\beta) = i\,\tilde{g}(\partial_\alpha, \partial_\beta) = 0$$

$$\omega(\partial_\alpha, \overline{\partial_\beta}) = i\,\tilde{g}(\partial_\alpha, \overline{\partial_\beta}) = i\,\tilde{g}_{\alpha\bar{\beta}}$$

$$\Rightarrow \boxed{\omega = i\,\tilde{g}_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta}$$ Important Formula

eg. $\mathbb{C}^n$:
$$\begin{aligned} \omega &= i\,\tilde{g}_{\alpha\bar{\alpha}}\, dz^\alpha \wedge d\bar{z}^\alpha \\ &= i \cdot \tfrac{1}{2} g_{\alpha\alpha} \cdot (-2i)\, dx^\alpha \wedge dy^\alpha \\ &= dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n \end{aligned}$$

(2) $\mathbb{P}^n_{\mathbb{C}} := \left(\mathbb{C}^{n+1} - 0\right)/\sim$

st. $(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \lambda z_1, \cdots, \lambda z_n) \ \forall \lambda \neq 0$

denote the pt by $[z] = [z_0, \cdots, z_n]$

or $(z_0 : z_1 \cdots : z_n)$

$\mathbb{P}^n_{\mathbb{C}} = \bigcup_{\alpha=0}^{n} U_\alpha$

$U_\alpha = \{ [z] \in \mathbb{P}^n \text{ st. } z_\alpha \neq 0 \} \ \ldots$

Fubini - Study metric : the fund. 2 form is

$$\omega_{FS} := \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log |Z|^2 \qquad |z|^2 = \sum_{i=0}^{n} z_i \bar z_i$$

$$= \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log\left(1 + |W|^2\right) \qquad \text{in a cor. system } (U_\alpha, W)$$

$$= \frac{\sqrt{-1}}{2\pi} \partial \frac{w_j d}{1 + |W|^2}$$

$$= \frac{\sqrt{-1}}{2\pi} \frac{\delta_{ij} \cdot dw^i \wedge d\bar w^j (1 + |W|^2) - w\, dw}{(1 + |W|^2)^2}$$

Rmk: Here we extend d to $\Lambda^\cdot(M) \otimes \mathbb{C}$

then $\Lambda^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}(M)$

$\Lambda^{p,q}(M) = \Lambda^p(T) \otimes \Lambda^q(\bar T)$

and $d = \partial + \bar\partial$

via $d(f\, dz^I) = \left(\underbrace{\frac{\partial f}{\partial z^\alpha} dz^\alpha}_{\partial} + \underbrace{\frac{\partial f}{\partial \bar z^\alpha} d\bar z^\alpha}_{\bar\partial} \right) \wedge dz^I$

so $0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2$

i.e. $\partial^2 = 0 = \bar\partial^2$ and $\partial\bar\partial = -\bar\partial\partial$.

$$\Rightarrow \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \cdot \frac{\left(\delta_{ij}(1 + |W|^2) - \bar W_i W_j\right)}{(1 + |W|^2)^2} dw^i \wedge d\bar w^j$$

why is a metric ?

clearly, by def $d\omega_{FS} = 0$.

Ex. Write $\omega_{FS} = i\, g_{\alpha\bar\beta}\, dw^\alpha \wedge d\bar w^\beta$

show that $g_{FS} := g_{\alpha\bar\beta}\, dw^\alpha \wedge d\bar w^\beta$ is a metric

and on $\mathbb{P}^1$, $\int_{\mathbb{P}^1} \omega_{FS} = 1$.

Ex. $\mathbb{P}^n$ is compact. (Hint : $S^1 \longrightarrow S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$)

let $f \in \mathbb{C}[z_0, \cdots, z_n]$ homogenous poly.

though f is NOT a function on $\mathbb{P}^n_{\mathbb{C}}$

$V(f) := \{ [z] \in \mathbb{P}^n_{\mathbb{C}} \mid f(z) = 0 \}$ is well defined

$M \subset \mathbb{P}^n_{\mathbb{C}}$ cut out by homo. poly's is called a <u>projective variety</u>. they are compact.

<u>Fact</u>: $M \overset{i}{\hookrightarrow} N$ cpx submfd, or hol. immersion **

then N Kähler $\Rightarrow$ M Kähler

pf: If g, ω Kähler str on N

then the metric $g|_M := i^* g$ has

fund. 2 form $\tilde{\omega} = \omega|_M = i^*\omega$

so $d\tilde{\omega} = d\, i^*\omega = i^* d\omega = 0$, M is Kähler $\square$.

existence

So Any cpx mfd inside $\mathbb{C}^n$, $\mathbb{P}^n$ are also Kähler.

<u>Fact</u>: ω fund. 2-form $\Rightarrow$ $\frac{\omega^n}{n!}$ = Vol. form.

So if M is compact Kähler, then

$[\omega], [\omega^2], \cdots [\omega^n]$ all $\neq 0$ in $H^{\cdot}_{DR}(M; \mathbb{R})$.

non-existence

$\exists$ cpx mfd with $b_2 = 0$ hence must be non-Kähler.

**: Definition of hol. map.

for $(M, J) \xrightarrow{f} (N, J)$ C^∞ map of almost cpx mfds.

f is (pseduo)holomorphic if

$$df \circ J = J \circ df$$

when M, N are cpx mfd. this is equiv to "f is holomorphic". (Ex.)

computation I.

$$d\omega(X,Y,Z) = X\,\omega(Y,Z) - Y\,\omega(X,Z) + Z\,\omega(X,Y)$$

$$- \omega(\cancel{[X,Y]},Z) + \omega(\cancel{[X,Z]},Y) - \omega(\cancel{[Y,Z]},X)$$

extend X,Y,Z to fields st $(\nabla_X Y)_p = 0$ etc. hence also $[X,Y]_p = 0$ ..

$$= X\,g(JY,Z) - \underbrace{Y\,g(JX,Z)}_{= +Y\,g(JZ,X)} + \underbrace{Z\,g(JX,Y)}_{= -Z\,g(JY,X)}$$

$$= g((\nabla_X J)Y + J\cancel{\nabla_X Y}, Z) + g(JY, \cancel{\nabla_X Z}) + \cdots$$

$$= g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) - g((\nabla_Z J)Y,X)$$

Notice here: $g((\nabla_a J)b,c) = -g((\nabla_a J)c,b)$

so, $\nabla J = 0 \Rightarrow d\omega = 0$.

conversely,

$$d\omega(X,JY,JZ) = g((\nabla_X J)\underline{JY},JZ) + g((\nabla_{JY}J)JZ,X) - g((\nabla_{JZ}J)JY,X)$$

$$\overset{"}{} - J(\nabla_X J)Y$$

$J^2 = -id$

$\Rightarrow (\nabla_X J)\circ J + J\circ \nabla_X J = 0$

computation II.

$$d\omega(X,Y,Z) - d\omega(X,JY,JZ)$$

$$= 2g((\nabla_X J)Y,Z) + g(\nabla_Y JZ,X) - g(J\nabla_Y Z,X)$$
$$- g(\nabla_Z JY,X) + g(J\nabla_Z Y,X)$$
$$- g(\nabla_{JY} Z,X) + g(J\nabla_{JY} JZ,X)$$
$$- g(\nabla_{JZ} Y,X) - g(J\nabla_{JZ} JY,X)$$

$$= 2g((\nabla_X J)Y,Z) + g(N(Y,Z),X)$$

Nijenhuis tensor: $N(Y,Z) := J[JY,JZ] - J[Y,Z] + [JY,Z] + [Y,JZ]$
the Neijenhuis tensor.

hence $\nabla J = 0 \iff d\omega = 0$ and $N \equiv 0$.

It is clear, for M cpx mfd, $N\equiv 0$ since $\left\{ \begin{array}{l} J\partial_{x_i} = \partial_{y_i} \\ J\partial_{y_i} = -\partial_{x_i} \end{array}\right.$ and $[\,,\,] \equiv 0$ for all coor. vf's.

- Theorem (Newlander–Nirenberg): PDE thm.

Final report problem

$N \equiv 0 \iff J$ is an int'ble cpx structure.

Computation III. (some unfinished attempt)

consider $L(Y,Z) := [JY, Z] - J[Y,Z]$

$(\neq \cancel{JY}Z - Z\underline{J}Y - J\cancel{Y}Z + \underline{J}ZY$

it is only a tensor. (check) in Y, but ↖ this makes no sense!

$$N(Y,Z) = L(Y,Z) + JL(Y,JZ)$$

$$\left(\begin{array}{l} [fY,Z] = fYZ - ZfY = f[Y,Z] - (Zf)Y \\ [Y,gZ] = YgZ - gZY = g[Y,Z] + (Yg)Z \end{array} \right)$$

so $L(fY,Z) = [JfY,Z] - J[fY,Z]$
$= f[JY,Z] - \cancel{(Zf)}JY - J(f[Y,Z] - \cancel{(Zf)}Y)$
$= f\,L(Y,Z)$.

$L(Y,gZ) = [JY,gZ] - J[Y,gZ]$
$= g[JY,Z] + [(JY)g]Z - J(g[Y,Z] + (Yg)Z)$
$= g\,L(Y,Z) + \underline{[(JY)g]Z - (Yg)(JZ)}$

If M is a cpx mfd. then $\begin{cases} J\partial x_i = \partial y_i \\ J\partial y_i = -\partial x_i \end{cases}$

$L(\partial x_i, \partial x_j) = 0$ bec. all $[\,,\,]$ of coor. vectors $\equiv 0$
↑
any combination

Q: Can we prove that Kähler ⇒ cpx
without using Newlander-Nirenberg thm?

Minimality I. Pluriharmonicity

let $M \xrightarrow{i} N$ cpx submfd of Kähler mfd.

$\exists$ basis of $T_pM \subset T_pN$

$$e_1, Je_1, \cdots e_m, Je_m ; \; e_{m+1}, Je_{m+1} \cdots e_n, Je_n .$$

$$B(e_i,e_i) + B(Je_i, Je_i) = (\nabla_{e_i} e_i + \nabla_{Je_i} Je_i)^N$$

but $B(u, Jv) = (\nabla_u Jv)^N = (J\nabla_u v)^N = J(\nabla_u v)^N$

$= JB(u,v)$ J is an isometry

i.e. B is J-bilinear (via symmetry)

so $B(u,u) + B(Ju,Ju) = 0$. i.e. i is a ~~pluri-harmonic~~ map.

$\Rightarrow$ minimal

Discussion:
We do not really need N to be Kähler.

Gauss Eq'n: $\bar{R}(x,y,z,w) = R(x,y,z,w)$

$$+ \langle B(x,z), B(y,w)\rangle - \langle B(x,w), B(y,z)\rangle$$

eg. $\bar{R}(X, JX, Y, JY) = R(X, JX, Y, JY)$

$$+ \langle B(X,Y), B(JX,JY)\rangle - \langle B(X,JY), B(JX,Y)\rangle$$

$$= R(X,JX,Y,JY) - \underline{2\|B(X,Y)\|^2}$$

eg. $\bar{R}(e,u,e,v) + \bar{R}(Je,u,Je,v)$

$$= R(e,u,e,v) + R(Je,u,Je,v)$$

$$+ \langle \underline{B(e,e)}, B(u,v)\rangle - \langle B(e,v), B(u,e)\rangle$$

$$+ \langle B(\underline{Je,Je}), B(u,v)\rangle - \langle B(Je,v), B(u,Je)\rangle$$

$$= R(e,u,e,v) + R(Je,u,Je,v) - \underline{2\langle B(e,u), B(e,v)\rangle}$$

Conclusion:

Pluriharmonic Map has "curvature decreasing property".
in a suitable sense. but not "exactly".

Thm (Wirtinger's inequality).

$p \in \overset{2m}{M} \subset \bar{M}$ Kähler, Then $\frac{\omega^m}{m!}\Big|_{T_pM} \le dV_p$ ← induced vol form on M
real oriented

"=" holds $\Leftrightarrow$ $T_pM \subset T_p\bar{M}$ is a cpx subspace. ← i.e. J inv.

pf: $\forall X, Y \in T_p\bar{M}$,

$$\omega(X,Y)^2 = \langle JX, Y\rangle^2 \le |JX|^2|Y|^2 = |X|^2|Y|^2$$

"=" holds $\Leftrightarrow$ $JX = \pm Y$, ie X, Y span a cpx 1-dim subspace of $T_p\bar{M}$.

Now consider $\omega' := \omega|_{T_pM}$. since it is skew sym 2-form

linear algebra $\Rightarrow$ $\exists$ ONB $e_1, \cdots, e_{2m}$ of T_pM st.

$$\omega' \sim \begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \lambda_m \\ & & & & & -\lambda_m & 0 \end{pmatrix}$$

where
$\lambda_k = \omega(e_{2k-1}, e_{2k}) = g(Je_{2k-1}, e_{2k})$
$k = 1, \cdots, m$
All $|\lambda_k| \le 1$.

ie. if $\theta_1, \cdots, \theta_{2m}$ are dual 1-form of $\{e_i\}$ on T_pM,

then $\omega' = \sum_{k=1}^m \lambda_k \,\theta_{2k-1} \wedge \theta_{2k}$

$\Rightarrow$ $\omega'^m = (m!) \cdot \lambda_1 \cdots \lambda_m \,\theta_1 \wedge \cdots \wedge \theta_{2m}$

ie. $\boxed{\frac{\omega'^m}{m!} = \lambda_1 \cdots \lambda_m \, dV_p}$

Thus, $\left|\frac{\omega'^m}{m!}\right| \le dV_p$, "=" holds $\Leftrightarrow$ $|\lambda_i| = 1 \ \forall i$

by above, $\Leftrightarrow$ $Je_{2k-1} = \pm e_{2k}$, ie. T_pM is J-invariant.

Also, "=" holds without $|\cdot|$ sign occurs when orientations are the same $\square$.

Proof of Theorem II: M cpx $\Rightarrow$ M min vol in $[M] \in H_{2m}(\bar{M}, \mathbb{Z})$.

for $M' \sim M$, $\mathrm{Vol}(M) = \int_M \frac{\omega^m}{m!} = \int_{M'} \frac{\omega^m}{m!} \le \int_{M'} dV_{M'} = \mathrm{Vol}(M')$.

"=" $\Leftrightarrow$ M' also cpx $\square$

Douglas' Solution to Plateau Problem

$\Gamma \subset \mathbb{R}^n$ Jordan curve

$\Delta \subset \mathbb{R}^2$ unit disk (closed)

$\varphi : \Delta \to \mathbb{R}^n$ piecewise C^1 if

- φ is C^0
- outside $\partial\Delta$ and finite pts and C^1 curves, φ is C^1

$b : \partial\Delta = S^1 \longrightarrow \Gamma$ is monotone if

$b^{-1}(p)$ is connected $\forall p \in \Gamma$

For 1):

competing class (fixing disk as top. type[1]; why needed?)

[2] why p-C^1 only?

$X_\Gamma := \{ \varphi : \Delta \to \mathbb{R}^n$ st. φ is piecewise C^1 and $\varphi|_{\partial\Delta} : S^1 \to \Gamma$ is monotone $\}$

Area functional

For 2): $z \mapsto (z^2, z^3)$

$A : X_\Gamma \to \mathbb{R}^+ \cup \{\infty\}$ by

$$A(\varphi) := \int_\Delta |\varphi_x \wedge \varphi_y| \, dx\, dy$$

Plateau Problem:

Assume that $G_\Gamma := \inf_{\varphi \in X_\Gamma} A(\varphi) < \infty$

Find $\varphi \in X_\Gamma$ st. $A(\varphi) = G_\Gamma$.

Direct Method in the Calculus of variations:

Take $\varphi_n \in X_\Gamma$ st. $\lim_{n\to\infty} A(\varphi_n) = G_\Gamma$

$\exists$? subsequence converges in X_Γ.

(i.e. compactness)

In general not possible due to diffeomorphism gp.
This is the origin of "Gauge theory" ∞-dim'l

Already in the 1-dim case: geodesics!

Dirichlet Integral (Energy)

$$D(\varphi) := \int_\Delta \left(|\varphi_x|^2 + |\varphi_y|^2\right) dx\,dy$$

Easy to see: invariant under conformal transf.

from $|v \wedge w|^2 = |v|^2 |w|^2 - |v \cdot w|^2 \leq \left[\frac{1}{2}\left(|v|^2 + |w|^2\right)\right]^2$

get $A(\varphi) \leq \frac{1}{2} D(\varphi)$ and $=$ iff

$$|\varphi_x| = |\varphi_y| \quad \text{and} \quad \varphi_x \cdot \varphi_y = 0$$

i.e. $\varphi : \Delta \to \mathbb{R}^n$ is an almost conformal map.

FACT 1: $G_\Gamma = \frac{1}{2} d_\Gamma$, $d_\Gamma := \inf_{\varphi \in X_\Gamma} D(\varphi)$

So for any $\varphi \in X_\Gamma$: pf in next page

$$D(\varphi) = d_\Gamma \iff A(\varphi) = G_\Gamma \text{ and } \varphi \text{ is a. conformal}$$

almost

Notice that in the Plateau problem, the solution minimal surface can have arbitrary parametrization via

$$\begin{tikzcd} & & \mathbb{R}^n \\ \Delta \arrow[urr, "\tilde{\varphi}"] \arrow[r, "\sigma"] & \Delta \arrow[ur, "\varphi"'] & \end{tikzcd} \qquad \sigma \in \mathrm{Diff}(\Delta)$$

For φ a C^1 immersion,

FACT 2: $\exists\, \sigma$ st. $\tilde{\varphi} = \varphi \circ \sigma$ is conformal (Final report)

All conformal parametrization of the same surface differ by the conformal gp $\mathrm{conf}(\Delta)$

Ex. $=$ Möbius transform $e^{i\theta} \cdot \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ ← finite dim'l Lie gp.

So it is equivalent to solve φ which minimize the Dirichlet integral.

FACT 3: For any $b \in C(\partial\Delta, \mathbb{R}^n)$, $X_b := \{\psi : \Delta \to \mathbb{R}^n \text{ p-}C^1,\ \psi|_{\partial\Delta} = b\}$

if $d_b := \inf_{\psi \in X_b} D(\psi) < \infty$, then $\exists!\ \psi_b \in X_b$ with $D(\psi_b) = d_b$.

It is given by har. funct with ∂-value b.

the pf is similar to Hodge theory and easier: Lawson p. 64–65

$$A(\varphi) = \int_\Delta |\varphi_x \wedge \varphi_y| \, dx\, dy \qquad a_\Gamma := \inf_{\varphi \in X_\Gamma} A(\varphi)$$

$$D(\varphi) = \int_\Delta \left(|\varphi_x|^2 + |\varphi_y|^2 \right) dx\, dy \qquad d_\Gamma := \inf_{\varphi \in X_\Gamma} D(\varphi)$$

Cor: $\underline{A(\varphi) \leq \frac{1}{2} D(\varphi)}$. "=" $\Leftrightarrow$ φ is a conformal. i.e. $|\varphi_x| = |\varphi_y|$. $\varphi_x \cdot \varphi_y = 0$

Proof of Fact 1 (based on Fact 2):

$a_\Gamma \leq \frac{1}{2} d_\Gamma$ is clear.

For $\geq$: Let $A(\varphi_n) \searrow a_\Gamma$, may assume that $\varphi_n \in C^1(\Delta^\circ)$ why?

Will reparametrize φ_n to $\tilde{\varphi}_n$ st.

$$\underset{A(\varphi_n)}{\underset{\shortparallel}{A(\tilde{\varphi}_n)}} + \frac{1}{n} \geq \frac{1}{2} D(\tilde{\varphi}_n), \text{ then } n \to \infty \text{ get } \geq.$$

To use Fact 2, need "immersion":

Let $\underset{\underset{0}{\shortparallel}}{\varphi_{n,r}} : \Delta \to \mathbb{R}^{n+2}$ $\quad (x,y) \mapsto (\varphi_n(x,y), rx, ry)$

immersion for $r \neq 0$

Get $\tilde{\varphi}_{n,r}$ conformal, set $r = \varepsilon$ small st

$$\frac{1}{2} D(\tilde{\varphi}_n) \leq \frac{1}{2} D(\tilde{\varphi}_{n,\varepsilon}) = A(\tilde{\varphi}_{n,\varepsilon}) = A(\varphi_{n,\varepsilon}) \leq A(\varphi_n) + \frac{1}{n}$$

※

- To "gauge" the conformal gp, pick $P_1, P_2, P_3 \in \Gamma$ distinct "marked points" $z_1, z_2, z_3 \in \partial\Delta$ and consider

$$X_\Gamma' = \{ \varphi \in X_\Gamma \mid \varphi(z_k) = P_k, \ k = 1, 2, 3 \}$$

we still have $d_\Gamma = \inf_{\varphi \in X_\Gamma'} D(\varphi)$.

Theorem (Prop 6 in Lawson P.67)

Let $M > d_\Gamma$, then $\mathcal{F} := \{ \varphi|_{\partial\Delta} : \varphi \in X_\Gamma', D(\varphi) \leq M \}$ is equicontinuous on $\partial\Delta \cong S^1$.

Cor. Arzela-Ascoli $\Rightarrow$ $\mathcal{F}$ is cpt in unif. conv. topology.

$\exists$ Douglas' solution.

Fix $z \in \partial\Delta$

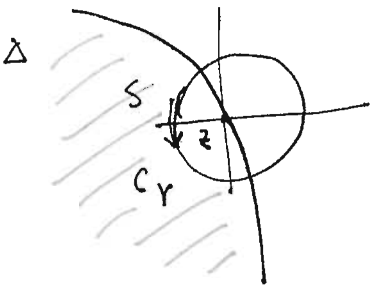
key estimate: $\forall\, 0<\delta<1$, $\forall \varphi \in X_\Gamma'$, $\exists \rho = \rho(\varphi)$, $\delta \le \rho \le \sqrt{\delta}$ and $D(\varphi) \le M$

s.t. $\ell(\varphi(C_\rho))^2 \le 2\pi\, \varepsilon(\delta)$. $\quad \varepsilon(\delta) := \dfrac{2M}{\log(1/\delta)}$.

pf: $D(\varphi) := \int_\Delta (|\varphi_x|^2 + |\varphi_y|^2)\, dx \wedge dy$ $\qquad s = r\theta$

$ds = \underline{dr \cdot \theta} + r \cdot d\theta$

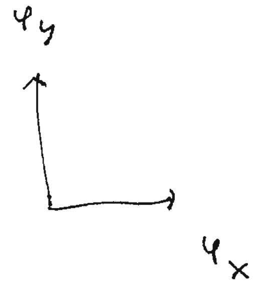
$$dx\wedge dy = r\, dr \wedge d\theta = \underline{dr \wedge ds}$$



$$\begin{cases} x = r\cos\theta = r\cos\left(\frac{s}{r}\right) \\ y = r\sin\theta = r\sin\left(\frac{s}{r}\right) \end{cases}$$

$$\varphi_s = \varphi_x \cdot \frac{\partial x}{\partial s} + \varphi_y \cdot \frac{\partial y}{\partial s}$$

$$= \varphi_x \cdot \left(-\sin\left(\tfrac{s}{r}\right)\right) + \varphi_y \cdot \left(\cos\left(\tfrac{s}{r}\right)\right)$$



$$|\varphi_s|^2 = |\varphi_x|^2 \sin^2\left(\tfrac{s}{r}\right) + |\varphi_y|^2 \cos^2\left(\tfrac{s}{r}\right)$$

$$-2\,\varphi_x \cdot \varphi_y \sin\left(\tfrac{s}{r}\right) \cdot \cos\left(\tfrac{s}{r}\right)$$

$$\left(\begin{array}{c} \wedge \\ 2|\varphi_x|\,|\varphi_y|\,|\sin\theta| \cdot |\cos\theta| \\ \wedge \\ |\varphi_x|^2 \cos^2\theta + |\varphi_y|^2 \sin^2\theta \end{array} \right)$$

total $\quad \underline{|\varphi_s|^2 \le |\varphi_x|^2 + |\varphi_y|^2}$

i.e. $\quad I := \int_\delta^{\sqrt{\delta}} \int_{C_r(z)} |\varphi_s|^2\, ds\, dr \le D(\varphi) \le M$.

$$= \int_\delta^{\sqrt{\delta}} \left(r \underline{\int_{C_r} |\varphi_s|^2 ds} \right) \frac{dr}{r} \quad = d\log r$$

this is conti' in r even if φ is only piecewise C^1.

since $\log r \nearrow$. by mean value of Rieman-Stieljes int.

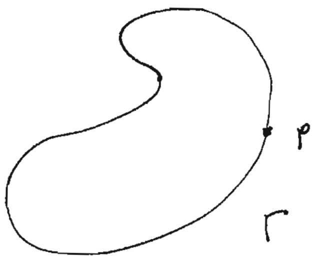
$$I = \rho \int_{C_\rho} |\varphi_s|^2 ds \int_\delta^{\sqrt{\delta}} d\log r \le M$$

$$\int_\delta^{\sqrt{\delta}} d\log r = \tfrac{1}{2}\log\left(\tfrac{1}{\delta}\right)$$

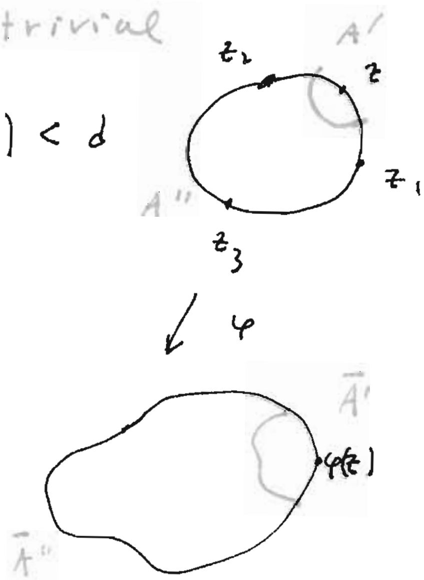
Now use Schwartz ineq. $\square$

(a) • Given $e > 0$ (small, say $< \min_{i \neq j} |p_i - p_j|$)

$\exists\, d > 0$ st. $\forall\, p \neq p' \in \Gamma$, $|p - p'| < d \Rightarrow$ a comp of $\Gamma - \{p, p'\}$ has diameter $< e$.



(b) • choose $\delta < 1$ st. $2\pi\, \varepsilon(\delta) < d^2$ and $\forall\, z \in \partial\Delta$
(small ~ 0) $|z - z_i| > \sqrt{\delta}$ at least 2 z_i's.
trivial



Now given $\varphi \in \mathcal{F} := X'_{\Gamma \le M}$

for any $z \in \partial\Delta$, $\exists\, \rho = \rho(\varphi)$ st. $\ell(C_\rho) < d$

(a) $\Rightarrow$ comp. $\bar{A}'$ has diam $< e$

(not $\bar{A}''$ bec. $A'' \supset z_i, z_j$ $i \neq j$

hence $\bar{A}'' \supset p_i \neq p_j$ and

$|p_i - p_j| > e$ by choice)

ie. for $|z' - z| < \delta$ (so $< \rho$)

get $|\varphi(z') - \varphi(z)| < e$

but δ is indep. of z & φ. hence the equicontinuity of $\mathcal{F}$. $\square$

Rmk (Ex). If $\Gamma \subset \mathbb{C} \cong \mathbb{R}^2$, this gives a pf of Riem mapping thm on $\Gamma = \partial\Omega$.

Minimal Graph v.s Plateau Solution.

$$\varphi = (x, y, f)$$

$$|\varphi_x|^2 + |\varphi_y|^2 = 1 + |f_x|^2 + 1 + |f_y|^2$$

$$= 2 + |f_x|^2 + |f_y|^2$$

$$|\varphi_x \wedge \varphi_y| = \sqrt{1 + f_x^2 + f_y^2}$$

if conformal.
ie.
$|\varphi_x| = |\varphi_y|$ ie. $|f_x| = |f_y|$
$\varphi_x \cdot \varphi_y = 0$. ie.
$(1, 0, f_x) \cdot (0, 1, f_y)$
$= f_x f_y = 0$.
can never be realized !

ie. for Plateau solution
we almost never use graph.

Global Stability of Minimal Hypersurface

$$\int \sqrt{1 + |\nabla f|^2}$$

$$\int \sqrt{1 + |\nabla f + \nabla h|^2}$$

1st try: Taylor exp.

$$\sqrt{1 + |\nabla f|^2 + \nabla f \cdot \nabla h + |\nabla h|^2}$$

$$= \sqrt{1 + |\nabla f|^2}\left(1 + \frac{1}{2}\,\frac{\nabla f \cdot \nabla h + |\nabla h|^2}{1 + |\nabla f|^2} * + \cdots\right)$$

$$= \text{original} + \frac{1}{2}\,\frac{\nabla f \cdot \nabla h}{\sqrt{1 + |\nabla f|^2}} + \text{positi.}$$

$$\sqrt{A + X} = \sqrt{A}\left(1 + \frac{X}{A}\right)^{\frac{1}{2}} = \sqrt{A}\left(1 + \frac{X}{2A} + \cdots\right)$$

gives 0 after integration

$$\int_\Omega \frac{\nabla f \cdot \nabla h}{\sqrt{1 + |\nabla f|^2}}$$

$$\operatorname{div}(F \cdot h) = (\operatorname{div} F)\, h + \underline{F \cdot \nabla h}$$

$$= \int_\Omega \operatorname{div}(Fh) - \int_\Omega (\operatorname{div} F)\, h$$

$$\int_\Omega \operatorname{div}(Fh) = \int_{\underline{\partial\Omega}} (F \cdot \vec{n})\, \underline{h} = 0, \qquad (\operatorname{div} F) = 0 \text{ lec. Minimal surface.}$$

For higher order term:
only term of order $|\nabla h|^2$ is

$$\frac{1}{2!}\,\frac{1}{2}\left(\frac{1}{2} - 1\right) = -\frac{1}{8}$$

small. controlled by $*$.

$$\sqrt{1 + |\nabla f|^2}\left(\frac{\nabla f \cdot \nabla h + |\nabla h|^2}{1 + |\nabla f|^2}\right)^2$$

$$= \frac{1}{\sqrt{1 + |\nabla f|^2}} \cdot \frac{|\nabla f \cdot \nabla h|^2}{1 + |\nabla f|^2} \le \frac{|\nabla f|^2 \cdot |\nabla h|^2}{\sqrt{\cdots}\,(1 + |\nabla f|^2)} \le \frac{|\nabla h|^2}{\sqrt{\cdots}}$$

2nd try Rigorous via 2nd variation.

$$A(t) = A(f + th) = \int_\Omega \sqrt{1 + |\nabla f + t\nabla h|^2}$$

$$A'(t) = \int_\Omega \frac{\langle \nabla f, \nabla h \rangle + t|\nabla h|^2}{\sqrt{1 + |\nabla f + t\nabla h|^2}}$$

$$A''(0) = \int_\Omega \frac{|\nabla h|^2}{\sqrt{1+|\nabla f|^2}} - \frac{\langle \nabla f, \nabla h \rangle^2}{\sqrt{1+|\nabla f|^2}^3}$$

$$= \int_\Omega \frac{1}{\sqrt{1+|\nabla f|^2}^3}\left((1 + |\nabla f|^2)|\nabla h|^2 - \langle \nabla f \cdot \nabla h \rangle^2 \right) \geq \int_\Omega \frac{|\nabla h|^2}{\sqrt{1+|\nabla f|^2}^3} > 0$$

so hyp surface is strictly stable (local minimum).

"Local stability" of Minimal submanifolds. in fact, locally "absolutely minimum".

Some General Notions in Minimal Surfaces

$$\vec{H} = \Delta\varphi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j\varphi)$$

To simplify the equation:

e.g. for $\varphi : \Delta \to \mathbb{R}^n$ graph case

$$(x,y) \mapsto (x, y, f_3(x,y), \dots, f_n(x,y))$$

$$\begin{cases} g_{11} = \varphi_x\cdot\varphi_x = 1+|f_x|^2 \\ g_{12} = \varphi_x\cdot\varphi_y = f_x\cdot f_y \\ g_{22} = \varphi_y\cdot\varphi_y = 1+|f_y|^2 \end{cases} \qquad f = (f_3, \dots, f_n)$$

$$g^{ij} = \frac{1}{g}\begin{bmatrix} 1+|f_y|^2 & -f_x\cdot f_y \\ -f_x\cdot f_y & 1+|f_x|^2 \end{bmatrix}$$

$$g = (1+|f_x|^2)\cdot(1+|f_y|^2) - (f_x\cdot f_y)^2$$

replace by f

$$\Delta\varphi = g^{ij}\partial_i\partial_j\varphi + \frac{1}{\sqrt{g}}\partial_j\varphi\cdot\underline{\partial_i(\sqrt{g}\,g^{ij}}$$

$$= \left\{ g^{ij}\partial_i\partial_j f + \frac{1}{\sqrt{g}}\partial_j f\cdot\partial_i(\sqrt{g}\,g^{ij})\right\} + \sum_{i,j}\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij})\,e_j$$

$$= \frac{1}{g}\left[(1+|f_y|^2)f_{xx} - 2(f_x\cdot f_y)f_{xy} + (1+|f_x|^2)f_{yy}\right]$$

j fixed: $\partial_i(\sqrt{g}\,g^{ij})$

$$= \frac{g^{kl}\partial_i g_{kl}}{2\sqrt{g}}\cdot g^{ij} - \sqrt{g}\,g^{ik}\,\partial_i g_{kl}\,g^{lj}$$

$=$ why this will be also equiv. to I. ?

since $\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j h) = \operatorname{div}(\nabla h) = \Delta h$

$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\cdot 1) = \Delta x_j$ (j fixed) get $\boxed{g^{ij}\partial_i\partial_j f + \partial_j f\,\Delta x_j = 0}$

Claim: this system of eq'n will show that $g^{ij}\partial_i\partial_j f = 0 \Rightarrow \Delta x_j = 0 \quad \forall j$

for component $\varphi^1 = x$, $\varphi^2 = y$, get

$$\sum_i \partial_i(\sqrt{g}\,g^{ij}) = 0 \qquad j = 1, 2.$$

so Eq'n is equiv. to

$$\begin{cases} \sum_{i,j} g^{ij}\partial_i\partial_j f = 0 & \text{I.} \\ \sum_i \partial_i(\sqrt{g}\,g^{ij}) = 0 \quad j=1,2 & \text{II.} \end{cases}$$

$$g_{ij} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}; \quad g^{ij} = \frac{1}{g}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

$$g = EG - F^2 = (1+|f_x|^2)(1+|f_y|^2) - (f_x \cdot f_y)^2$$

$$0 = \partial_i(\sqrt{g}\, g^{il}) = \partial_i\left(\frac{G^{il}}{\sqrt{g}}\right)$$

$$\boxed{\begin{aligned} &l=1: \\ &0 = \partial_x\left(\frac{1+|f_y|^2}{\sqrt{g}}\right) + \partial_y\left(\frac{-f_x\cdot f_y}{\sqrt{g}}\right) \\ &l=2 \\ &0 = \partial_x\left(\frac{-f_x\cdot f_y}{\sqrt{g}}\right) + \partial_y\left(\frac{1+|f_x|^2}{\sqrt{g}}\right) \end{aligned}}$$

* in case $n=2$:

$$g = 1 + |f_x|^2 + |f_y|^2 + |f_x \wedge f_y|^2$$

* in case of hyp. surf.

$$g = 1 + |Df|^2$$

In general, $g \overset{?}{=} 1 + \sum |f_i|^2 + \sum_{i<j} |f_i \wedge f_j|^2 + \cdots$

$$0 = \frac{1}{\sqrt{g}}\, 2 f_y \cdot f_{xy} - \frac{1+|f_y|^2}{2\sqrt{g}^3}\left\{\begin{aligned} &2 f_x\cdot f_{xx}(1+|f_y|^2) \\ &+ 2 f_y\cdot f_{xy}(1+|f_x|^2) \\ &- 2(f_x\cdot f_y)\left[\begin{aligned}&(f_{xx}\cdot f_y) \\ &+(f_x\cdot f_{xy})\end{aligned}\right] \end{aligned}\right\}$$

$$- \frac{1}{\sqrt{g}}\left(f_{xy}\cdot f_y + f_x\cdot f_{yy}\right)$$

$$+ \frac{(f_x\cdot f_y)}{2\sqrt{g}^3}\left\{\begin{aligned} &2 f_x\cdot f_{xy}(1+|f_y|^2) + 2 f_y\cdot f_{yy}(1+|f_x|^2) \\ &-2(f_x\cdot f_y)[f_{xy}\cdot f_y + f_x\cdot f_{yy}] \end{aligned}\right\}$$

Let $p = f_x$, $q = f_y$, $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

$g^{ij}\partial_i\partial_j \varphi = 0$ is $(1+|q|^2)r - 2(p\cdot q)s + (1+|p|^2)t = 0$

$$\begin{aligned} = &\; 2 q\cdot s\left(1+|p|^2+|q|^2+|p|^2|q|^2-(p\cdot q)^2\right) \\ &- (1+|q|^2)\left(p\cdot r(1+|q|^2) + q\cdot s(1+|p|^2) - p\cdot q(q\cdot r + p\cdot s)\right) \\ &- (q\cdot s + p\cdot t)\left(1+|p|^2+|q|^2+|p|^2|q|^2-(p\cdot q)^2\right) \\ &+ p\cdot q\left(p\cdot s(1+|q|^2) + q\cdot s(1+|p|^2) - p\cdot q(q\cdot s + p\cdot t)\right) \end{aligned}$$

$= \cancel{(1+|q|^2}\,)$ Too Complicated !?

Take inner product with φ_x, φ_y, get (since $\vec{H} \perp M$)

$$\begin{cases} \Delta x + f_x S + f_x^2 \Delta x + f_x f_y \Delta y = 0 \\ \Delta y + f_y S + f_x f_y \Delta x + f_y^2 \Delta y = 0 \end{cases}$$

ie.

$$\begin{cases} (1+f_x^2)\Delta x + f_x f_y \Delta y = -f_x S \\ f_x f_y \Delta x + (1+f_y^2)\Delta y = -f_y S \end{cases}$$

$\det = g \neq 0$. hence $S = 0 \Rightarrow \Delta x = 0 = \Delta y$.

$$\boxed{g^{ij}\partial_i\partial_j f^l = 0} \quad l = 3 \dots n$$

Eq'n for minimal graph of general codimensions

Fact:

$$\det(I + aa^T) = 1 + |a|^2$$

pf: 1 is an eigenvalue of multi $n-1$

$$(I + aa^T)a = a + aa^Ta = (1+|a|^2)a$$

ie eigenvalue $= 1 + |a|^2$
eigenvector $= a$. $\square$

For hypersurfaces:
can even simplified:

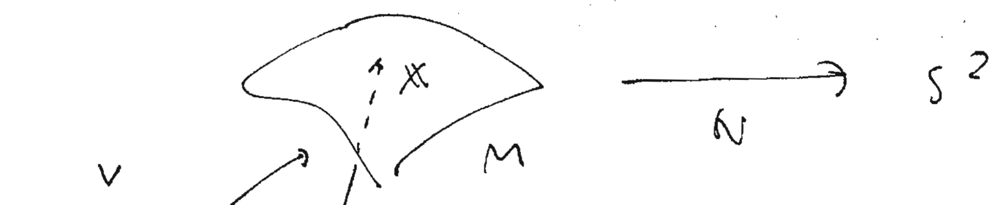
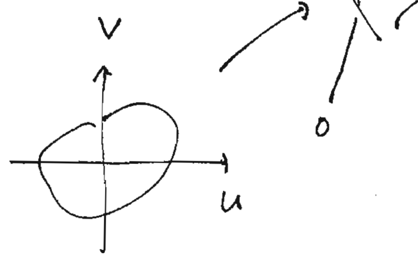
$$\partial_i\left(\frac{\partial_i f}{\sqrt{1+|Df|^2}}\right)$$

$$= \frac{\partial_i\partial_i f}{\sqrt{1+|Df|^2}} - \frac{\partial_i f \cdot 2\partial_i\partial_j f \cdot \partial_j f}{2\sqrt{1+|Df|^2}^{\,3}}$$

$$= \frac{1}{\sqrt{1+|Df|^2}^{\,3}}\Big[\underbrace{(1+|Df|^2)\delta_{ij} - \partial_i f\,\partial_j f}_{\text{basically} \sim g^{ij}}\Big]\partial_i\partial_j f$$

Famous Divergence form in 2nd order PDE.

Mean Curvature formula for hyp. surf. graphs :

local coor. (u, v), may think $X = X(u,v)$

coor. vector $\partial_u, \partial_v = X_u, X_v$

$$B(\partial_i, \partial_j) = \left(\nabla^{\mathbb{R}^3}_{\partial_i}\partial_j\right)^N = X_{ij}^N$$

i.e. $B_{ij} = X_{ij} \cdot N$

So $H = \operatorname{Tr} B = g^{ij} B_{ij} = g^{ij} X_{ij} \cdot N$ $\quad \left(= \Delta X \cdot N \ \text{ one way} \right)$

$$= g^{ij}\left(\cancel{X_i \cdot N}\right)_j - g^{ij} X_i \cdot N_j = -\langle dX, dN\rangle$$

$\|$

$$\boxed{-\sum_{\alpha=1}^{3} N^\alpha_\alpha \ \text{ on } M}$$

rewrite this is standard coor. in $\mathbb{R}^3$ e_1, e_2, e_3

hold for general hypersurfaces.

⊙ in case $X(x,y) = (x, y, f(x,y))$

$$N = \frac{(-f_x, -f_y, 1)}{\sqrt{1+|\nabla f|^2}}$$

$N^3_3 = 0$ bec. f is unrelated to x_3

$$\Rightarrow \quad H = -(N^1_1 + N^2_2) = \sum_{\alpha=1}^{2}\left(\frac{f_\alpha}{\sqrt{1+|\nabla f|^2}}\right)_\alpha \quad \#.$$

Rmk: for $\nu \in \Gamma(N)$, 2nd fund. form. op.

$A^\nu : T_M \to T_M \ : \ A^\nu(v) := \nabla^T_v \nu$ is self adjoint in fact:

$$\langle A^\nu(v), w\rangle = \langle \nabla^T_v \nu, w\rangle = \langle \nabla_v \nu, w\rangle = v\langle \cancel{\nu, w}\rangle - \langle \nu, \nabla_v w\rangle = -\langle \nu, B(v,w)\rangle$$

Mean Curvature Eq^n v.s harmonic Coordinates :

$$\vec{H} = \Delta \varphi = g^{ij}\partial_i\partial_j\varphi + \frac{1}{\sqrt{g}}\partial_j\varphi\, \underline{\partial_i\left(\sqrt{g}\, g^{ij}\right)}$$

$$= g^{ij}\partial_i\partial_j\varphi + \partial_j\varphi\, \underline{\Delta x_j}$$

all these are in tangent part but $\vec{H} \perp M$. hence must $\vec{H} = (\vec{S})^\perp$ trivially.

$$0 = \partial_k\varphi \cdot \vec{H} = \varphi_k \cdot \vec{S} + g_{kj}\Delta x_j$$

i.e. $\quad \Delta x_j = -g^{jk}(\varphi_k \cdot \vec{S})$ $\quad$ or.

$$\vec{H} = \vec{S} - \underline{\varphi_j \cdot g^{jk}(\varphi_k \cdot \vec{S})} = (\vec{S})^\perp$$

projection. back side

Minimal graphs :

$$\frac{\partial}{\partial t}\langle \nabla f + t\nabla h, \nabla f + t \nabla h\rangle$$

$$A(f) = \int_\Omega \sqrt{1+|\nabla f|^2}$$

$$\frac{d}{dt} A(f+th)\Big|_{t=0} = \int_\Omega \frac{2\langle \nabla f, \nabla h\rangle}{2\sqrt{1+|\nabla f|^2}}$$

$F\cdot\nabla h + (\mathrm{div} F)h = \mathrm{div}(Fh)$ $\Rightarrow$

$$= -\int_\Omega \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) h$$

Eq'n for minimal graph $f : \Omega \to \mathbb{R}$ in $\mathbb{R}^{n+1}$:

$$(*) \quad \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \sum_i D_i\left(\frac{D_i f}{\sqrt{1+|\nabla f|^2}}\right) = 0 .$$

However, the solution is not nec. a Plateau Solution

eg.

Ω

this can also gives non-uniqueness of Plateau Problem.

Moreover (see Gilbarg-Trudinger p. 352. Thm 14.14)

Thm (Jenkins / Serrin 1968)

Let $\Omega \subset \mathbb{R}^n$ be bounded $C^{2,\gamma}$ domain, then the Dirichlet prob $(*)$ with $f = \varphi$ on $\partial\Omega$ solv'ble $\forall \varphi \in C^{2,\gamma}(\bar\Omega) \Leftrightarrow H|_{\partial\Omega} \geq 0$. (ie convex for $n=2$).