2021. 9. 23.

Textbook: Clay - AMS, Hori, Vata, et al. Mirror Symmetry, 2003

"Part 1 (150 pp) prelimaries in math" ← assumed.

Part 2. prelimories in physics — path integrals / super symmetry

Part 3 & 4 proof

Part 5 (next semester)

Geometry and Topological Field Theory

↑

Calabi - Yau manifolds

(M, g): oriented Riemannian manifold

$dvol_p \neq 0$ for all $p \in M$.

complex analogue

M: complex manifold, $n = \dim_{\mathbb{C}} X$

Ω: holomorphic volume form i.e. $\Omega \in \Gamma(M, K_M)$ $\Rightarrow K_M \simeq \mathcal{O}_M \Rightarrow c_1(M) = 0$.

$\Omega_p \neq 0$ for all $p \in M$

Rieman surface $n=1$ $M = \mathbb{CP}^1$

$M = \mathbb{C}/\Lambda$

general type (classify by genus: g)

$n=2$ κ: Kodiara dimension

$\kappa = -\infty$ Enrique theorem

$\kappa = 0$ ← not necessarily flat e.g. $K3$, $\sum_{i=0}^{3} x_i^4 = 0$.

$$\begin{tikzcd} E \arrow[r, dashed] & M \arrow[d] \\ \mathbb{C}/\Lambda_s \arrow[u] & S \end{tikzcd}$$

$\kappa = 1$

$\kappa = 2$ general type.

$n=3$ ~1980, Mori, minimal model program. X

$\kappa(X)$ $h^0(X, K_X^{\otimes m}) \sim m^{\kappa(X)}$ for large m.

$\kappa = -\infty$ ← ~1990 abundance theorem, Miyaoka, Kawamata, "uniruled"

$\underline{\kappa = 0}$ Yau's theorem on Calabi - conjecture (1976)

get stuck!

CY 3!

$c_1(X)_{\mathbb{R}} = 0 \Rightarrow \exists!$ Ricci flat Kähler metric in each Kähler class

Corollary $\tilde{X} = \tilde{A} \times \tilde{B} \times \tilde{C}$ — CY $\qquad X = \tilde{X}/\pi_1(X)$

$\tilde{A} \cong \mathbb{C}^n$ flat, $\qquad \tilde{B}$ hyperkähler Sp, $\qquad \tilde{C}$ CY SU

$A = \mathbb{C}^n/\Lambda$

$SU(3) \Rightarrow \pi_1 = 0$

Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{22} & & 0 & \\ 1 = h^{30} & & \underline{h^{21}} & & h^{12} & & h^{30} = 1 \\ & 0 & & \underline{h^{11}} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$h^{11} = h^1(X, \Omega^1_X)$

$h^{21} = h^1(X, T_X)$ Kähler-Specier theorem

$\Rightarrow (h^{11}, h^{21})$: "coordinate" of space of Einstein metrics (Ricci flat)

$\kappa = 1$, $\kappa = 2$: $\quad X \dashrightarrow Z^{\kappa(X)}$ Iitaka fibration.

fiber has $\kappa = 0$.

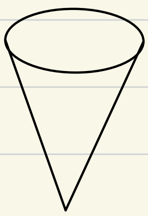
$\kappa = 3$?

The meta goal of this course :

To understand the role of CY3 in physics / math through the study of TFT.

$L \quad X \quad \{y\}$ Wilson

$|L^{\otimes n}| : X \longrightarrow Z$ Oguiso



K_X : Kähler cone
"
ample

$h^{11} =: \mathrm{Pic}(X) \geq$ "12"

$\Rightarrow \exists\ L$ to do dimension reduction.

QFT $\rightsquigarrow$ conformal FT $\rightsquigarrow$ TFT

has special "cases" theories can be understood by math!

Topological twisted theories : A-model

B-model

half-twisted ......

String theory, Witten 1980, Bott

"QFT" choice of M^d (special : Euclidean or Minkowski space)

g

Fields V vector bundle ∇: connection $\leftarrow$ Gauge fields

$$\begin{tikzcd} V \arrow[d] \\ M \arrow[u, bend left, "X = s"] \end{tikzcd}$$

$X = s$: matter fields

$M \xrightarrow{X} N$, σ-model.

maps $\longleftrightarrow$ fields

"path integrals" $\equiv$ integration over the space of fields.

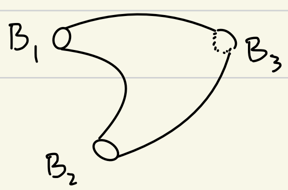
"Quantum gravity" $\equiv$ integration also on "g"

$Z = \int DX\, e^{-S(X)}$ or $e^{-iS(X)}$ $\quad S(X)$: action functional on fields

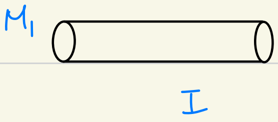
Operator formalism.

$d \geq 1$ $\quad \partial M = \coprod B_i$ $\quad \mathcal{H}_i$: Hilbert space (boundary values) on B_i.

path integral : $\bigotimes_i \mathcal{H}_i \longrightarrow \mathbb{C}$



Example : Let $M = M_1 \times I$, $I = [0, T]$



$$U(T) : \mathcal{H} \longrightarrow \mathcal{H}^* \simeq \mathcal{H}$$

$$U(T_2)\, U(T_1) = U(T_2 + T_1) \quad \Rightarrow \quad U(T) = e^{-TH}$$

QFT makes sense only up to $d \le 6$.

almost rigorous up to $d \le 1$.

$d = 2$ requires "mirror symmetry".

QFT in $d = 0$

$X : M \longrightarrow \mathbb{R}$ is just a variable. ($M = pt$)

$$Z = \int_{\mathbb{R}} dx\, e^{-S(x)}$$

For example, $\int dx\, e^{-\left(\frac{\alpha}{2}x^2 + i\varepsilon x^3\right)} =: Z(\alpha, \varepsilon)$ for ε small.

$$Z(\alpha, 0) = \sqrt{\frac{2\pi}{\alpha}}$$

$$= \int dx \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}x^2} \frac{(-i\varepsilon x^3)^n}{n!}$$

parity contraction

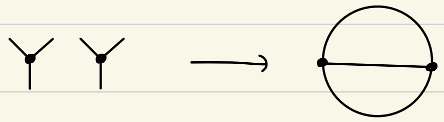
"Feymann diagrams"

$$f(\alpha, J) := \int e^{-\frac{\alpha}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{J^2}{2\alpha}} \xrightarrow{\frac{\partial}{\partial J}} \frac{J}{\alpha} e^{\frac{J^2}{2\alpha}} \xrightarrow{\frac{\partial}{\partial J}} \frac{1}{\alpha} e^{\frac{J^2}{2\alpha}} + \left(\frac{J}{\alpha}\right)^2 e^{\frac{J^2}{2\alpha}}$$

$$\left.\frac{\partial^r J}{\partial J^r}\right|_{J=0} = \int dx \cdot x^r \cdot e^{-\frac{\alpha}{2}x^2} = \left(\frac{1}{\alpha}\right)^{\frac{r}{2}} \cdot \#\text{ of contractions}$$

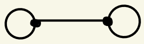
1^{st} correction term $n=2$

$$\frac{(i\varepsilon)^2}{2!}\int dx\; x^3\cdot x^3\cdot e^{-\frac{\alpha}{2}x^2} \;=\; \frac{(-i\varepsilon)^2}{2}\left(\frac{1}{\alpha}\right)^3\cdot 15$$



$3!$

$6+9=15$



3^2

propagator weighted by $\frac{1}{\alpha}$

<u>Exercise 1</u>

$$Z(\alpha,\varepsilon) = e^{\sum_{\Gamma} n_{\Gamma}}$$

connected graph, 3-regular

$$n_{\Gamma} = \frac{(-3!\,i\,\varepsilon)^V}{a^E}\;\frac{1}{\mathrm{Aut}(\Gamma)}$$

V = # vertices of Γ

E = # edges of Γ

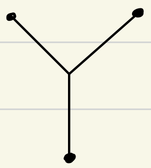
Multivariable case:

$$S(x^1,\dots,x^N,M,C) = \frac{1}{2}M_{ij}\,x^i x^j + C_{ijk}\,x^i x^j x^k$$

$M = (M_{ij})$, $C = (C_{ijk})$; M_{ij} positive definite.

$$Z(M,C=0) = \int dx^1\cdots dx^N\; e^{-\frac{1}{2}M_{ij}x^i x^j} = \frac{(2\pi)^{N/2}}{(\det M)^{1/2}}$$

For $C \neq 0$: small.



j k i

weight : $-C_{ijk}$

$\rightsquigarrow$ $Z(M,C)$ = set the result as in <u>Ex 1</u>.

propagator  i j weight M^{ij}

Bosmic

Fermion Grassmannian

$$\theta_1 \theta_2 = -\theta_2 \theta_1$$

Y. Manin book : super determinant , Berzenin integral.

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$\Lambda^{m/n}$: super space

$= \underbrace{\mathbb{R}^m}_{\text{even degree}} \times \underbrace{\Lambda^n}_{\text{odd degree}}$

Euclidean	Grussman
Bosonic	Fermionic
x_i	θ_j $\theta^2=0$

All analytic functions are linear $a+b\theta$. $\theta_1 \cdots \theta_k$

Differentiation is easy, using $\mathbb{Z}_2$-graded rule

e.g. left hand rule $\frac{\partial}{\partial\theta_1}\theta_1\theta_2 = \theta_2$

$\frac{\partial}{\partial\theta_2}\theta_1\theta_2 = -\theta_1$

Integration : require translation invariant $\Rightarrow \int_\Lambda 1\, d\theta = 0$, $\underline{\int_\Lambda \theta\, d\theta = 1}$ (normalize)

define it!

$$\int_\Lambda (\theta+\underset{=}{\eta})\, d(\theta+\underset{=}{\eta}) = \int_\Lambda \theta\, d\theta + \eta \int_\Lambda d\theta \Rightarrow \int_\Lambda d\theta = 0.$$

constant

Change of variable formula :

$$b = \int_\Lambda (a+b\theta)\, d\theta = \int_\Lambda \big(a+b(p\xi+q)\big)\, \underline{p}\, d\xi$$

$\theta = p\xi + q$, $p, q \in \mathbb{R}$ expect p^{-1} !?

We require that $d\theta = \underline{\left(\frac{d\theta}{d\xi}\right)^{-1}} d\xi$

Berezinian (compare to Jacobian)

General case : $\Lambda^{m/n} \circlearrowright$ change variable.

$$(x,\theta) \longrightarrow (y,\xi)$$

$$J = \left(\frac{\partial(x,\theta)}{\partial(y,\xi)}\right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\mathrm{Ber}(J) = \underline{s\cdot\det}(J)$

super determinant

unique determined by supertrace $T: V_+ \oplus V_- \longrightarrow V_+ \oplus V_-$

$\mathrm{str}(T) = \mathrm{tr}_{V_+}(T) - \mathrm{tr}_{V_-}(T)$

Practically, use Gauss elimination in matrix form, if D^{-1} exists

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$$

$\mathrm{Ber}(J) := \det(A - BD^{-1}C)\cdot \det(D)^{-1}$.

<u>Exercise 2</u> Prove change of variable formula.

x^i : even $\qquad x^i \psi^a = \psi^a x^i$

ψ^a : odd $\qquad \psi^a \psi^b = -\psi^b \psi^a$

$(a = 1, 2 \dots n)$

$$\int \psi^1 \cdots \psi^n \, d\psi^1 \cdots d\psi^n = 1 .$$

$$Z = \int \prod_i dx^i \prod_a d\psi^a \, e^{-S(x,\psi)}$$

<u>Example</u> $S(\psi) = \frac{1}{2} \sum \underbrace{M_{ij}}_{M_{ij} = -M_{ji}} \psi^i \psi^j$

$$Z = \int \prod_k d\psi^k \, e^{-\frac{1}{2} M_{ij} \psi^i \psi^j} = Pf(M) \text{ , where } Pf(M)^2 = \det(M) .$$

The first non-trivial case with both x, ψ:

$$Z = \int dx \, d\psi^1 d\psi^2 \, \underbrace{e^{-(S_0(x) + \psi^1 \psi^2 S_1(x))}}_{\parallel \atop e^{-S_0(x)} \cdot \sum_{n=0}^{\infty} \frac{(\psi^1\psi^2)^n}{n!} S_1(x)^n} = \int dx \, e^{-S_0(x)} \cdot S_1(x)$$

Special case : $S(x, \psi^1, \psi^2) = \underbrace{\frac{1}{2} h'(x)^2}_{S_0(x)} - \underbrace{h''(x)}_{S_1(x)} \psi^1 \psi^2$, $h(x)$: polynomial.

Infinitesimal supersymmetry : $\delta x = \varepsilon^1 \psi^1 + \varepsilon^2 \psi^2$, $\varepsilon_1, \varepsilon_2$: odd.

$$\delta \psi^1 = \varepsilon^2 h' \qquad (*)$$

$$\delta \psi^2 = -\varepsilon^1 h'$$

<u>Fact</u> $\delta S = 0$: $S = \frac{1}{2} h'(x)^2 - h'' \psi^1 \psi^2$

$$\delta S = h'(x) h''(x) \underbrace{\delta x}_{\varepsilon^1\psi^1 + \varepsilon^2\psi^2} - h'' \left(\varepsilon^2 h' \psi^2 - \underbrace{\psi^1 \varepsilon^1}_{\overset{\parallel}{-\varepsilon^1 \psi^1}} h' \right) = 0 .$$

<u>Exercise 2</u> Show the invariance of S, $dx \, d\psi^1 d\psi^2$ under $(*)$.

Supersymmetric localization principle:

If $h'(x)=0$ for all $x\in\mathbb{R}$, then $Z=0$.

idea: use $\hat{x} = x - \dfrac{\psi^1\psi^2}{h'}$, $\hat{\psi}^1 = \alpha\psi^2$, $\hat{\psi}^2 = \psi^1+\psi^2$.

$\alpha=\alpha(x)\neq 0$ for all x.

to eliminate ψ^1 in S.

Then, we get
$$\begin{aligned} S(\hat{x}, 0, \hat{\psi}^2) &= \frac{1}{2} h'(\hat{x})^2 + 0 \\ &= \frac{1}{2}\left(h'(x) - h''(x)\frac{\psi^1\psi^2}{h'}\right)^2 \\ &= \frac{1}{2}h'(x)^2 - h''(x)\psi^1\psi^2 = S(x,\psi^1,\psi^2) \end{aligned}$$

$$\int dx\, d\psi^1 d\psi^2\, e^{-S(x,\psi_1,\psi_2)} = \int d\hat{x}\, d\hat{\psi}^1 d\hat{\psi}^2\, \underbrace{\mathrm{Ber}\left(\frac{\partial(x,\psi^1,\psi^2)}{\partial(\hat{x},\hat{\psi}^1,\hat{\psi}^2)}\right)}_{\text{Jacobian of } J} e^{-S(\hat{x},0,\hat{\psi}^2)}$$

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1+\frac{h''\psi^1\psi^2}{(h')^2} & -\frac{\psi^2}{h'} & \frac{\psi^1}{h'} \\ \alpha'\cdot\psi^1 & \alpha & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} A-BD^{-1}C & B \\ 0 & D \end{pmatrix}$$

$$\mathrm{Ber} = \alpha^{-1}\cdot\left(1+\frac{h''\psi^1\psi^2}{(h')^2} - \frac{\alpha'}{\alpha}\frac{\psi^1\psi^2}{h'}\right) = \underline{\underline{\alpha^{-1}}} + \underbrace{\frac{h''\psi^1\psi^2}{\alpha h'^2} - \frac{\alpha'\psi^1\psi^2}{\alpha^2 h'}}_{= \left(\frac{1}{\alpha h'}\right)'\psi^1\psi^2}$$

$\int = 0$

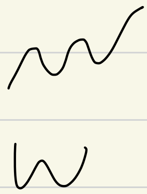
$\underline{\left(\frac{1}{\alpha h'}\right)'\psi^1\psi^2}$ ok!

$\int = 0$ with some boundary condition.

check!

If $h'(x_c) = 0$ for some x_c, then $Z = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|} = \sum_{x_c} \pm 1 = \begin{cases} 1 \\ -1 \end{cases}$ odd degree, 0 even degree

$h(x)$:

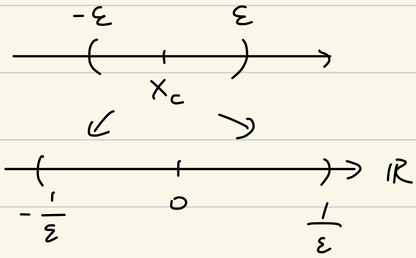


$\Rightarrow 1$

signed degree

$\Rightarrow 0$

idea: At x_c, use scaling (blow-up) of x coordinate.



$$h(x) = h(x_c) + \frac{\alpha_c}{2}(x - x_c)^2 + \cdots \qquad \alpha_c = h''(x_c)$$

$$= \frac{1}{2}\alpha_c^2 (x - x_c)^2 + \alpha_c \psi^1 \psi^2 + \cdots$$

$$\Rightarrow Z = \sum_{x_c} \int_{x_c - \varepsilon}^{x_c + \varepsilon} \frac{dx\, d\psi^1 d\psi^2}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha_c^2 (x - x_c)^2 + \alpha_c \psi^1 \psi^2 + \cdots}$$

. normalized volume

$$= \sum_{x_c} \frac{\alpha_c}{|\alpha_c|}$$

Check: $Z = \frac{1}{\sqrt{2\pi}} \int dx\, e^{\frac{1}{2}h'^2} h''$

$y = h'(x)$ ↓

$$= \text{s-deg}\, h' \cdot \frac{1}{\sqrt{2\pi}} \int dy\, e^{-\frac{1}{2}y^2} = \text{s-deg}\, h'$$

signed degree

Complex case: (0-dimensional) Landau-Ginzburg model.

LG

$$S(z, \bar z, \psi_1, \psi_2, \bar\psi_1, \bar\psi_2) = |\partial W|^2 - \partial^2 W \psi_1 \psi_2 - \overline{\partial^2 W}\, \bar\psi_1 \bar\psi_2, \quad W(z): \text{holomorphic in } z.$$

on $\Lambda^{2/4} \otimes \mathbb{C}$

4 real supersymmetry: $\delta, \bar\delta$

$$\begin{cases} \delta z = \epsilon^1 \psi_1 + \epsilon^2 \psi_2 & \delta \bar z = 0 \\ \delta \psi_1 = \epsilon^2 \partial W & \delta \bar\psi_1 = 0 \\ \delta \psi_2 = -\epsilon^1 \overline{\partial W} & \delta \bar\psi_2 = 0 \end{cases} \qquad \begin{cases} \bar\delta \bar z = \bar\epsilon^1 \bar\psi_1 + \bar\epsilon^2 \bar\psi_2 & \bar\delta z = 0 \\ \bar\delta \bar\psi_1 = \bar\epsilon^2 \partial W & \bar\delta \psi_1 = 0 \\ \bar\delta \bar\psi_2 = -\bar\epsilon^1 \partial W & \bar\delta \psi_2 = 0 \end{cases}$$

For $\epsilon' = \epsilon^2$, we get $\delta^2 = 0$. Similarly, $\bar{\epsilon}' = \bar{\epsilon}^2$, we get $\bar{\delta}^2 = 0$.

Assume all critical points of W are non-degenerate.

$$W(z) = W(z_c) + \frac{\overbrace{W''(z_c)}^{=\alpha}}{2!}(z - z_c)^2 + \cdots$$

localization: (if you believe it!?)

$$Z = \frac{1}{2\pi}\int e^{-S}\, dz\, d\bar{z}\, d\psi_1\, d\psi_2\, d\bar{\psi}_1\, d\bar{\psi}_2$$

$$-|\alpha(z-z_c)|^2 + \alpha\psi_1\psi_2 + \bar{\alpha}\bar{\psi}_1\bar{\psi}_2 + \cdots$$

$$= \frac{1}{2\pi}\sum_{W'(z_c)=0} |\alpha|^2 \int e^{-|\alpha(z-z_c)|^2}\, dz\, d\bar{z} = \sum_{W'(z_c)=0} 1 .$$

For general correlation function $\langle f \rangle = \int f\, dx\, d\psi\, e^{-S}$.

e.g. $f = z^i \bar{z}^\ell$ $\rightsquigarrow$ no supersymmetry exists to fix $\langle f \rangle$!

For $f(z)$ holomorphic, $\bar{\delta}$ supersymmetry "fixes $\langle f \rangle$".

$$\Rightarrow \langle f \rangle = \sum_{W'(z_c)=0} f(z_c)$$

Similarly, for antiholomorphic $g(\bar{z})$, use δ. $\Rightarrow \langle g \rangle = \sum_{W'(z_c)=0} g(\bar{z}_c)$.

<u>Multivariable case:</u>

$$S(z_i, \bar{z}_i, \psi_1^i, \bar{\psi}_1^i, \psi_2^i, \bar{\psi}_2^i) = \sum_{i=1}^{N} |\partial_i W|^2 - \sum_{i,j}\left(\partial_i\partial_j W \psi_1^i \psi_2^j + \overline{\partial_i\partial_j W}\, \bar{\psi}_1^i \bar{\psi}_2^j\right)$$

$(i = 1, 2 \ldots N)$ 6N variables $W(z_1 \ldots z_N)$: holomorphic (polynomial)

$\rightsquigarrow$ 4N real supersymmetry

localization $\Rightarrow$ $\langle f(z) \rangle$, $\langle g(\bar{z}) \rangle$

correlation function for bosonic fields only.

Taking $\overline{\epsilon}_i^1 = \overline{\epsilon}_i^2$, get $\overline{\delta}^2 = 0$.

$$\overline{\delta}(f \overline{\psi}_1^i) = f\, \partial_j W\, \overline{\epsilon}_j\, \overline{\psi}_1^i$$

$$R = \overline{\delta} \text{ cohomology ring} := \frac{\overline{\delta}\text{-closed}}{\overline{\delta}\text{-exact}} \simeq \mathbb{C}[z_1 \dots z_n] \Big/ \langle \partial_1 W, \dots, \partial_N W \rangle$$

chiral ring in physics.   Jacobi ring   make sense $\langle \overline{\delta} \Lambda \rangle = 0$.

Singularity Theory

$W(z_1, \dots, z_N) = 0$ defines an isolated hypersurface singularity.

quasi-homogeneous polynomial : $W(\lambda^{q_1} z_1 \dots \lambda^{q_N} z_N) = \lambda W(z)$, $\lambda \in \mathbb{C}$

Exercise 3 Poincaré polynomial : $p(t) := \sum_{x_\alpha \in R^{\text{homog.}}} t^{wt(x_\alpha)}$

Show $p(t) = \prod_{i=1}^{N} \dfrac{1 - t^{1-q_i}}{1 - t^{q_i}}$

$\Rightarrow \dim R = ?$

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Last time correction

$$x = \hat{x} + \frac{\hat{\psi}_1 \hat{\psi}_2}{h'(\hat{x})} \qquad \hat{x} = x - \frac{\psi_1\psi_2}{h'(x)}$$

$$\psi_1 = \hat{\psi}_1 \quad \Longleftrightarrow \quad \hat{\psi}_1 = \psi_1$$

$$\psi_2 = \hat{\psi}_2 - \hat{\psi}_1 \qquad \hat{\psi}_2 = \psi_1 + \psi_2$$

h: analytic , $h' \neq 0$, $h'(\hat{x}) = h'\left(x - \frac{\psi_1\psi_2}{h'(x)}\right) = h'(x) - \frac{\psi_1\psi_2}{h'(x)} h''(x) + \cdots$

$$J = \left(\begin{array}{c|cc} 1 - \frac{h''(\hat{x})\hat{\psi}_1\hat{\psi}_2}{h'(\hat{x})^2} & -\frac{\hat{\psi}_2}{h'} & -\frac{\hat{\psi}_1}{h'} \\ \hline 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right) \qquad \mathrm{Ber} = 1 - \frac{h''(\hat{x})\hat{\psi}_1\hat{\psi}_2}{h'(\hat{x})^2}$$

$$S(\hat{x}, 0, \hat{\psi}_2) = S(x, \psi_1, \psi_2)$$

$$\int e^{-\frac{1}{2}S(x,\psi_1,\psi_2)} dx\, d\psi_1\, d\psi_2 = \int e^{-\frac{1}{2}h'(\hat{x})^2}\left(1 - \frac{h''(\hat{x})\hat{\psi}_1\hat{\psi}_2}{h'(\hat{x})^2}\right) d\hat{x}\, d\hat{\psi}_1\, d\hat{\psi}_2$$

$$= -\int e^{-\frac{1}{2}h'(\hat{x})^2} \frac{h''}{(h')^2} dt$$

$$\left(\frac{e^{-\frac{1}{2}h'^2}}{h'}\right)' = \frac{e^{-\frac{1}{2}h'^2}}{h'} \cdot 2h'h'' - \frac{e^{-\frac{1}{2}h'^2} h''}{h'^2}$$

$$\Lambda^{n|m} = \mathbb{R}^n_{even} \oplus \mathbb{R}^m_{odd} \ ; \quad \mathbb{R}_e ,\ \mathbb{R}_o$$

$$\Lambda(\underbrace{\theta_1, \theta_2, \theta_3, \dots}_{\text{Grassman variable.}}) = \Lambda^{even} \oplus \Lambda^{odd}$$

QFT in d=1 (i.e. quantum mechanics)

$X: M \longrightarrow \mathbb{R}$

$I=[a,b]$, S^1_β, $\mathbb{R}$

$$S = \int L\,dt = \int \Big(\underbrace{\tfrac{1}{2}\dot{x}^2}_{動能} - \underbrace{V(x)}_{位能}\Big)\,dt \qquad \dot{x} = \frac{dx}{dt}$$

$$\delta S = \int \left(\dot{x}\,\delta\dot{x} - \frac{dV}{dx}(x)\,\delta x\right) dt \underset{\text{with boundary condition}}{=} -\int \Big(\underbrace{\ddot{x} + V'(x)}_{=\,0}\Big)\,\delta x\,dt$$

Euler-Lagrange equation.

E. Noether's procedure :

S has translation invariant $t \longmapsto t+\alpha$.

Variation of parameter on "$\alpha(t)$"

$$X_s = x(t+s\alpha) \Rightarrow \delta x = \frac{d}{ds} x_s \Big|_{s=0} = \dot{x}\,\alpha$$

$(s \in \mathbb{R})$

$$\Rightarrow (\delta x)' = \ddot{x}\alpha + \dot{x}\alpha'$$

$$\Rightarrow \delta S \underset{\substack{\text{at } x(t) \\ \text{solving the E-L equation}}}{=} \int \dot{x}\left(-V'(x)\,\alpha + \dot{x}\alpha'\right) - V'(x)\,\dot{x}\,\alpha = 2\int dt\, \dot{\alpha}\left(\tfrac{1}{2}\dot{x}^2 + V(t)\right) = 0$$

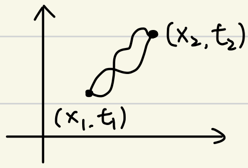
$$\Rightarrow \tfrac{1}{2}\dot{x}^2 + V(t) = \text{constant}$$

Thus, $H := \frac{1}{2}\dot{x}^2 + V(x)$ is constant called Noether's charge with respect to t.

=: Hamiltonian.

$$Z(x_2,t_2,x_1,t_1) = \int DX(t)\, e^{iS(x)}$$

"X": all possible path from (x_1,t_1) to (x_2,t_2).



$\uparrow$ This could be defined.

$$\Rightarrow Z_{t_2,t_1} = \underset{x_1}{\mathcal{H}} \longrightarrow \underset{x_2}{\mathcal{H}} =: L^2(\mathbb{R},\mathbb{C}) \quad \text{by} \int_{\mathbb{R} \ni x_1} Z(x_2,t_2,x_1,t_1)\, f(x_1)\, dx_1$$

Time invariance $\Rightarrow$ e^{-itH} for some operator H.

How to compute H? We have

<u>Theorem</u> $H = \frac{1}{2}p^2 + V(x)$ with the rule (quantization rule)

$$\underset{\text{動量}}{\underline{p}} = \frac{\partial L}{\partial x} = \dot{x} \longmapsto p = -i\frac{d}{dx}$$

conjugate mechanism

$$x \longmapsto x\,\cdot$$

(multiply x)

Classically: Poisson bracket $\{f,g\}$. We have $\{x,p\} = 1$.

Now, $[x,p] = xp - px = i$. (測不準原理!?)

We check this "theorem" by examples: $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$.

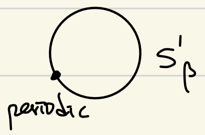
$$H = \frac{1}{2}(p^2+x^2) \underset{[x,p]=i}{=} \frac{1}{2}(p+ix)(p-ix) + \frac{1}{2} =: a^\dagger a + \frac{1}{2}$$

$$\begin{cases} a = \frac{1}{\sqrt{2}}(p-ix) \\ a^\dagger = \frac{1}{\sqrt{2}}(p+ix) \end{cases}$$

On $M = S^1_\beta$ (periodic with period β) and apply the Wick rotation $t \mapsto -i\tau$.
(analytic continuation)

Then, $Z(\beta) = \int_{\mathbb{R}} Z_\beta(x_1, x_1)\, dx_1 = \operatorname{tr} e^{-\beta H}$.

use eigenfunction expression.



From H: $[a, a^\dagger] = aa^\dagger - a^\dagger a = \frac{1}{2}(p - ix)(p + ix) - \frac{1}{2}(p + ix)(p - ix)$
$= i\,(px - xp) = 1$.

$[H, a] = a^\dagger a a - a a^\dagger a = -a$ (decreasing operator)
$[H, a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger$ (increasing operator)

For $H\psi = \lambda\psi$ $(\lambda \geq 0)$, $Ha\psi = (aH - a)\psi = (\lambda - 1)\, a\psi$.
$Ha^\dagger\psi = (\lambda + 1)\, a^\dagger\psi$.

$|0\rangle$ ground state $:=$ $a\underbrace{|0\rangle}_{\text{vector}} = 0$. Then, $H|0\rangle = \frac{1}{2}|0\rangle$.

$\mathcal{H}$ is spanned by $|n\rangle = (a^\dagger)^n |0\rangle$ with $\lambda = E_n = n + \frac{1}{2}$.

Then, $\operatorname{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n + \frac{1}{2})} = \dfrac{e^{-\beta/2}}{1 - e^{-\beta}} = \dfrac{1/2}{\sinh(\beta/2)}$

<u>Remark</u>: $a\psi_0 = 0$. $\left(-i\frac{d}{dx} - ix\right)\psi_0(x) = 0$.

From path integral, $Z(\beta) = \int_{X(t+\beta) = X(t)} \text{"}DX(t)\text{"}\, e^{-S_E(x)}$

$S_E(x) = \frac{1}{2}\int dt\left(\dot{x}^2 + x^2\right) \underset{\text{integration by part.}}{=} \frac{1}{2}\int dt\; x\underbrace{\left(-\frac{d^2}{dt^2} + 1\right)}_{H} x$

$H f_n = \lambda_n f_n$, $\lambda_n := 1 + \left(\frac{2\pi n}{\beta}\right)^2$, $n \in \mathbb{Z}$.

In this Fourier coordinate system, we get $X(t) = \sum_{n\in\mathbb{Z}} x_n f_n(t)$.

$$\Rightarrow Z(\beta) = \int \text{"} \prod_n \frac{dx_n}{\sqrt{2\pi}} \text{"} \, e^{-\frac{1}{2}\sum \lambda_n x_n^2} = \text{"} \prod_n \frac{1}{\sqrt{\lambda_n}} \text{"}$$

$$= \underbrace{\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2}}_{??} \cdot \underbrace{\prod_{n=1}^{\infty} \left(1 + \left(\frac{2\pi n}{\beta}\right)^{-2}\right)^{-1}}_{= \frac{\beta/2}{\sinh \beta/2}}$$

Zeta function regularization

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Let $\zeta_1(s) = \left(\frac{\beta}{2\pi}\right)^{-s} \cdot \zeta(s)$. Then, $\zeta_1'(0) = 2\log\frac{\beta}{2\pi} \underbrace{\zeta(0)}_{= -\frac{1}{2}} + 2\underbrace{\zeta'(0)}_{= -\frac{1}{2}\log(2\pi)}$

$$e^{\zeta_1'(0)} = \frac{1}{\beta} \quad \rightsquigarrow \quad \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2} = \frac{1}{\beta}.$$

σ-model on S^1_R

$$S^1_\beta \xrightarrow{X} S^1_R$$

$S^1_\beta = M$ i.e. $X \sim X + R$

operator formalism:

$$S(x) = \int \frac{1}{2}\dot{x}^2 \, dt, \quad H = \frac{1}{2}p^2 = -\frac{1}{2}\frac{d^2}{dx^2}$$

$$\psi_n = e^{\frac{2\pi i n x}{R}}, \quad E_n = \frac{2\pi^2 n^2}{R^2}$$

$$Z(\beta) = \operatorname{tr} e^{-\beta H} = \sum_{n=-\infty}^{\infty} e^{-\frac{2\pi \beta n^2}{R^2}}$$

path-integral formalism: Let $X_m(\tau)$ be of winding number $= m$.

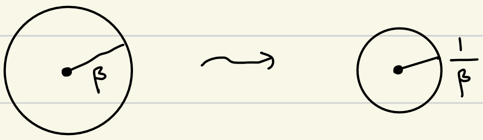
Then, $X_m(\tau) = \frac{m\tau R}{\beta} + X_0(\tau)$

$$Z(\beta) = \int DX \, e^{-\int_0^\beta \frac{1}{2}\dot{x}^2 d\tau} = \sum_{m=-\infty}^{\infty} \int DX_m \, e^{-S_E(X_m)}$$

$$= \sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}} \int DX_0 \, e^{-\int_0^\beta X_0\left(-\frac{1}{2}\frac{d^2}{dt^2}\right)X_0 \, dt} = \frac{R}{\sqrt{2\pi\beta}} \sum_{-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}}$$

As before, we get $\vartheta(t) = t^{1/2}\,\vartheta\left(\frac{1}{t}\right)$

$$\vartheta(t) = \sum_{-\infty}^{\infty} e^{-\pi m^2 t}$$



"T-duality".

2021. 10. 4

Super symmetric QM

$x(t)$, $\overbrace{\psi(t), \bar{\psi}(t)}^{\text{"}\mathbb{R}\text{-independent" !}}$

$\psi = \psi_1 + i\psi_2$

$$L = \frac{1}{2}\dot{x}(t)^2 - \frac{1}{2}h'^2 + \frac{i}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) - h''\bar{\psi}\psi .$$

Let $\begin{cases} \delta x = \varepsilon\bar{\psi} - \bar{\varepsilon}\psi \\ \delta\psi = \varepsilon(i\dot{x} + h') \\ \delta\bar{\psi} = \bar{\varepsilon}(-i\dot{x} + h') \end{cases}$ for later use, we let $\varepsilon = \varepsilon(t)$.

$$\Rightarrow \delta L = \dot{x}\left(\varepsilon\dot{\bar{\psi}} - \bar{\varepsilon}\dot{\psi}\right) + \dot{x}\left(\dot{\varepsilon}\bar{\psi} - \dot{\bar{\varepsilon}}\psi\right) - hh''\left(\varepsilon\bar{\psi} - \bar{\varepsilon}\psi\right)$$
$$+ \frac{i}{2}\Big(\bar{\varepsilon}(-i\dot{x} + h')\dot{\psi} + \bar{\psi}\dot{\varepsilon}(i\dot{x} + h') + \bar{\psi}\varepsilon(i\ddot{x} + h''\dot{x})$$
$$- \dot{\bar{\varepsilon}}(-i\dot{x} + h')\psi - \bar{\varepsilon}(-i\ddot{x} + h''\dot{x})\psi - \dot{\bar{\psi}}\varepsilon(-i\dot{x} + h')\Big)$$
$$- h'''\left(\varepsilon\bar{\psi} - \bar{\varepsilon}\psi\right)\psi\bar{\psi} - h''\bar{\varepsilon}(-i\dot{x} + h')\psi - h''\bar{\psi}\varepsilon(i\dot{x} + h')$$
$$= \frac{d}{dt}(\cdots) - i\dot{\varepsilon}\bar{\psi}\underbrace{(i\dot{x} + h')}_{=\,Q} - i\dot{\bar{\varepsilon}}\psi\underbrace{(-i\dot{x} + h')}_{=\,\bar{Q}} .$$

super charges

Given $\varepsilon_1, \varepsilon_2$, we have $[\delta_1, \delta_2] = 2i\left(\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1\right)\frac{d}{dt}$ ... via E-L equation.

"δ" : "square root" of $\frac{d}{dt}$.

Quantization

conjugate momentum $\quad p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}$$

using integration by parts

$$L = \cdots + i\bar{\psi}\dot{\psi} .$$

or $\psi_n \cdots \psi_1$

$$\int \psi_1 \cdots \psi_n \, d\psi_1 \cdots d\psi_n = 1$$

right derivative !?

$$\text{Poisson } \{\ \} \begin{cases} \text{Boson } [\ ,\ ] \\ \text{Fermion } \{a,b\} = ab+ba \end{cases}$$

We require that $[\hat{x},\hat{p}] = i$ (classical)

$\hookrightarrow \{\hat{\psi},\hat{\pi}\} = i$ $\quad \{\psi,\bar{\psi}\} = 1$

$$H = p\dot{x} + \pi\dot{\psi} - L \longmapsto \frac{1}{2}p^2 + \frac{1}{2}h'^2 + \frac{1}{2}h''\underbrace{(\bar{\psi}\psi - \psi\bar{\psi})}_{=[\bar{\psi},\psi]}$$

They are "operators".

Representation on the "Hilbert space of states"

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F = L^2(\mathbb{R},\mathbb{C})\underbrace{|0\rangle}_{} \oplus L^2(\mathbb{R},\mathbb{C})\bar{\psi}|0\rangle$$

A vector with $\psi|0\rangle = 0$.

Quantize:

$$\begin{cases} x \mapsto x\cdot = \hat{x} \\ p \mapsto -i\frac{d}{dx} = \hat{p} \\ \psi \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \hat{\psi} \\ \bar{\psi} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \hat{\bar{\psi}} \end{cases}$$

$$\psi\bar{\psi} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\bar{\psi}\psi \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: F$$

Fermion number operator

$$Q \longmapsto Q = \bar{\psi}(ip + h')$$

$$\bar{Q} \longmapsto \bar{Q} = \psi(-ip + h') = Q^T$$

Hermitian conjugate.

check: $[H, Q] = 0 = [H, \bar{Q}]$.

For $F = \bar{\psi}\psi$, $[F,\psi] = F\psi - \psi F = \bar{\psi}\psi\psi - \underbrace{\psi\bar{\psi}}_{=(1-\bar{\psi}\psi)}\psi = -\psi$

$$[F,\bar{\psi}] = \bar{\psi}$$

$\Rightarrow [F,Q] = Q$, $[F,\bar{Q}] = -\bar{Q}$, $Q,\bar{Q}$: exchange $\mathcal{H}^B$, $\mathcal{H}^F$.

$\{Q,Q\}=0$, $\{\bar{Q},\bar{Q}\}=0$ since $Q^2=0=\bar{Q}^2$

Key computation: $\{Q,\bar{Q}\}=2H$ (check it!)

$$\begin{gathered}\{Q,\bar{Q}\} \\ \| \\ Q\bar{Q}+\bar{Q}Q=(Q+\bar{Q})^2\end{gathered}$$

$\rightsquigarrow$ Some kind of "Hodge theory" $Q=d$, $\bar{Q}=d$, $2H=\Delta$. !?

$\mathcal{H}^B$: even degree form

$\mathcal{H}^F$: odd degree form

$\mathcal{H}_{(n)} := \overset{\neq 0}{\lambda_n}$ eigenspace of H , $\lambda_0=0$.

$\Rightarrow \mathcal{H}^B_{(n)} \xrightarrow{Q+\bar{Q}} \mathcal{H}^F_{(n)}$ is an isomorphism if $n\neq 0$.

For $M=S^1_\beta$, $\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \mathrm{str}\, e^{-\beta H} = \mathrm{tr}\,(-1)^F e^{-\beta H}$

called Witten index. independent of β.

$$Z(\beta)=\mathrm{tr}\, e^{-\beta H}$$

Supersymmetry ground state, i.e. $H\Phi=0 \Leftrightarrow Q\Phi=0=\bar{Q}\Phi$

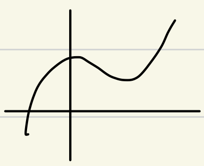
Wick rotation $t\mapsto i\tau$ on S^1_β

$\Phi = f_1(x)|0\rangle + f_2(x)\bar{\Psi}|0\rangle \Rightarrow f_1'+h'f_1=0$ i.e. $f_1(x)=c_1e^{-h(x)}$

$-f_2'+h'f_2=0$ $f_2(x)=c_2e^{h(x)}$

We need L^2 condition, suppose h is polynomial:

I



X : not L^2

II.

$\Phi=e^{-h(x)}|0\rangle$

$\mathrm{tr}\,(-1)^F=1$

III.

$\Phi=e^{h(x)}\bar{\Psi}|0\rangle$

$\mathrm{tr}\,(-1)^F=-1$.

$\Rightarrow$ At most 1 ground state.

pertribative analysis : $h'(x_i) = 0$.

All of these generalize to multivariables, complex case : L-G model.

$h = -\mathrm{Re}\, W(z^1 \dots z^m)$

$\Rightarrow$ super symmetry ground state $\overset{1-1}{\longleftrightarrow}$ critial points of W.

Sigma Model for QFT $d=1$

$$\phi : T \longrightarrow M$$

1-dimensional space, e.g. $[0, t]$

(M, g) : Riemannian manifold

Boson $\phi(t) = (x^i(t))$

Fermion $\psi, \bar{\psi} \in \Gamma(T, \phi^* TM \otimes \mathbb{C})$ $\quad \psi = \sum \psi^i \frac{\partial}{\partial x^i}$, $\bar{\psi}$

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{\sqrt{-1}}{2} g_{ij} \left(\bar{\psi}^i \underline{D_t} \psi^j - D_t \bar{\psi}^i \psi^j \right) - \frac{1}{2} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l$$

covariant derivative

$$D_t \psi^j = \dot{\psi}^j + \Gamma^j_{lm} \dot{x}^l \psi^m$$

Super symmetry :
$$\delta x^i = \varepsilon \bar{\psi}^i - \bar{\varepsilon} \psi^i$$
$$\delta \psi^i = \varepsilon \left(\sqrt{-1}\, \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k \right)$$
$$\delta \bar{\psi}^i = \bar{\varepsilon} \left(-\sqrt{-1}\, \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k \right)$$

If $\varepsilon, \bar{\varepsilon}$: constant, Fermion, then $\underline{\delta \int L \, dt = 0}$.

Homework 1, part 1.

$\Rightarrow$ conserved super charges : $Q = \sqrt{-1}\, g_{ij} \bar{\psi}^i \dot{x}^j$, $\bar{Q} = -\sqrt{-1}\, g_{ij} \psi^i \dot{x}^j$

Phase rotation : $\psi^i \longmapsto e^{\sqrt{-1}\gamma} \psi^i$ fixes L.

$\Rightarrow$ charge $F = g_{ij} \bar{\psi}^i \psi^j$

Quantization:

$$\hat{p}_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\dot{x}^j + \Gamma^j_{im}\, g_{kj}\, \bar{\psi}^k \psi^m$$

$$\pi_i = \frac{\partial L}{\partial \dot{\psi}^i} = \sqrt{-1}\, g_{ij}\, \bar{\psi}^j$$

same reason as $M=\mathbb{R}$.

So $Q = \sqrt{-1}\,\bar{\psi}^i \hat{p}_i$, $\bar{Q} = -\sqrt{-1}\,\psi^i \hat{p}_i$

canonical relations: $[\hat{x}^i, \hat{p}_j] = \sqrt{-1}\,\delta^i_j$

$$[\hat{\psi}^i, \hat{\pi}_j] = \sqrt{-1}\,\delta^i_j$$

$$\{\hat{\psi}^i, \hat{\bar{\psi}}^j\} = g^{ij}$$

$\mathcal{H} = \Omega^{L^2}(M)\otimes\mathbb{C}$, $\langle \omega_1, \omega_2\rangle = \int_M \bar{\omega}_1 \wedge *\omega_2$.

$\hat{x}^i = x^i\cdot$

$\hat{p}_i = -\sqrt{-1}\,\nabla_{\frac{\partial}{\partial x^i}}$

$\hat{\psi}^i = g^{ij}\,\iota_{\frac{\partial}{\partial x^j}}$

on $\mathbb{Z}$-graded.

$\hat{\bar{\psi}}^i = dx^i\wedge$

$\Rightarrow |0\rangle = 1$, $F = \sum_i dx^i \wedge \iota_{\frac{\partial}{\partial x^i}} = k$ on $\Omega^k(M)$.

$Q = \sum_i dx^i \wedge \nabla_i \equiv d$

$\bar{Q} = \sum g^{ij}\,\iota_{\frac{\partial}{\partial x^j}} \cdot \nabla_{\frac{\partial}{\partial x^i}} \equiv d^*$ $\left(:= (-1)^{?} * d * \right)$

$H = \frac{1}{2}\{Q, \bar{Q}\} = \frac{1}{2}(dd^* + d^*d) = \frac{1}{2}\Delta$.

Homework 1, part 3.

Exercise Show that "in physical sense" for $T = S^1_\beta$, let $\beta \to 0$

$$\chi(M) = \operatorname{tr}(-1)^F e^{-\beta H} = \frac{1}{(2\pi)^{n/2}} \int_M \operatorname{Pf}(-R)$$

Gauss - Bonnet - Chern.

2021. 10. 7

"<u>Theorem</u>" (Witten 1981 Nuclear Physics B)

For $T \longrightarrow (M, g)$ non-linear σ-model,

canonical quantization $\longmapsto$ Hodge-deRham complex.

"<u>Theorem</u>" (Witten JDG 1982)

Introduce potential h (Morse function)

$$d_h := d + dh \wedge \cdot = e^{-h} d e^{h} \rightsquigarrow d_h, d_h^*, \Delta_h$$

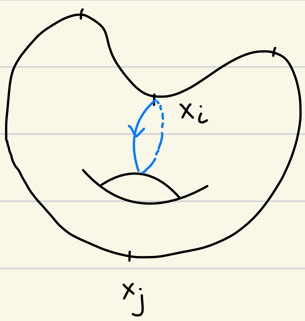
Then, for λh, $\lambda \to \infty$, we get the "Morse complex".

$\nabla h(x_i) = 0 \rightsquigarrow$ Morse index $=$ # of negative eigenvalues of $Hh(x_i)$.

$M = \coprod (\text{cells})$: CW-complex

Question: boundary map d? product structure?

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d} \underbrace{C^{k}}_{\mathbb{R}^m,\ m = \#\text{ of critical point with Morse index } = k} \xrightarrow{d} C^{k+1} \longrightarrow \cdots$$



$$\frac{d\gamma}{dt} = \nabla h \circ \partial$$

"Floer" 1987

"Weil conjecture"

⋮

rigorous proof:

"Weiping Zhang" or "Ziming Ma" Ph.D. Thesis.

$T \longrightarrow (M, g)$, h $\quad\quad \psi = \psi^i \frac{\partial}{\partial x^i}$, $\bar\psi \in \Gamma(T, \phi^* TM \otimes \mathbb{C})$

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{\sqrt{-1}}{2} g_{ij} \left(\bar\psi^i \nabla_t \psi^j - \nabla_t \bar\psi^i \psi^j \right) - \frac{1}{2} R_{ijkl} \psi^i \bar\psi^j \psi^k \bar\psi^l \leftarrow L_0$$

$$- \frac{1}{2} g^{ij} \partial_i h \partial_j h - \nabla_i (\partial_j h) \bar\psi^i \psi^j \quad \leftarrow \Delta L$$

Super symmetry : $\begin{cases} \delta x^i = \varepsilon \bar\psi^i - \bar\varepsilon \psi^i \\ \delta \psi^i = \varepsilon \left(\sqrt{-1} \dot{x}^i - \Gamma^i_{jk} \bar\psi^j \psi^k + g^{ij} \partial_j h \right) \\ \delta \bar\psi^i = \bar\varepsilon \left(-\sqrt{-1} \dot{x}^i - \Gamma^i_{jk} \bar\psi^j \psi^k + g^{ij} \partial_j h \right) \end{cases}$

supercharges : $Q = \bar\psi^i (\sqrt{-1} p_i + \partial_i h)$, $\quad p_i = -\sqrt{-1} \nabla_i = \frac{\partial L}{\partial \dot{x}^i}$

$\bar{Q} = \psi^i (-\sqrt{-1} p_i + \partial_i h)$

Fermion rotation $\longmapsto F = g_{ij} \bar\psi^i \psi^j$

Quantization : $\begin{cases} Q = d + dh \wedge \cdot = e^{-h} d e^{h} =: d_h \\ \bar{Q} = d_h^* \end{cases}$

Hamiltonian : $H = \frac{1}{2} \{ Q, \bar{Q} \} = \frac{1}{2} \Delta_h$

$H_Q^* \simeq H_{dR}^*(M)$ for all h. (chain isomorphism)

Now, let h be Morse , $x_1, \dots, x_N$ be critical points of h.

Rescaling : $h \longmapsto \lambda h$, $\lambda \gg 0$, $\quad 2H_\lambda = \Delta_\lambda = \Delta + \lambda^2 |\nabla h|^2 + \lambda \nabla_i \partial_j h [\bar\psi^i, \psi^j]$.

Pertubation analysis at x_i : $h(x) = h(x_i) + \frac{1}{2} \sum \underline{C_I} (x^I)^2 + \cdots$

Morse lemma , C_I : eigenvalues of Hess$(h)(x_i)$.

$$H_\lambda \sim \frac{1}{2} \sum_{I=1}^{n} \left(p_I^2 + \lambda^2 C_I^2 (x^I)^2 + \lambda C_I [\bar\psi^I, \psi^I] \right)$$

$H_\lambda \Phi = 0 \underset{\text{pertubative ground state}}{\rightsquigarrow} \Phi_i^{(0)}$

$\overset{1-1}{\longleftrightarrow}$ critical point $x_1 \dots x_N$.

Recall (Last time)

$T \to \mathbb{R}$, h

$$H\Psi = 0 \iff Q\Psi = 0 = \bar{Q}\Psi \qquad \begin{cases} Q = \bar{\psi}(\sqrt{-1}\,p + h') \\ \bar{Q} = \psi(-\sqrt{-1}\,p + h') \end{cases}$$

$$\rightsquigarrow \Psi = f_1(x)\,|0\rangle + f_2(x)\,\bar{\psi}\,|0\rangle$$

$$\Rightarrow f_1 = c_1 e^{-h(x)}, \quad f_2 = c_2 e^{h(x)}$$

Example Harmonic oscilator : $h(x) = \frac{\omega}{2}x^2$

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 + \frac{\omega}{2}[\bar{\psi}, \psi].$$

$$\Psi_{\omega>0} = e^{-\frac{1}{2}\omega x^2}\,|0\rangle$$

$$\Psi_{\omega<0} = e^{-\frac{1}{2}|\omega| x^2}\,\bar{\psi}\,|0\rangle$$

$$\Rightarrow \Psi_i^{(0)} = e^{-\lambda \sum |C_I| (x^I)^2} \Big(\prod_{C_j<0} \bar{\psi}^j \Big) \underbrace{|0\rangle}_{= 1 \text{ constant function}}$$

$$\Rightarrow \Psi_i \in \Omega^{\mu_i}(M) \otimes \mathbb{C}, \quad \mu_i = \text{Morse index at } x_i.$$

↑ pertubative ground state

"Theorem" (Witten, Floer, ...)

Let $C^\mu = \bigoplus_{\mu_i = \mu} \mathbb{C}\Psi_i$, then

$$0 \to C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} C^2 \to \cdots \to C^n \to 0 \qquad \text{Morse-Witten complex}$$

is given by $Q\Psi_i = \sum_{\mu_j = \mu_i + 1} \Psi_j \langle \Psi_j, Q\Psi_i \rangle + (\cdots)$ — small terms corresponding to non-zero energy state.

$\lambda \gg 0$

$$\langle \Psi_j, Q\Psi_i \rangle = \sum_\gamma n_\gamma e^{-\lambda(h(x_j) - h(x_i))},$$

γ : sum over all gradient line connecting x_i, x_j.

$n_\gamma = \pm$ depend on orientation of $\Psi_j \wedge * \Psi_i$

(WKB approximation : $\varepsilon f''(x) + a(x) f' + b(x) = 0$ solution approximation as $\varepsilon \to 0$?
generalization to many case in physics)

$\Downarrow$

<u>Lemma</u> $\langle \Phi_j , Q \Phi_i \rangle = \dfrac{1}{h(x_i) - h(x_j) + o(\lambda^{-1})} \lim_{T\to\infty} \langle \Phi_j , e^{-TH} [Q, h] e^{-TH} \Phi_i \rangle$.

The limit term is $\displaystyle \int_{\substack{\phi(-\infty) = x_i \\ \phi(\infty) = x_j}} D\phi\, D\psi\, D\bar{\psi}\, e^{-S_E}\, \bar{\psi}^I \partial_I h \Big|_{\tau=0}$

$[Q, h] = dh \wedge$

with fast descreasing condition

on $\dfrac{d\phi}{d\tau}$, $\psi(\tau)$, $\bar{\psi}(\tau)$ as $\tau \to \pm\infty$.

$$S_E = \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2} |\dot{\phi}|^2 + \frac{\lambda^2}{2} |\nabla h|^2 + g_{ij} \bar{\psi}^i D_t \psi^j + \lambda (\nabla_i \partial_j h) \bar{\psi}^i \psi^j + \frac{1}{2} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l \right)$$

Boson part :

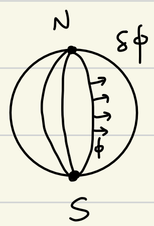
$$S_B = \int_{-\infty}^{\infty} \frac{1}{2} |\dot{\phi} + \lambda \nabla h|^2 \mp \underbrace{\lambda \int_{-\infty}^{\infty} \underline{\dot{\phi}} \cdot \nabla h}_{= \mp \lambda (h(x_j) - h(x_i))} > 0 .$$

$\underline{\dot{\phi}} = \dfrac{d\phi}{d\tau}$

$\underline{\mp}$ choose to make it > 0

<u>Definition</u> Instanton := minimizer i.e. $\dot{\phi}^i \pm \lambda \nabla h = 0$. gradient lines

How many ? (We want $< \infty$)



$D_\pm (\delta\phi) := D_\tau (\delta\phi) \pm \lambda \underline{H_h} (\delta\phi)$

Hessian operator : $T_x M \longrightarrow T_x M$.

For the Fermion bilinear part:

$$S_{\psi\bar\psi} = \int_{-\infty}^{\infty} d\tau < \bar\psi , D_+ \psi > = -\int_{-\infty}^{\infty} d\tau < D_- \bar\psi , \psi >$$

path integral $\neq 0 \Rightarrow \bar\psi$: 0-mode $-$ ψ : 0-mode ?

$=:$ index of $D_- = 1$

|

localize to instantons.

Since S_E is invariant under supersymmetry $(t \mapsto -\sqrt{-1}\tau)$

$[Q, h] = \bar\psi^I \partial_I h$ invariant under δ_ε (i.e. $\bar\varepsilon = 0$) given by Q.

$$\delta\bar\psi^i = \bar\varepsilon(\cdots)$$
$$\delta x^i = \varepsilon\bar\psi^i - \underbrace{\bar\varepsilon}_{0}\psi^i$$

$$\rightsquigarrow \delta\psi^i = \underline{\underline{\varepsilon}}\left(-\frac{dx^i}{d\tau} + \lambda g^{ij}\partial_j h - \underbrace{\Gamma^i_{jk}\bar\psi^j\psi^k}_{0}\right)$$

coming from δx^i.

The δ_ε fixed loci $\Rightarrow \dot\phi = \lambda\nabla h$.

<u>Lemma</u> (§10.5.2) index $D_- = \mu_j - \mu_i$. (reading)

Choose h generic such that $\ker D_+ = \operatorname{coker} D_- = 0$ along any γ.

$\Rightarrow \gamma$ instanton $x_i \rightsquigarrow x_j$ with $\mu_j - \mu_i = 1$, then $\underline{\ker D_- = 1}$.

given by time shift $\tau \mapsto \tau + \tau_1$
$\gamma \mapsto \gamma_{\tau_1}$

Now, calculate the path integral, using "mode expansion"

$$\bar\psi^i = \bar\psi_0 \frac{d\gamma_{\tau_1}}{d\tau} + \underbrace{\cdots}_{\text{non-zero modes.}}$$

$\bar{\Psi}_0$: zero mode gives $e^{-\lambda(h(x_i)-h(x_j)) \underline{\int_{-\infty}^{\infty} d\tau_1 \int d\bar{\Psi}_0 \bar{\Psi}^i \partial_i h \big|_{\tau=0}}}$

$$\int_{-\infty}^{\infty} d\tau_1 \frac{dh(\gamma(\tau_1))}{d\tau_1} = h(x_j) - h(x_i) .$$

2021. 10. 14

QFT in 1+1 dim

Free theory : $\Sigma = \mathbb{R} \times S^1 \xrightarrow{X} \mathbb{R} = M$

$t, s \quad X(t,s)$

$s \sim s + 2\pi$

$$S = \frac{1}{2\pi} \int_\Sigma \underbrace{\frac{1}{2} \left[\overset{\text{"}+1\text{"}}{\left(\frac{\partial X}{\partial t}\right)^2} - \overset{\text{"}-1\text{"}}{\left(\frac{\partial X}{\partial s}\right)^2} \right]}_{L} dt\, ds$$

$$\delta S = \frac{1}{2\pi} \int_\Sigma \left[\frac{\partial X}{\partial t} \left(\frac{\partial}{\partial t} \delta X \right) - \frac{\partial X}{\partial s} \left(\frac{\partial}{\partial s} \delta X \right) \right] dt\, ds$$

$$= \frac{-1}{2\pi} \int_\Sigma \delta X \left(\frac{\partial^2 X}{\partial t^2} - \frac{\partial^2 X}{\partial s^2} \right) dt\, ds$$

integration by parts

$\rightsquigarrow$ Euler - Lagrange equation $(\partial_t^2 - \partial_s^2) X = 0$.

$$\Rightarrow X(t,s) = \underbrace{f(t-s)}_{\text{right move}} + \underbrace{g(t+s)}_{\text{left move}}$$

Noether's charges at "equation of motion" :

$$p = \frac{1}{2\pi} \int_{S^1} j^t\, ds \quad , \quad j^t = \partial_t X \text{ and } j^s = -\partial_s X .$$

It comes from shifting in X : $\delta X = \alpha(t,s)$.

Also, the (t,s)-translation symmetry :

$$\frac{d}{d\varepsilon} X(t + \varepsilon c^t, s + \varepsilon c^s) \Big|_{\varepsilon=0} = \sum X_\mu c^\mu := \delta_c X$$

$$\Rightarrow \underset{(*)}{\delta S} = \frac{1}{2\pi} \int_\Sigma T^\nu_\mu \partial_\nu c^\mu = 0 \text{ for all } c = (c^\mu) \Longleftrightarrow \partial_\nu T^\nu_\mu = 0 .$$

Then, we get conserved charges (by $\int_{S^1}$)

$$H = \frac{1}{2\pi}\int_{S^1} T^t{}_t \, ds = \frac{1}{2\pi}\int_{S^1} \frac{1}{2}(X_t^2 + X_s^2)\, ds$$ Hamiltonian

$$P = \frac{1}{2\pi}\int_{S^1} T^t{}_s \, ds = \frac{1}{2\pi}\int_{S^1} X_t X_s \, ds$$ World sheet momentum

$$\frac{dP}{dt} = \frac{1}{2\pi}\int_{S^1} \frac{1}{2}\frac{d}{ds}(X_s)^2 \, ds = 0.$$

$$\frac{dH}{dt} = \frac{1}{2\pi}\int_{S^1} \frac{d}{ds}(X_s X_t)\, ds = 0.$$

(*): $$\delta_c S = \frac{1}{2\pi}\int_\Sigma X_t \frac{\partial}{\partial t}(X_\mu c^\mu) - X_s \frac{\partial}{\partial s}(X_\mu c^\mu)$$

$$= \frac{1}{2\pi}\int_\Sigma \underline{X_t X_{tt} c^t} + \underline{X_t X_{ts} c^s} - \underline{X_s X_{st} c^t} - \underline{X_s X_{ss} c^s}$$

$$+ \underline{X_t X_t c^t_t} + X_t X_s c^s_t - X_s X_t c^t_s - \underline{X_s X_s c^s_s}$$

$$\left(\frac{1}{2\pi}\int \frac{d}{dt}\left(\frac{1}{2}(X_t^2 - X_s^2)\right) c^t = \frac{1}{2\pi}\int -\frac{1}{2}(X_t^2 - X_s^2)\, c^t_t \right)$$

$$= \frac{1}{2\pi}\int_\Sigma \underbrace{\frac{1}{2}(X_t^2 + X_s^2)}_{= T^t{}_t} c^t_t + \underbrace{X_t X_s}_{= T^t{}_s} c^s_t - \underbrace{X_s X_t}_{= T^s{}_t} c^t_s - \underbrace{\frac{1}{2}(X_t^2 + X_s^2)}_{= T^s{}_s} c^s_s$$

How to Quantize?

Idea: Treat S^1 as ∞-many degree of freedom via Fourier series.

$$X(t,s) = X_0(t) + \sum_{n\neq 0} X_n(t)\, e^{ins}, \quad X_{-n} = \overline{X_n}.$$

$$S = \int dt \left[\frac{1}{2}\dot{X}_0 + \sum_{n=1}^{\infty} \left(|\dot{X}_n|^2 - n^2 |X_n|^2 \right) \right]$$ X_n: the n-th sector. $(n \in \mathbb{Z})$

Sector x_0 : $p_0 = \dot{x}_0 \longmapsto \hat{p}_0 = -\sqrt{-1}\frac{d}{dx_0}$

$H_0 = \frac{1}{2}p_0^2$

$|k\rangle_0$ has energy $\frac{1}{2}k^2$, $k \in \mathbb{R}$.

$\shortparallel$

e^{ikx_0} (not L^2, ...)

Sector x_n : Lagrangian $x_n = \frac{1}{\sqrt{2}}(x_{1n} + \sqrt{-1}\, x_{2n})$.

$$L_n = \left(\frac{1}{2}\dot{x}_{1n}^2 - \frac{n^2}{2}x_{1n}^2\right) + \left(\frac{1}{2}\dot{x}_{2n}^2 - \frac{n^2}{2}x_{2n}^2\right)$$: two harmonic oscillators.

$p_{1n} = \dot{x}_{1n}$ $\qquad p_n = p_{1n} + \sqrt{-1}\, p_{2n}$

$p_{2n} = \dot{x}_{2n}$

$$H_n = \left(\frac{1}{2}\hat{p}_{1n}^2 + \frac{n^2}{2}\hat{x}_{1n}^2\right) + \left(\frac{1}{2}\hat{p}_{2n}^2 + \frac{n^2}{2}\hat{x}_{2n}^2\right)$$

$$= n\left(a_{1n}^\dagger a_{1n} + \frac{1}{2}\right) + n\left(a_{2n}^\dagger a_{2n} + \frac{1}{2}\right)$$

$$a_{in}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_{in}}{\sqrt{n}} + \sqrt{-1}\sqrt{n}\,\hat{x}_{in}\right)$$

$$a_{in} = \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_{in}}{\sqrt{n}} - \sqrt{-1}\sqrt{n}\,\hat{x}_{in}\right)$$

Canonical relations: $[a_{in}, a_{jn}^\dagger] = \delta_{ij}$, others $= 0$.

$\rightsquigarrow$ get creation / annihilation operators

$a_{in}^\dagger \qquad a_{in}$

Go back to "complex coordinates":

$$\alpha_n = \sqrt{\frac{n}{2}}\,(a_{1n} + \sqrt{-1}\, a_{2n}) = \frac{\sqrt{n}}{2}\left(\frac{p_{1n}+\sqrt{-1}\,p_{2n}}{\sqrt{n}} - \sqrt{-1}\sqrt{n}\,(x_{1n}+\sqrt{-1}\,x_{2n})\right)$$

$/\!/ \ \frac{1}{\sqrt{2}}(p_n - \sqrt{-1}\, n\, x_n)$

$$\tilde{\alpha}_n = \sqrt{\frac{n}{2}}\,(a_{1n} - \sqrt{-1}\, a_{2n})$$

$/\!/ \ \frac{1}{\sqrt{2}}(\bar{p}_n + \sqrt{-1}\, n\, \bar{x}_n) = \sqrt{\frac{n}{2}}\,(a_{1n}^\dagger - \sqrt{-1}\, a_{2n}^\dagger)$

$$\Rightarrow \begin{cases} \alpha_{-n} := a_n^\dagger \\ \tilde{\alpha}_{-n} := \tilde{a}_n^\dagger \end{cases}$$

The canonical relation is $[\alpha_n, \alpha_{-n}] = n = [\tilde{\alpha}_n, \tilde{\alpha}_{-n}]$, others $= 0$.

$\rightsquigarrow H_n = \alpha_n^\dagger \alpha_n + \tilde{\alpha}_n^\dagger \tilde{\alpha}_n + n.$ $(n=1,2,3\ldots)$

$|0\rangle_n$: the vector killed by $\alpha_n, \tilde{\alpha}_n$ $\Rightarrow H_n|0\rangle_n = n|0\rangle_n \rightsquigarrow$ get $\mathcal{H}_n$.

Consider $H = \sum_{n\geq 0} H_n$

$$\mathcal{H} := \bigotimes_{n\geq 0} \mathcal{H}_n$$

$$\text{“} \, |k\rangle = |k\rangle_0 \otimes \bigotimes_{n\geq 1} |0\rangle_n \, \text{”}$$

$$H = \sum_{n\geq 0} H_n = \frac{1}{2} p_0^2 + \sum_{n\geq 1} (\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n) + \text{“}\sum_{n\geq 1} n\text{”}$$

$\overset{\text{“}}{=} \zeta(-1) = \frac{-1}{12}$: the energy of $|0\rangle$

general states are obtained by applying $\alpha_{-n}, \tilde{\alpha}_{-n}$ on $|k\rangle$.

Since $[H, x_0]f = -\frac{1}{2}(x_0 f)'' + \frac{1}{2} x_0 f'' = -f' = -\sqrt{-1}\, p_0 f$.

$-\sqrt{-1}\frac{\partial x_0}{\partial t} = [H, x_0] = -\sqrt{-1}\, p_0$, $[H, p_0] = 0$ $\Rightarrow$ $x_0(t) = x_0 + t p_0$.

(equation of motion on operators)

Also, $\underset{-\sqrt{-1}\frac{\partial \alpha_n}{\partial t}}{\overset{\shortparallel}{[H, \alpha_n]}} = -n\alpha_n$ $\Rightarrow \begin{cases} \alpha_n(t) = e^{-\sqrt{-1}nt}\alpha_n \\ \tilde{\alpha}_n(t) = e^{-\sqrt{-1}nt}\tilde{\alpha}_n \end{cases}$

$$x_n = \frac{\tilde{\alpha}_{-n} - \alpha_n}{\sqrt{2}\sqrt{-1}\, n}$$

$$\Rightarrow X(t,s) = x_0 + t p_0 + \frac{\sqrt{-1}}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \Big(\underbrace{\alpha_n e^{-\sqrt{-1}n(t-s)}}_{R} + \underbrace{\tilde{\alpha}_n e^{-\sqrt{-1}n(t+s)}}_{L} \Big)$$

the general solution of operators in t.

$$\underline{e^{\sqrt{-1}\,k\,X(t,s)}} \ ?$$

<u>Definition</u> (Normal ordering) For $n \geqslant 1$,

$$:\alpha_{-n}\alpha_n: \, = \, :\alpha_n\alpha_{-n}: \, = \alpha_{-n}\alpha_n$$

$$:x_0 p_0: \, = \, :p_0 x_0: \, = x_0 p_0$$

$$:e^{\sqrt{-1}\,k\,X(t,s)}: \, = U^{\dagger}\, e^{\sqrt{-1}\,k x_0}\, e^{\sqrt{-1}\,k p_0}\, U,$$

where $U := e^{\sqrt{-1}\,k \frac{\sqrt{-1}}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \tilde{z}^{-n}\right)}$, $\quad z := e^{i(t-s)}$, $\tilde{z} := e^{i(t+s)}$

$$X(t_1,s_1)\cdot X(t_2,s_2) = \, :X(t_1,s_1)\,X(t_2,s_2): \, = -\sqrt{-1}\, t_1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{z_2}{z_1}\right)^n + \left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n\right].$$

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$$p = \frac{1}{2\pi}\int_{S^1} X_t X_s\, ds = -\sqrt{-1}\sum_{n\neq 0} n \underbrace{\dot{X}_n}_{= p_n} X_{-n}$$

$$p_n = \frac{\tilde{\alpha}_{-n} + \alpha_n}{\sqrt{2}}$$

$$X_n = \frac{\alpha_{-n} - \tilde{\alpha}_n}{\sqrt{2}\sqrt{-1}\, n}$$

$$\overset{\text{Quantize}}{\rightsquigarrow} = -\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n$$

$$H = \frac{1}{2\pi}\int_{S^1} \frac{1}{2}\left(X_t^2 + X_s^2\right) ds \quad , \text{ similarly.}$$

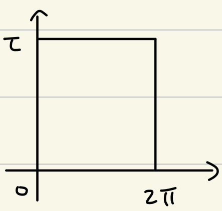
<u>Definition</u>

$$H_R = \frac{1}{2}(H - P) = \frac{1}{4}p_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24}$$

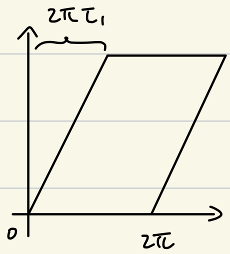
$$H_L = \frac{1}{2}(H + P) = \frac{1}{4}p_0^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}$$

$$X(t,s) = X_0 + t p_0 + \frac{\sqrt{-1}}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n \underset{\substack{\uparrow \\ R}}{e^{-\sqrt{-1}n(t-s)}} + \tilde{\alpha}_n \underset{\substack{\uparrow \\ L}}{e^{-\sqrt{-1}n(t+s)}}\right)$$

$$Z(\beta) = \operatorname{tr} e^{-\beta H}$$



$\rightsquigarrow 2\pi\tau_2 = \beta$



$$\tau := \tau_1 + \sqrt{-1}\,\tau_2$$
$$q := e^{2\pi\sqrt{-1}\,\tau}$$

Since "P" comes from rotation (translation) in s-direction.

$$\Rightarrow Z(\tau_1, \tau_2) = \operatorname{tr} \underline{e^{-2\pi\sqrt{-1}\tau_1 P}}\, e^{-2\pi\tau_2 H} = \operatorname{tr} e^{2\pi\sqrt{-1}\tau H_R} e^{-2\pi\sqrt{-1}\bar{\tau} H_L}.$$

HW: why?

$$\Rightarrow Z(\tau, \bar{\tau}) = \operatorname{tr} q^{H_R}\bar{q}^{H_L} \quad \text{on} \quad \mathcal{H} = \mathcal{H}_0 \otimes \bigotimes_{n=1}^{\infty} \mathcal{H}_n^R \otimes \bigotimes_{n=1}^{\infty} \mathcal{H}_n^L$$

$$(\alpha_{-n}\alpha_n)\, \alpha_{-n}^{\ell}|0\rangle_n = \ell\cdot n\left(\alpha_{-n}^{\ell}|0\rangle_n\right)$$

$\Uparrow$

$H_n|0\rangle_n = n|0\rangle_n$

$|k\rangle$, $k \in \mathbb{R}$

α_{-n}, $\tilde{\alpha}_n$ by $[\alpha_n, \alpha_{-n}] = n$.

$$\mathrm{tr}\ q^{\alpha_{-n}\alpha_n}\Big|_{\mathcal{H}_n^R} = \sum_{\ell=0}^{\infty} q^{\ell n} = \frac{1}{1-q^n}$$

$$\mathrm{tr}\ \bar{q}^{\tilde{\alpha}_{-n}\tilde{\alpha}_n}\Big|_{\mathcal{H}_n^L} = \frac{1}{1-\bar{q}^n}$$

$$\left(\int_{\mathbb{R}} e^{-2\pi\tau_2\frac{k^2}{2}}dk = \frac{1}{\sqrt{\tau_2}}\right)$$

$$\Rightarrow Z(\tau,\bar\tau) = (q\bar q)^{-\frac{1}{24}}\ \underline{\mathrm{tr}\,(q\bar q)^{p_0^2/4}}\Big|_{\mathcal{H}_0}\ \prod_{n=1}^{\infty}\frac{1}{|1-q^n|^2}$$

$$\to e^{-2\pi\tau_2\left(-\frac{1}{2}\frac{d^2}{dx^2}\right)}$$

but $e^{ikx} \notin L^2$

$\rightsquigarrow$ cut on finite interval

$$\longleftarrow\!\!(\;\; V \;\;)\!\!\longrightarrow$$

$$= \frac{\text{"}V\text{"}}{|\eta(\tau)|^2}\,\frac{1}{\sqrt{\tau_2}}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$: Dedekind η-function is modular

$\Rightarrow SL(2,\mathbb{Z})$-invariance of $Z(\tau,\bar\tau)$. (check it!)

i.e. conformal invariant.

To make it "more rigorous"

$$\Sigma = \mathbb{R}\times S^1 \xrightarrow{X} S^1_R \qquad X \sim X + 2\pi R$$

- Target momentum : $p = \frac{1}{2\pi}\int_{S^1} X_t\, ds = \dot{X}_0(t) \longmapsto p_0 = -\sqrt{-1}\,\frac{d}{dX_0}$

How we have discrete spectrum : $p = \frac{\ell}{R}$, $\ell \in \mathbb{Z}$.

- Another target "top" charge

$$w = \frac{1}{2\pi}\int_{S^1} X_s\, ds = mR, \quad m\in\mathbb{Z}, \quad \text{winding number.}$$

$\mathcal{H} = \bigoplus_{\ell,m}\mathcal{H}_{(\ell,m)}$. given by $\alpha_{-n}, \tilde\alpha_{-n}$ acting on $|\ell,m\rangle$,

$|\ell,m\rangle$ is killed by $\alpha_n, \tilde\alpha_n$ for $n>0$.

$$\underline{p_0}\,|\ell,m\rangle := \frac{\ell}{K}|\ell,m\rangle$$

$\parallel$

$e^{i\frac{\ell}{K}x_0}$: shift moment

$[x_0, p_0] = \sqrt{-1}$

define w_0 such that $w_0|\ell,m\rangle = mR|\ell,m\rangle$

$\exists\ e^{imR\hat{x}_0}$ shift winding number

i.e. $[\hat{x}_0, w_0] = \sqrt{-1}$. what is this $\hat{x}_0$?

$$X(t,s) = X_R(t-s) + X_L(t+s)$$

$$p_R = \frac{p_0 - w_0}{\sqrt{2}}$$
$$p_L = \frac{p_0 + w_0}{\sqrt{2}}$$

operator level: $$= \frac{1}{2}\left(X_0 - \hat{X}_0\right) + \frac{t-s}{\sqrt{2}}\frac{p_0 - w_0}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-\sqrt{-1}n(t-s)}$$
$$+ \frac{1}{2}\left(X_0 + \hat{X}_0\right) + \frac{t+s}{\sqrt{2}}\frac{p_0 + w_0}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-\sqrt{-1}n(t+s)}$$

Hence, $$H_R = \frac{1}{2}(H+P) = \frac{1}{2}p_R^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - \frac{1}{24}$$
$$H_L = \frac{1}{2}(H-P) = \frac{1}{2}p_L^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}$$

$$\Rightarrow Z(\tau,\bar{\tau},R) = \frac{1}{|\eta(\tau)|^2}\sum_{\ell,m} q^{\frac{1}{4}\left(\frac{\ell}{R}-mR\right)^2}\,\bar{q}^{\frac{1}{4}\left(\frac{\ell}{R}+mR\right)^2}$$

T-duality: $R \longmapsto \frac{1}{R}$ $\qquad Z(\tau,\bar{\tau},R) = Z(\tau,\bar{\tau},\frac{1}{R})$

$$\mathcal{H}^{R}_{(\ell,m)} \mapsto \hat{\mathcal{H}}^{1/R}_{(m,\ell)}$$
$$(p_R, p_L) \longmapsto (-\hat{p}_R, \hat{p}_L)$$
$$(\alpha_n, \tilde{\alpha}_n) \longmapsto (-\hat{\alpha}_n, \hat{\tilde{\alpha}}_n)$$
$$\hat{X}(t,s) = -X_R(t-s) + X_L(t+s).$$
$\Rightarrow \hat{X}_0$ is the 0-mode.

Line IV. 15-21. ✓

Path integral point of view:

$$(\underbrace{\Sigma_g}_{}, h) \xrightarrow{X} S^1_R.$$

Riemann surface of genus g

Let $\varphi := \frac{X}{R}$, period: 2π.

Action $$S(\varphi) = \frac{1}{4\pi}\int_\Sigma R^2|d\varphi|_h^2$$

Let $$S'(\varphi, \underline{B}) = \frac{1}{4\pi}\int_\Sigma \frac{1}{R^2}|B|_h^2 + \frac{\sqrt{-1}}{2\pi}\int_\Sigma B\wedge d\varphi$$
1-form

Then, $S(\varphi) = S'(\varphi, B)$ for $B = \sqrt{-1}R^2 * d\varphi$ since $\alpha \wedge (*\alpha) = |\alpha|_h^2$.

or by taking minimal value with B via completing square.

Do path integral : $\int DB \underbrace{\int D\varphi}_{\text{over } \varphi_0 \text{ and } n_i.} e^{-S'(\varphi, B)}$

where $d\varphi = \underbrace{d\varphi_0}_{\text{exact}} + \sum_{i=1}^{2g} 2\pi n_i \omega^i$, ω^i : basis of $H^1(\Sigma, \mathbb{Z})$

γ_i : dual basis of $H_1(\Sigma, \mathbb{Z})$

i.e. $\varphi_0 : \Sigma \longrightarrow \mathbb{R}$

φ shifts by $2\pi n_i$ along γ_i.

$\int_\Sigma B \wedge d\varphi_0 = \int_\Sigma dB \cdot \varphi_0$ in order for the "integral" to be invariant under $\varphi_0 \mapsto \varphi_0 + \underbrace{\psi_0}_{\text{constant}}$

$\Rightarrow dB = 0$

$\Rightarrow B = d\vartheta_0 + \sum_{i=1}^{2g} a_i \omega^i$ $\Rightarrow \int_\Sigma B \wedge d\varphi = 2\pi \sum_{i,j} a_i n_j \underbrace{\int_\Sigma \omega^i \omega^j}_{= J^{ij}}$, $n^i := n_j J^{ij} \in \mathbb{Z}$.

$\vartheta_0 : \Sigma \longrightarrow \mathbb{R}$

$$= 2\pi \sum_{i=1}^{2g} a_i n^i .$$

Poisson summation formula : $\sum_{n \in \mathbb{Z}} e^{ian} = 2\pi \sum_{m \in \mathbb{Z}} \delta(a - 2\pi m)$.

Fourier transform of e^{iax}.

$\rightsquigarrow$ B has contribution in $\int DB$ only when $a_i \in 2\pi \cdot \mathbb{Z}$.

(or $\mathbb{Z}$???)

$\Rightarrow B = d\vartheta_0 + 2\pi \sum_{i=1}^{2g} \underbrace{m_i}_{\in \mathbb{Z}} \omega^i =: \underbrace{d\vartheta}_{\text{period } 2\pi}$.

$e^{-S'(\varphi, B)} \longmapsto e^{-S'(\vartheta)}$, $S'(\vartheta) = \frac{1}{4\pi} \int_\Sigma \frac{1}{R^2} |d\vartheta|_h^2$

This is the T-duality we expect for : $R \longmapsto \frac{1}{R}$

with $R\, d\varphi = \frac{\sqrt{-1}}{R} * B = \sqrt{-1}\left(\frac{1}{R}\right) * d\vartheta$ since $*^2 = -1$.

$$\begin{matrix} \varphi_t & \longleftrightarrow & \vartheta_s \\ \varphi_s & & \vartheta_t \end{matrix}$$

exchanges momentum and winding number.

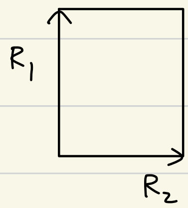
<u>Exercise</u> $e^{i\vartheta}$ is the shift operator of winding number.

2021. 10. 21

__Last time__ $(\Sigma_g, h) \xrightarrow{X} S^1_R$ T-duality $R \mapsto \frac{1}{R}$

$$\varphi = \frac{X}{R}$$

σ-model on T^2 : $\Sigma \xrightarrow{X} T^2 = M$,



If $T^2 = S^1_{R_1} \times S^1_{R_2}$ rectangular torus , $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

parameter (R_1, R_2) is equivalent to $\begin{cases} A = \dfrac{\text{area}}{(2\pi)^2} = R_1 R_2 & \text{symplectic structure (Kähler class)} \\ \sigma = i\dfrac{R_1}{R_2} & \text{complex moduli} \end{cases}$

$$(A, \sigma) \longmapsto \left(\frac{R_1}{R_2}, R_1 R_2\right) =: (A', \sigma') = (\sigma, A)$$

apply T-duality
to the 2-nd factor

This is the early apperance of "mirror symmetry" via T-duality.

__Conjecture__ (Strominger - Yau - Zaslow, 1996, MS is T-duality)

X. CY 3-fold (projective / $\mathbb{C}$), $\exists$ special Lagrangian torus fibration

$$\begin{tikzcd} T^3 \arrow[r, hook] & X \arrow[d] \\ & B \end{tikzcd} \quad \overset{\text{take dual torus}}{\rightsquigarrow} \quad \begin{tikzcd} \check{T}^3 \arrow[r, hook] & \check{X} \arrow[d] \\ & B \end{tikzcd}$$

(change A & B model : A model e.g. GW or Qcoh
B model e.g. KS theory

__Reference__ : Freed, 5 lectures on super symmetry.

General tori:

complex structure $\sigma = \sigma_1 + i\sigma_2 \in \mathbb{C}$

Kähler structure (complexified) $\rho = \underbrace{\frac{B}{2\pi}}_{\text{B-field}} + iA$

$B \in H^2(M, \mathbb{R}) / 2\pi H^2(M, \mathbb{Z})$

$$Z := \int DX\, e^{-S + i\int_\Sigma x^* B}$$

<u>Exercise</u> Formulate S correctly on T^2 (general torus). Compute Z with B-field and show the T-duality to exchange σ and ρ.

<u>Free Dirac Fermion</u> (spinor / $\mathbb{C}$)

$$\begin{array}{ccl} Cl_{1,1}^{\mathbb{C}} & & \text{Clifford bundle at } T_p\Sigma \otimes \mathbb{C} = \langle e^t, e^s \rangle \otimes \mathbb{C} \\ \downarrow & & (e^t)^2 = 1,\ (e^s)^2 = -1 \\ \Sigma = \mathbb{R} \times S^1 & \text{. Minkowski} & e^t e^s = -e^s e^t \\ (t, s) & & (\text{deformation of } \Lambda^* T\Sigma) \end{array}$$

$Cl_{1,1}^{\mathbb{C}} \simeq \operatorname{End} S$, $\quad S = S_- \oplus S_+$

S: Spinor representation $\simeq \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \in \Gamma(\Sigma, S)$

$$e^t \longmapsto \gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e^s \longmapsto \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S(\psi) = \frac{1}{2\pi} \int_\Sigma i\bar\psi \underbrace{\gamma^\mu \partial_\mu}_{= \not{\partial}} \psi \, dt\, ds, \qquad \begin{aligned} \bar\psi &:= \psi^\dagger \gamma^t \\ &= (\bar\psi_- \ \bar\psi_+) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\bar\psi_+ \ \bar\psi_-) \end{aligned}$$

$$= \frac{1}{2\pi} \int_\Sigma i (\bar\psi_+ \ \bar\psi_-) \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\psi_-)_t \\ (\psi_+)_t \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\psi_-)_s \\ (\psi_+)_s \end{pmatrix} \right) dt\, ds$$

inner product on S via $\langle \psi_1, \psi_2 \rangle = i\psi_1^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi_2$

$\Rightarrow \not{\partial}$: self-adjoint.

$$\Rightarrow \delta S = \frac{1}{2\pi}\int_{\Sigma} 2i\, \delta\bar{\psi}\, \not{\partial}\psi$$

$\Rightarrow$ equation of motion $\quad 0 = \not{\partial}\psi = \begin{pmatrix} 0 & \partial_t - \partial_s \\ \partial_t + \partial_s & 0 \end{pmatrix}\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$

$$\begin{cases} \psi_- = \psi_-(t,s) = f(t-s) & : \text{Right move} \\ \psi_+ = \psi_+(t,s) = g(t+s) & : \text{Left move} \end{cases}$$

Rotations : $\psi_\pm \longmapsto e^{-i\alpha}\psi_\pm$: vector rotation

$\psi_\pm \longmapsto e^{\mp i\beta}\psi_\pm$: axial rotation

$\rightsquigarrow$ conserved quantity $\quad F_V = \frac{1}{2\pi}\int_{S^1}\left(\bar{\psi}_-\psi_- + \bar{\psi}_+\psi_+\right)ds$

$$F_A = \frac{1}{2\pi}\int_{S^1}\left(-\bar{\psi}_-\psi_- + \bar{\psi}_+\psi_+\right)ds$$

Space-time translation
(World sheet)

$$H = \frac{1}{2\pi}\int_{S^1}\left(-i\bar{\psi}_-\partial_s\psi_- + i\bar{\psi}_+\partial_s\psi_+\right)ds$$

($\because$ no "∂_t" by Dirac equation.)

$$P = \frac{1}{2\pi}\int_{S^1}\left(i\bar{\psi}_-\partial_s\psi_- + i\bar{\psi}_+\partial_s\psi_+\right)ds$$

Decomposition into Fourier coordinates.

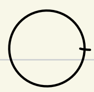
$\exists$ 4 possible boundary conditions :

periodic / anti-periodic $\qquad (a,\tilde{a}) = (0,0)$, $(0,\tfrac{1}{2})$, $(\tfrac{1}{2},0)$, $(\tfrac{1}{2},\tfrac{1}{2})$.

$\uparrow$ Ramond Sector $\qquad \uparrow$ Neveu-Schwarz sector

R-R $\qquad$ R-NS $\qquad$ NS-R $\qquad$ NS-NS

For the N-N sector : $\psi_- = \sum \psi_n(t)\, e^{ins}$



$S^1_{R=1}$

$\psi_+ = \sum \tilde{\psi}_n(t)\, e^{-ins}$ (convention in textbook.)

$\psi_-^\dagger = \bar{\psi}_- = \sum \bar{\psi}_n(t)\, e^{ins}$

$\psi_+^\dagger = \bar{\psi}_+ = \sum \bar{\tilde{\psi}}_n(t)\, e^{-ins}$

$\Rightarrow \quad \bar{\psi}_n = \psi_{-n}^\dagger$

$\bar{\tilde{\psi}}_n = \tilde{\psi}_{-n}^\dagger$

$$\Rightarrow S = \int \sum_{n\in\mathbb{Z}} \left(i\bar{\psi}_{-n}(\partial_t + in)\psi_n + i\bar{\tilde{\psi}}_{-n}(\partial_t + in)\tilde{\psi}_n \right) dt$$

$$\Rightarrow \pi_n = \frac{\partial L}{\partial(\partial_t \psi_n)} = i\bar{\psi}_{-n}$$

$$\tilde{\pi}_n = \frac{\partial L}{\partial(\partial_t \tilde{\psi}_n)} = i\bar{\tilde{\psi}}_{-n}$$

$\psi_n \rightsquigarrow \pi_n$

$\tilde{\psi}_n \rightsquigarrow \tilde{\pi}_n$

Quantization: $\{\psi_n, \bar{\psi}_m\} = \delta_{n+m,0}$

$\{\tilde{\psi}_n, \bar{\tilde{\psi}}_m\} = \delta_{n+m,0}$

others $= 0$.

For all n, ψ_n, $\bar{\psi}_{-n}$ is represented in a 2-dimensional space. (even for $n=0$)

$H_n(t) = n\bar{\psi}_{-n}\psi_n$, $|0\rangle_n$ killed by ψ_n if $n>0$

$\bar{\psi}_{-n}$ if $n<0$

$H_n(t) = n\bar{\tilde{\psi}}_{-n}\tilde{\psi}_n$, similarly get $\widetilde{|0\rangle}_n$

$$|0\rangle := \bigotimes_{n\geq 0} |0\rangle_n \otimes \widetilde{|0\rangle}_n$$

$$H = \sum_{n\in\mathbb{Z}} \left(n\bar{\psi}_{-n}\psi_n + n\bar{\tilde{\psi}}_{-n}\tilde{\psi}_n \right) = \sum_{n\in\mathbb{Z}} n : \bar{\psi}_{-n}\psi_n : + n : \bar{\tilde{\psi}}_{-n}\tilde{\psi}_n : + \underbrace{\frac{1}{6}}_{=\sum_{n=1}^{\infty}(-2n) = \frac{1}{6}}$$

i.e. $|0\rangle$ has energy $E_0 = \frac{1}{6}$.

Now, there are 4 such ground states: $\psi_0|0\rangle$, $\bar{\tilde{\psi}}_0|0\rangle$, $\psi_0\bar{\tilde{\psi}}_0|0\rangle$.

Similarly and easier: $P = \sum_{n\in\mathbb{Z}} -n : \bar{\psi}_{-n}\psi_n : + n : \bar{\tilde{\psi}}_{-n}\tilde{\psi}_n :$ $\square$

For "twisted" boundary condition, $\psi_-(t, s+2\pi) = e^{2\pi i a}\psi_-(t,s)$

$\psi_+(t, s+2\pi) = e^{2\pi i \tilde{a}}\psi_+(t,s)$

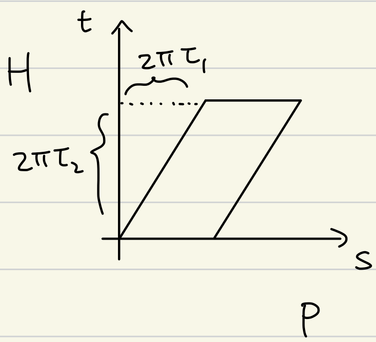
$(a,\tilde{a}) = (0,0)$
$(\frac{1}{2}, 0)$
$(0, \frac{1}{2})$
$(\frac{1}{2}, \frac{1}{2})$.

$$H_R = \frac{1}{2}(H-P) = \sum_{r\in\mathbb{Z}+a} r : \bar{\psi}_{-r}\psi_r : + \frac{1}{2}\left(\{a\} - \frac{1}{2}\right)^2 - \frac{1}{24}$$

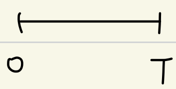
$$H_L = \frac{1}{2}(H+P) = \sum_{r\in\mathbb{Z}+\tilde{a}} r : \bar{\tilde{\psi}}_{-r}\tilde{\psi}_r : + \frac{1}{2}\left(\{\tilde{a}\} - \frac{1}{2}\right)^2 - \frac{1}{24}$$

Partition function:

$t \mapsto i\tau$



$$\mathrm{tr}\, e^{-2\pi i \tau_1 P} e^{-2\pi \tau_2 H}$$



$$\rightsquigarrow \mathrm{tr}\, e^{-iTH} = \int DX\, e^{iS(X)}$$

2021.10.25

Free Dirac Fermion

Twisted boundary conditions for $a, \tilde{a} \in \mathbb{R}$.

$\psi_-(t, s+2\pi) = e^{2\pi i a} \psi_-(t,s)$ for $\overline{\psi}_\pm$ use the complex conjugate condition.

$\psi_+(t, s+2\pi) = e^{-2\pi i \tilde{a}} \psi_-(t,s)$

i.e. $\psi_-' = e^{-ias} \psi_-(t,s)$ $\Rightarrow$ periodic, but with action

$\psi_+' = e^{i\tilde{a}s} \psi_+(t,s)$

$$S = \frac{1}{2\pi} \int_\Sigma \sqrt{-1} \left(\overline{\psi}_-' (\partial_t + \partial_s + ia) \psi_-' + \overline{\psi}_+' (\partial_t - \partial_s + i\tilde{a}) \psi_+' \right) dt\, ds$$

i.e. Dirac Fermion coupled to flat connection $/ S^1$ with holonomies $e^{2\pi i a}$, $e^{-2\pi i \tilde{a}}$.

(U(1) Gauge field.)

$$Cl_{1,1}^{\mathbb{C}} = \mathrm{End}(\underbrace{S_-}_{\otimes \cdots} \oplus \underbrace{S_+}_{\otimes \cdots})$$

$\Rightarrow$ Fourier expansions of $\psi_\pm$ are slightly different:

$$\psi_- = \sum_{r \in \mathbb{Z}+a} \psi_r(t) e^{irs}, \quad \overline{\psi}_- = \sum_{r' \in \mathbb{Z}-a} \overline{\psi}_{r'}(t) e^{ir's} \Rightarrow \psi_r^\dagger = \overline{\psi}_{-r}$$

$$\psi_+ = \sum_{\tilde{r} \in \mathbb{Z}+\tilde{a}} \psi_{\tilde{r}}(t) e^{-i\tilde{r}s}, \quad \overline{\psi}_+ = \sum_{\tilde{r} \in \mathbb{Z}-\tilde{a}} \overline{\tilde{\psi}}_{\tilde{r}}(t) e^{-i\tilde{r}s} \Rightarrow \tilde{\psi}_{\tilde{r}}^\dagger = \overline{\tilde{\psi}}_{-\tilde{r}}$$

The original action becomes

$$S = \int \left(\sum_{r \in \mathbb{Z}+a} \sqrt{-1}\, \overline{\psi}_{-r} (\partial_t + \sqrt{-1} r) \psi_r + \sum_{r \in \mathbb{Z}+\tilde{a}} \sqrt{-1}\, \overline{\tilde{\psi}}_{-\tilde{r}} (\partial_t + \sqrt{-1}\tilde{r}) \tilde{\psi}_{\tilde{r}} \right) dt$$

get quantization: $\{ \psi_r, \overline{\psi}_{r'} \} = \delta_{r+r', 0}$

$\{ \tilde{\psi}_r, \overline{\tilde{\psi}}_{\tilde{r}'} \} = \delta_{\tilde{r}+\tilde{r}', 0}$

2-dimensional representation for $r' = -r$ $(\tilde{r}' = -\tilde{r})$

For sector $r \in \mathbb{Z} + a$, $H_r = r \overline{\psi}_{-r} \psi_r$

$|0\rangle_r$ killed by ψ_r if $r \geq 0$ ($r=0$ case up to choices!)

$\overline{\psi}_{-r}$ if $r < 0$

For sector $\tilde{r} \in \mathbb{Z} + \tilde{a}$, $H_{\tilde{r}} = \tilde{r} \overline{\tilde{\psi}}_{-\tilde{r}} \tilde{\psi}_{\tilde{r}}$, $|0\rangle_{\tilde{r}}$ similarly.

$\rightsquigarrow$ ground state $|0\rangle_{a,\tilde{a}} = \bigotimes$ all of them!

(If $a \neq 0$ and $\tilde{a} \neq 0$, then the ground state is unique!)

Energy (Hamiltonian)

$$E_0(a,\tilde a) = \sum_{r\in\mathbb{Z}+a,\,<0} r + \sum_{\tilde r\in\mathbb{Z}+\tilde a,\,<0} \tilde r$$

$$H = \sum_{r\in\mathbb{Z}+a} r\,\overline{\psi}_{-r}\psi_r + \sum_{\tilde r\in\mathbb{Z}+\tilde a} \tilde r\,\overline{\tilde\psi}_{-\tilde r}\tilde\psi_{\tilde r}$$

$r\geq 0$: done

$r<0$: via $\overline{\psi}_{-r}\psi_r + \psi_r\overline{\psi}_{-r} = 1$.

Consider $\zeta(s,x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^s}$ do analytic continuation.

$$\rightsquigarrow E_0 = -\frac{1}{12} + \frac{1}{2}\left(\{a\}-\frac{1}{2}\right)^2 + \frac{1}{2}\left(\{\tilde a\}-\frac{1}{2}\right)^2$$

$\{a\} = a - [a]$

P is easier, $(\sum r + \sum \tilde r) = 0$

$-\sum : \; : + \sum : \; :$

$$H_R = \frac{1}{2}(H-P) = \sum_{r\in\mathbb{Z}+a} r :\overline{\psi}_{-r}\psi_r: + \frac{1}{2}\left(\{a\}-\frac{1}{2}\right)^2 - \frac{1}{24}$$

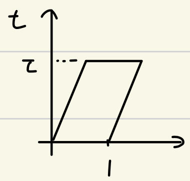
$$H_L = \frac{1}{2}(H+P) = \sum_{\tilde r\in\mathbb{Z}+\tilde a} \tilde r :\overline{\tilde\psi}_{-\tilde r}\tilde\psi_{\tilde r}: + \frac{1}{2}\left(\{\tilde a\}-\frac{1}{2}\right)^2 - \frac{1}{24}$$

$$F_R = \frac{1}{2}(F_V - F_A) = \sum_{r\in\mathbb{Z}+a} :\overline{\psi}_{-r}\psi_r: + \left(\{a\}-\frac{1}{2}\right)$$

$$F_L = \frac{1}{2}(F_V + F_A) = \sum_{\tilde r\in\mathbb{Z}+a} :\overline{\tilde\psi}_{-\tilde r}\tilde\psi_{\tilde r}: + \left(\{\tilde a\}-\frac{1}{2}\right)$$

$$\left(\begin{array}{ll} \text{since } \psi_\pm \mapsto e^{-i\alpha}\psi_\pm & \rightsquigarrow F_V = \frac{1}{2\pi}\int_{S^1}(\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)\,ds \\ \psi_\pm \mapsto e^{\mp i\beta}\psi_\pm & F_A = \frac{1}{2\pi}\int_{S^1}(-\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)\,ds \end{array}\right)$$

Next step: Partition functions : tr...



Consider $\tau = \tau_1 + \sqrt{-1}\,\tau_2$

$\xi = \frac{1}{2\pi}(s + \sqrt{-1}\,t) \mod (1,\tau)$

$\xi \mapsto \xi + 1$ (i.e. $s \mapsto s + 2\pi$) get factors $e^{2\pi i a}$; $e^{-2\pi i\tilde a}$.

Assume: $\xi \mapsto \xi + \tau$ $\left(\begin{array}{l}\text{i.e. } s\mapsto s+2\pi\tau_1 \\ t \mapsto t + 2\pi\tau_2\end{array}\right)$ get factors $e^{2\pi i b}$; $e^{-2\pi i\tilde b}$ for some $b,\tilde b\in\mathbb{R}$.

$R(-)$; $R(+)$

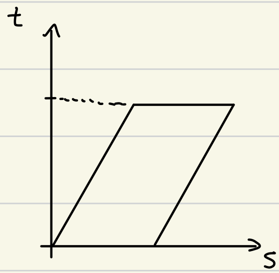
i.e. we consider periodic Dirac Fermions coupled to flat connection

$$A^{0,1} = \frac{\pi\sqrt{-1}}{\tau_2}(b - \tau a)\,d\bar\xi \quad \text{on } S_-, \qquad \tilde A^{1,0} = \frac{\pi\sqrt{-1}}{\tau_2}(\tilde b - \bar\tau\tilde a)\,d\xi \quad \text{on } S_+$$

<u>Homework</u> Check this! and solve the general solution of the form.

If $b=0=\tilde{b}$, the solution is given:
$$\left(\begin{array}{ll} \psi_- = \sum_{r\in\mathbb{Z}+a} \psi_r e^{-ir(t-s)} , & \bar{\psi}_- = \sum_{r\in\mathbb{Z}-a} \bar{\psi}_r e^{-ir(t-s)} \\ \psi_+ = \sum_{\tilde{r}\in\mathbb{Z}+\tilde{a}} \tilde{\psi}_{\tilde{r}} e^{-i\tilde{r}(t+s)} , & \bar{\psi}_+ = \sum_{\tilde{r}\in\mathbb{Z}-\tilde{a}} \bar{\tilde{\psi}}_{\tilde{r}} e^{-i\tilde{r}(t+s)} \end{array}\right)$$

$$[H,\psi_r] = -r\,\psi_r$$
$$\bar{\psi} \quad = -r\,\bar{\psi}$$
$$\tilde{\psi} \quad = -r\,\tilde{\psi}$$
$$\bar{\tilde{\psi}} \quad = -r\,\bar{\tilde{\psi}}$$



time evolution $t \longmapsto t+2\pi\tau_2 \longmapsto e^{-2\pi\tau_2 H}$

it induces $e^{-2\pi i\tau_1 P}$ as before (i.e. $s \longmapsto s-2\pi\tau_1$)

and now with twisted contribution $e^{-2\pi i b F_R}\; e^{2\pi i \tilde{b} F_L}$

$$\Rightarrow Z = \mathrm{tr}\left(\underline{e^{-2\pi\sqrt{-1}(b-\frac{1}{2})F_R}\; e^{2\pi\sqrt{-1}(\tilde{b}-\frac{1}{2})F_L}}\; e^{-2\pi\sqrt{-1}\tau_1 P}\; e^{-2\pi\tau_2 H}\right)$$

anti-periodic boundary condition of Fermions in path integral, coming from anti-commutativity $\psi_1\psi_2 = -\psi_2\psi_1$.

Since Z should be invariant under $b \longmapsto b+1$, $\tilde{b} \longmapsto \tilde{b}+1$ if $(\tilde{a},\tilde{b}) = \pm(a,b)$.

2021. 10. 28.

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + \sqrt{-1}\,\theta^\pm \partial_\pm \qquad \overline{Q}_\pm = -\frac{\partial}{\partial \overline{\theta}^\pm} - \sqrt{-1}\,\theta^\pm \partial_\pm$$

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - \sqrt{-1}\,\overline{\theta}^\pm \partial_\pm \qquad \overline{D}_\pm = -\frac{\partial}{\partial \overline{\theta}^\pm} + \sqrt{-1}\,\theta^\pm \partial_\pm$$

$$x^\pm = x^0 \pm x^1 \quad , \quad \partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$

$$\{Q_\pm, \overline{Q}_\pm\} = -2\sqrt{-1}\,\partial_\pm$$
$$\{D_\pm, \overline{D}_\pm\} = 2\sqrt{-1}\,\partial_\pm$$
all others $= 0$.

Systematic way to write down supersymmetry Lagrangians

$\mathbb{R}^{1,1} \ni (t, s) = (x^0, x^1)$ (or $\mathbb{R}^2$)

$+ \quad -$ Minkowski

$\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$ complex fermion (e.g. spinors)

(1 1) (0,2)

Call this a (2,2) superspace. $\rightsquigarrow$ $2^4 = 16$ choices of "basis".

"superfields" $\mathcal{F}(x, \theta) = f_0 + \theta^+ f_+ + \theta^- f_- + \overline{\theta}^+ f_+' + \overline{\theta}^- f_-' + \theta^+\theta^- f_{+-} + \cdots$

i.e. functions such that $f_*(x^0, x^1)$ decays fast at infinity.

<u>Definition</u> Chiral superfield Φ : $\overline{D}_\pm \Phi = 0$.

<u>Homework1</u> $\Phi = \phi(y^\pm) + \theta^\alpha \underline{\psi_\alpha(y^\pm)} + \theta^+\theta^- F(y^\pm)$

$y^\pm = x^\pm - i\,\theta^\pm \overline{\theta}^\pm$ Fermion function

Others: Anti-Chiral $\overline{\Phi}$: $D_\pm \overline{\Phi} = 0$ ($\Leftrightarrow$ Φ : Chiral)

Twisted-Chiral U : $\overline{D}_+ U = 0 = D_- U$

Twisted-anti-Chiral $\overline{U}$: $D_+ \overline{U} = 0 = \overline{D}_- \overline{U}$

super symmetric action :

Let $\delta = \varepsilon_+ Q_- - \varepsilon_- Q_+ - (\bar{\varepsilon}_+ \bar{Q}_- - \bar{\varepsilon}_- \bar{Q}_+)$

What kind of Lagrangians are δ-invariant ?

- D-term : $\int \underbrace{d^2x}_{dx^0 dx^1} \underbrace{d^4\theta}_{d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+} K(\mathcal{F}_i)$ is always δ-invariant, K : any C^∞ function on $\mathcal{F}_1, \mathcal{F}_2, \ldots$ "decay fast in $x^\pm$".

$\bar{\varepsilon}_-$ - coefficient in $\delta(-)$: $\bar{\varepsilon}_- \bar{Q}_+ K = -\bar{\varepsilon}_- \Big(\underbrace{\frac{\partial K}{\partial \bar{\theta}^+}}_{\text{no } \bar{\theta}^+} + \sqrt{-1}\, \theta^+ \partial_+ K \Big)$

no $\bar{\theta}^+$ ⇝ integral $= 0$

$\sqrt{-1}\,\theta^+ \partial_+ K$ ⇝ integral $= 0$ by FTC, decay fast.

- F-term : $\int d^2x \underbrace{d^2\theta}_{= d\theta^+ d\theta^-} \overset{\text{holomorphic function}}{W}(\underbrace{\Phi_i}_{\text{Chiral superfield}})$ is also δ-invariant.

i.e. we set $\bar{\theta}^\pm = 0$.

$\bar{\varepsilon}_+$ - coefficient in $\delta(-)$: Note that $\bar{Q}_- = \bar{D}_- - 2\sqrt{-1}\,\theta^- \partial_-$.

$\bar{D}_- \underbrace{W}_{\text{holomorphic} \Rightarrow \text{ expand it!}}(\Phi_i) = 0$.

$\Rightarrow \bar{\varepsilon}_+ 2\sqrt{-1}\, \theta^- \underbrace{\frac{\partial W(\Phi_i)}{\partial x^-}}_{\text{total derivative}} \xrightarrow{\text{integral}} 0$

- Twisted F-term : $\int d^2x \underbrace{d^2\bar{\theta}}_{= d\bar{\theta}^- d\bar{\theta}^+} \overset{\text{holomorphic}}{\widetilde{W}}(\underbrace{U_i}_{\text{twisted Chiral superfield}})$ is also δ-invariant.

Super Calculus

Poincaré lemma $D_+ \mathcal{F} = 0 \Rightarrow \mathcal{F} = D_+ \mathcal{G}$.

proof: $D_+ \mathcal{F} = 0 \Rightarrow 2\sqrt{-1}\, \partial_+ \mathcal{F} = D_+ \bar{D}_+ \mathcal{F} \Rightarrow \mathcal{F} = \frac{D_+}{2\sqrt{-1}} \underbrace{\int_{-\infty}^{x^+} \bar{D}_+ \mathcal{F}\, dx_1^+}_{= \mathcal{G}}$. #

Basic Examples

One Chiral superfield $\Phi = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm)$, $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$

Taylor expansion at $x^\pm$
$$= \phi - i\theta^\pm\bar{\theta}^\pm \partial_\pm \phi - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+ \partial_+\partial_-\phi + \theta^\pm\psi_\pm - i\theta^\pm\theta^\mp\bar{\theta}^\mp \partial_\mp \psi_\pm + \theta^+\theta^- F.$$

$$(\psi_1\psi_2)^\dagger = \psi_2^\dagger\psi_1^\dagger \Rightarrow \bar{\Phi} = \bar{\phi} + i\theta^\pm\bar{\theta}^\pm\partial_\pm\bar{\phi} - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_+\partial_-\bar{\phi} - \bar{\theta}^\pm\bar{\psi}_\pm - i\bar{\theta}^\pm\theta^\mp\bar{\theta}^\mp\partial_\mp\bar{\psi}_\pm + \bar{\theta}^-\bar{\theta}^+\bar{F}.$$

Kinetic D-term

$$S_{kin} = \int d^2x\, d^4\theta\, \bar{\Phi}\Phi$$

$$= \int d^2x \left(-\bar{\phi}\,\partial_+\partial_-\phi + \partial_\pm\bar{\phi}\,\partial_\mp\phi - \partial_+\partial_-\bar{\phi}\phi + i\bar{\psi}_\pm\partial_\mp\psi_\pm - i\partial_\mp\bar{\psi}_\pm\psi_\pm + |F|^2 \right)$$

Integration by parts
$$= \int d^2x\, \underbrace{\frac{1}{2}\left(|\partial_0\phi|^2 - |\partial_1\phi|^2\right)}_{\text{free boson}} + \underbrace{2i\bar{\psi}_\pm\partial_\mp\psi_\pm}_{\text{free fermion}} + \underbrace{|F|^2}_{\text{auxilary field}}$$

F-terms:

$$S_W = \int d^2x\, d^2\theta\, \left(W(\Phi) + \bar{W}(\bar{\Phi})\right)$$ ← make it real!

$$= \int d^2x \left(W'(\phi)F - W''(\phi)\psi_+\psi_- + \bar{W}'(\bar{\phi})\bar{F} - \bar{W}''(\phi)\bar{\psi}_-\bar{\psi}_+ \right)$$

$\Rightarrow$ Action: $S = S_{kin} + S_W$

$$= \int d^2x \left(\text{free scalar} + \text{free Dirac Fermion} - \overset{\text{potential}}{|W'(\phi)|^2} + \overbrace{\left(W''(\phi)\psi_+\psi_- + \bar{W}''(\phi)\bar{\psi}_-\bar{\psi}_+\right)}^{\text{Yukawa term}} + \underbrace{|F + \bar{W}'(\bar{\phi})|^2}_{=0} \right)$$

To simply, set $F = -\bar{W}'(\bar{\phi})$. reason: (1) equation of motion. (2) integrate out F.

This recovers all the previous examples except the non-linear σ-model: $T \to M$.

Question Can δ in Φ be written in $\delta\phi$, $\delta\psi_\pm$, $\delta\bar{\psi}_\pm$, δF?

It might have problem if the superfield $\mathcal{F}$ is constrant.

For chiral superfield, it is OK! Since $Q_\pm$, $\bar{Q}_\pm$ anti-commute with $\bar{D}_\pm$

i.e. $\delta\Phi$ is still chiral.

Conserved current and charges:

Homework
$$Q_\pm = \int dx^1 G^0_\pm := \int dx^1 \left(2(\partial_\pm \bar{\phi})\psi_\pm \mp i \bar{\psi}_\mp \bar{W}'(\bar{\phi})\right)$$

$$\bar{Q}_\pm = \int dx^1 \bar{G}_\pm := \int dx^1 \left(2\bar{\psi}_\pm(\partial_\pm \phi) \pm i \psi_\mp W'(\phi)\right)$$

global symmetry

$$F_A = \int dx^1 J^0_A = \cdots$$

$$F_V = \int dx^1 J^0_V =$$

$\hookrightarrow$ F term is invariant only for monomial, $W(\Phi) = c \cdot \Phi^k$.

2021. 11. 1.

Last time Supersymmetry Lagrangian D, F - terms.
Global rotation symmetry (R-symmetry)
Axial rotation symmetry $e^{i\alpha F_A}: \mathcal{F} \longmapsto e^{i\alpha q_A} \cdot \mathcal{F}(x^\mu, e^{\mp i\alpha}\theta^\pm, e^{\mp i\alpha}\bar{\theta}^\pm)$
Vector rotation symmetry $e^{i\alpha F_V}: \mathcal{F} \mapsto e^{i\alpha q_V} \cdot \mathcal{F}(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm)$

q_A, q_V : charge.

On components :
Axial : $\phi \mapsto \phi$
$\psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm$
(for Chiral superfield, $\Phi = \phi + \theta^i\psi_i + \theta^+\theta^- F$)

Set $q_A = 0 \Rightarrow \theta^4, \theta^2$ are both invariant.

$$Q_\pm = \int dx^1 G^0_\pm = \int dx^1 (\cdots) \Rightarrow \begin{cases} Q_\pm \longmapsto e^{\mp i\alpha} Q_\pm \\ \bar{Q}_\pm \mapsto e^{\mp i\alpha}\bar{Q}_\pm \end{cases}$$

Vector : $\psi_\pm \mapsto e^{-i\alpha}\psi_\pm$, D-term is OK. (θ^4-invariant)
$\underset{\theta^+\theta^-}{\overset{\shortparallel}{\theta^2}} \mapsto e^{-2i\alpha}\theta^2$, F-term is invariant only if $W(\Phi) \mapsto e^{2i\alpha}W(\Phi)$
i.e. $q_V = 2$.

$\Rightarrow W(\Phi) = c\,\Phi^k$ i.e. $q_V = \frac{2}{k}$ for Φ.

for components $\phi \mapsto e^{\frac{2}{k}i\alpha}\phi$
$\psi_\pm \mapsto e^{(\frac{2}{k}-1)i\alpha}\psi_\pm$

$$F_V = \int dx^1 J^0_V = \int dx^1 \left(\frac{2i}{k}\left((\partial_0\bar{\phi})\phi - \bar{\phi}(\partial_0\phi)\right) - \left(\frac{2}{k}-1\right)\left(\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_-\right)\right)$$

Also, $\begin{cases} Q_\pm \mapsto e^{-i\alpha}Q_\pm \\ \bar{Q}_\pm \mapsto e^{i\alpha}\bar{Q}_\pm \end{cases}$

The case for a twisted Chiral superfield U can be studied similarly.

$$S = -\int dx^2 d\theta^4\, \bar{U}U + \int d^2x \underset{d\bar{\theta}^- d\bar{\theta}^+}{\underset{\shortparallel}{\underline{d^2\tilde{\theta}}}} \left(\tilde{W}(U) + \bar{\tilde{W}}(U)\right)$$

..... reading !!

Notice: Chiral superfield $\longleftrightarrow$ Twisted Chiral superfield

θ^- $\longleftrightarrow$ $-\bar{\theta}^-$ (i.e. $\bar{D}_- \longleftrightarrow D_-$)

$\Rightarrow Q_- \longleftrightarrow \bar{Q}_-$; $F_V \longleftrightarrow F_A$.

N=(2,2) supersymmetry QFT

Start with a classical supersymmetry FT (for a few fields)

$\rightsquigarrow$ 4 supercharges $Q_\pm$, $\bar{Q}_\pm$. $\qquad \delta = \epsilon_+ \mathcal{Q}_- - \bar{\epsilon}_+ \bar{\mathcal{Q}}_- - \epsilon_- \mathcal{Q}_+ + \bar{\epsilon}_- \bar{\mathcal{Q}}_+$

Noether charges. for time $\frac{\partial}{\partial x^0} \rightsquigarrow H$

for space $\frac{\partial}{\partial x^1} \rightsquigarrow P$

for Lorentz rotation $x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0} \rightsquigarrow M$

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

R-rotations : F_A, F_V.

$\rightsquigarrow$ "Quantum theory" (if we know how to produce it!)

If all the symmetries are preserved, i.e. "no anomely$^{!?}$".

then conserved charges $\mapsto$ symmetry transformation in QT

e.g. in QT: $\delta\theta = [\hat{\delta}, \theta]$ with $\hat{\delta} = i\epsilon_+ Q_- - i\bar{\epsilon}_+ \bar{Q}_- - i\epsilon_- Q_+ + i\bar{\epsilon}_- \bar{Q}_+$

original

$Q_\pm^2 = \bar{Q}_\pm^2 = 0$

$\{Q_\pm, \bar{Q}_\pm\} = H \pm P$ $\qquad$ others $= 0$

$\{iF_A, Q_\pm\} = \mp i Q_\pm$

$\{iF_V, Q_\pm\} = -i Q_\pm$ (also for $\bar{Q}_\pm$)

Another version: ① without F_V, then we allow $\{\bar{Q}_+, \bar{Q}_-\} = Z$ "central charge".
② without F_A, then allow $\{Q_-, \bar{Q}_+\} = \tilde{Z}$.

$N=(2,2)$ supersymmetry algebra $\mathbb{Z}_2$-(outer) automorphism

$$Q_- \leftrightarrow \bar{Q}_-$$
$$F_V \leftrightarrow F_A$$
$$Z \leftrightarrow \tilde{Z}$$

<u>Definition</u> Two supersymmetry algebra are mirror to each other if they are related by this rule.

<u>Non-linear σ-model</u>

Classical theory: Let $\{\Phi^1, \dots, \Phi^n\}$ be Chiral superfield. $\Phi^i = \phi^i + \theta^\alpha \psi^i_\alpha + \theta^2 F^i$.

Chiral "multiplet"

Let $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi, \bar{\Phi})$ Assume $(g_{i\bar{j}}) > 0$, i.e. $ds^2 = g_{i\bar{j}} dz^i d\bar{z}^j$ defines a Kähler metric locally in $\mathbb{C}^n$.

formal assignment to $\bar{\Phi}$.

<u>Fact</u> Levi-Civita $\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}} = \overline{\Gamma^{\bar{i}}_{\bar{j}\bar{k}}}$, others $= 0$.

$$L_{kin} = \int d^4\theta\, K(\Phi, \bar{\Phi}) = -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^j + i\, g_{i\bar{j}} \bar{\psi}^j_\mp D_\pm \psi^i_\mp + R_{i\bar{j}k\bar{\ell}} \psi^i_+ \psi^k_- \bar{\psi}^j_- \bar{\psi}^\ell_+$$

$$+ g_{i\bar{j}} (F^i - \Gamma^i_{\ell k} \psi^\ell_+ \psi^k_-)(\bar{F}^j - \Gamma^{\bar{j}}_{\bar{\ell}\bar{k}} \bar{\psi}^\ell_- \bar{\psi}^k_+).$$

<u>Homework</u> Show above equality!!

This can be defined globally for $\phi: \Sigma \longrightarrow M$: Kähler, $\psi_\pm \in \Gamma(\Sigma, \phi^* T \otimes S_\pm)$

$ \overset{\shortparallel}{T} \qquad (\phi^i) \qquad \bar{\psi}_\pm \in \Gamma(\Sigma, \phi^* \bar{T} \otimes S_\pm)$.

We have global supersymmetry, but we can only check it locally!

$$L_W = \frac{1}{2} \int d^2\theta \left(W(\Phi) + \overline{W(\Phi)} \right) = \frac{1}{2}(F^i \partial_i W + \bar{F}^j \partial_{\bar{j}} \bar{W}) - \frac{1}{2} \partial_i \partial_j W \psi^i_+ \psi^j_- - \frac{1}{2} \partial_{\bar{i}\bar{j}} \bar{W} \bar{\psi}^i_- \bar{\psi}^j_+$$

holomorphic, $\exists$ only if M is non-compact.

Set $F^i = \Gamma^i_{jk} \psi^j_+ \psi^k_-$ (so does $\bar{F}^j \dots$)

$$\Rightarrow L = -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\bar{j}} + i\, g_{i\bar{j}}\bar{\psi}^{\bar{j}}_{\mp} D_{\pm}\psi^i_{\mp} + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\bar{\psi}^{\bar{j}}_-\bar{\psi}^{\bar{l}}_+$$
$$-\frac{1}{4}g^{\bar{i}j}\partial_{\bar{i}}\bar{W}\partial_j W - \frac{1}{2}D_i(\partial_j W)\psi^i_+\psi^j_- - \frac{1}{2}D_{\bar{i}}(\partial_{\bar{j}}\bar{W})\bar{\psi}^{\bar{i}}_-\bar{\psi}^{\bar{j}}_+$$

$$\delta\int d^2x\, L = \int d^2x\left(\partial_\mu\varepsilon_+ G^\mu_- - \partial_\mu\varepsilon_- G^\mu_+ + \partial_\mu\bar{\varepsilon}_-\bar{G}^\mu_+ - \partial_\mu\bar{\varepsilon}_+\bar{G}^\mu_-\right)$$

A direct generalization of previous homework works for superfield.

$$\Rightarrow Q_\pm = \int dx^1\, G^0_\pm = \int dx^1\left(2g_{i\bar{j}}(\partial_\pm\bar{\phi}^{\bar{j}})\psi^i_\pm \mp \frac{i}{2}\psi^i_\mp\partial_i\bar{W}\right)$$
$$\bar{Q}_\pm = \int dx^1\, \bar{G}^0_\pm = \cdots$$

D-term: S_{kin} always $U(1)_V$, $U(1)_A$-invariant by setting charge $=0$.

F-term: S_W : $U(1)_A$ is OK by setting R-charge $=0$ to Φ^i.

For $U(1)_V$, an assignment is possible $\Leftrightarrow$ $W(\lambda^{q_i}\Phi^i) = \lambda^2 W(\Phi^i)$

2021. 11. 4.

"Anomaly"

Toy model: $S = \int_{T^2} d^2 z \left(i \overbrace{\overline{\psi}_+ \underline{D_z} \psi_+}^{\langle \overline{\psi}_+ , D_z \psi_+ \rangle} + i \overline{\psi}_- D_{\bar{z}} \psi_- \right)$

Dirac operator with hermitian connection on E.

Hermitian line bundle.

$\psi_\pm \in \Gamma(T^2, \widehat{E} \otimes S_\pm)$

$\overline{\psi}_\pm \in \Gamma(T^2, E^* \otimes S_\pm)$

It is invariant under $V: e^{-i\alpha}$

$A: \psi_\pm \longmapsto e^{\mp i\beta} \psi_\pm$.

Let $k := c_1(E) \in \mathbb{Z}$. Say $k > 0$. Then,

A-S index theorem $\Rightarrow$ $\dim\ker D_{\bar{z}} - \underbrace{\dim\operatorname{coker} D_{\bar{z}}}_{\substack{\shortparallel \\ \ker D_{\bar{z}}^{\dagger}}} = \int_{T^2} ch(E) \underbrace{\hat{A}(T^2)}_{\substack{\shortparallel \\ 1}} = k$.

(adjoint $= \dagger$)

$\Rightarrow \int D\psi D\overline{\psi}\, e^{-S[\psi, \overline{\psi}]} = 0$.

(Look at fermion integral on zero mode!)

To get non-zero correlation functions, we should consider

$\langle \psi_-(z_1) \cdots \psi_-(z_k) \overline{\psi}_+(w_1) \cdots \overline{\psi}_+(w_k) \rangle$ at general points $z_1 \dots z_k, w_1 \dots w_k$.

$\overset{*}{\circ}$

$\Big\downarrow A$ still V-invariant

$e^{2ik\beta} \langle \cdots \rangle$ not A-invariant unless $k = 0$, i.e. $c_1(E) = 0$.

For σ-model: $\phi : \underset{\substack{\shortparallel \\ \Sigma}}{T^2} \longrightarrow M$, $E = \phi^* T_M^{1,0}$ anomaly free requires $\langle \phi^* c_1(M), \Sigma \rangle = 0$.

e.g. if $c_1(M) = 0$, Calabi-Yau

Notice that the term $R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \overline{\psi}_-^{\bar{j}} \overline{\psi}_+^{\bar{l}}$ does not affect R-symmetry. (classically)

In quantum theory, it corresponds to a perturbation term in the "large radius expansion".

$$\overline{\Psi}_- = \sum_{n=1}^{\infty} b_n \overline{\varphi}_-^n \quad , \quad \Psi_- = \sum_{\alpha=1}^{k} C_{0\alpha} \varphi_-^{0\alpha} + \sum_{n=1}^{\infty} C_n \varphi_-^n$$

$$\Psi_+ = \sum_{n=1}^{\infty} \tilde{b}_n \varphi_+^n \quad , \quad \overline{\Psi}_+ = \sum_{\alpha=1}^{k} \tilde{C}_{0\alpha} \overline{\varphi}_+^{0\alpha} + \sum_{n=1}^{\infty} \tilde{C}_n \overline{\varphi}_+^n$$

In terms of eigenfunctions of $D_{\bar z}^\dagger D_{\bar z}$, $D_z^\dagger D_z$.

$b_n, \tilde{b}_n, C_{0\alpha}, \tilde{C}_{0\alpha}, C_n, \tilde{C}_n$: fermion coordinates.

$$\int D\Psi D\overline{\Psi}\, e^{\underline{-S[\Psi,\overline{\Psi}]}} = \int \prod_{\alpha=1}^{k} dC_{0\alpha} d\tilde{C}_{0\alpha} \prod_{n=1}^{\infty} db_n d\tilde{b}_n dC_n d\tilde{C}_n$$

Need insertions to remedy it!

no $C_{0\alpha}$, nor $\tilde{C}_{0\alpha}$

Thus, we have 4 possibilities.

(1) CY σ-model.	$U(1)_A$	$U(1)_V$
(2) σ-model with $C_1 \neq 0$.	$\times$	$U(1)_V$
(3) LG-model on CY (non-compact) with general W.	$U(1)_A$	$\times$
(4) LG-model on CY with quasi-homogeneous W.	$U(1)_A$	$U(1)_V$

Supersymmetry algebras : $Q_- \longleftrightarrow \overline{Q}_-$ mirror symmetry.

$F_V \longleftrightarrow F_A$

$(Z \longleftrightarrow \tilde{Z})$

<u>Question</u>: Why Ricci flat on (M, g)?

Renormalization

σ-model $(\Sigma, h) \longrightarrow (M, g)$: Kähler.

classical action $S = \int_\Sigma \sqrt{h}\, dx^2 \left(g_{i\bar j} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \overline{\phi}^{\bar j} + i\, g_{i\bar j} \overline{\Psi}^{\bar j} \gamma^\mu D_\mu \Psi^j + R_{i\bar j k \bar l} \Psi_+^i \Psi_-^k \overline{\Psi}_-^{\bar j} \overline{\Psi}_-^{\bar l} \right)$

is invariant under scaling: $h_{\mu\nu} \mapsto \lambda^2 h_{\mu\nu}$, $\phi^i \mapsto \phi^i$, $\underline{\gamma^\mu \mapsto \lambda^{-1} \gamma^\mu}$, $\Psi_\pm \longmapsto \lambda^{-\frac{1}{2}} \Psi_\pm$

(scalar transform) since $\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$.

Quantum level ? Still requires $c_1(M)=0$ to keep R-symmetry.

In fact, $[\omega] \mapsto [\omega] - \log\lambda \cdot c_1(M)$ so this is necessary.

$\omega \longmapsto$?

§13.3 (reading) For $\Sigma = T^2$, CY/LG corresponds as "mirror symmetry" can be checked on supersymmetry ground state via T-duality.

harmonic forms.

"idea": Consider $\underset{h}{T^2} \longrightarrow \underset{\text{Kähler}}{(M.g)}$, $k = \int_{T^2} \phi^* c_1(M)$.

$f(h.g) := \langle \underbrace{\psi_-^{\otimes k}}_{\psi_-(z_1)\cdots\psi_-(z_k)} \bar{\psi}_+^{\otimes k} \rangle$ in general $\neq 0$.

It satisfies (1) $f(h.g) = f(\lambda^2 h.g)\lambda^k$

(2) $\underline{f}(h.g) = n_h e^{-A_g}$, A_g = area of $\phi(T)$ in $(M.g) = \int_{T^2} \phi^*\omega$.

has contributions only from holomorphic maps ϕ.

$\Rightarrow f(h.g) = n_{\lambda^2 h}\, e^{-(A_g - k\log\lambda)} = f(\lambda^2 h.g')$ for another metric g' s.t. $A_{g'} = A_g - k\log\lambda$.

Bosonic σ-model on Riemannian manifolds

$$S = \int g_{IJ}(\phi)\,\partial_\mu \phi^I \partial^\mu \Phi^J\, d^2x \sqrt{h}$$

$\phi^I = \phi_0^I + \xi^I$ expansion near a point $\phi_0 \in M$.

Consider Riemann Normal coordinate at ϕ_0 : $g_{IJ}(\phi) = \delta_{IJ} - \frac{1}{3} R_{IJKL}(\phi_0)\xi^K\xi^L + O(|\xi|^3)$

Recall 0 & 1-dim QFT:

$$\langle \theta \rangle = \frac{1}{Z(M.C)} \int d^n x\, e^{-\frac{1}{2}x_i M_{ij} x_j + C_{ijkl} x_i x_j x_k x_l}\, \theta(x_1, \dots x_n)$$

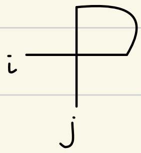
2 point function at $c=0$: propagator

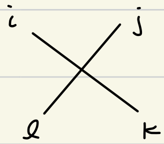
i.e. $M_{ij}\langle x_j . x_k\rangle_{(0)} = \delta_{ik}$.

$$\langle x_i , x_j\rangle_{(0)} = \frac{1}{Z(M,0)}\int d^n x \, e^{-\frac{1}{2}x_k M_{kl} x_l} x_i x_j = M^{ij}$$

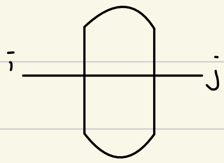
Feynman diagram $\langle x_i , x_j\rangle =$ i_____j 0-loop

$\underset{M^{ij}}{i \underline{\qquad\quad} j}$ edge

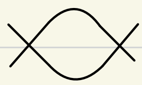
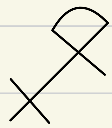
\+  1-loop

 vertex

$-C_{ijkl}$

\+  2-loop

\+ ...

$\langle x_i . x_j . x_k . x_l\rangle =$  $+ \Big($  $+$  $\Big) + \cdots$

$\langle x_i x_j x_k x_l\rangle_{(0)}$ $\langle x_i x_j x_k x_l\rangle_{(1)}$

Homework Compute 2 point, 4 point functions up to 1-loop.

Analogue in 2D QFT: (sketch)

$$\Delta \langle \xi^I(x) , \xi^J(y)\rangle_{(0)} = \delta(x-y)\,\delta^{IJ} \xRightarrow[\text{transform}]{\text{Fourier}} \langle \xi^I(x) , \xi^J(y)\rangle_{(0)} = \int \frac{d^2k}{(2\pi)^2}\frac{e^{ik(x-y)}}{k^2}\delta^{IJ}$$

$\Delta = -\sum \partial^\mu \partial_\mu$

log divergence is removed by cut-off

Now, the action is

$$S = \int \left(\frac{1}{2}\partial^\mu \xi^I \partial_\mu \xi^J - \frac{1}{6}R_{IJKL}(\phi_0)\partial^\mu \xi^I \partial_\mu \xi^J \xi^K \xi^L + O(|\xi|^3)\right) d^2x\sqrt{h} .$$

$$\rightsquigarrow \langle \xi^I(x) . \xi^J(y)\rangle_{(1)} = \int \frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2}\left(\delta^{IJ} + \frac{1}{3}\int_{\mu \le |k| \le \Lambda_{UV}} \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}R_{IJ}(\phi_0)\right)$$

get $\frac{1}{2\pi}\log\frac{\Lambda_{UV}}{\mu}$ UV - ultraviolet 紫外線
μ 紅外線 !?

Homework Similarly

$$\langle \xi^{I_1}(x_1)\, \xi^{I_2}(x_2)\, \xi^{I_3}(x_3)\, \xi^{I_4}(x_4) \rangle_{(1)}$$

$$= -\frac{1}{3} \int \prod_{i=1}^{4} \frac{d^2 p_i}{(2\pi)^2} \frac{e^{ip_i x_i}}{p_i^2} (2\pi)^2 \delta(p_1+p_2+p_3+p_4) \cdot$$

$$\left(p_3 \cdot p_4 \left\{ R_{I_1 I_2 I_3 I_4} + \frac{1}{6\pi} \log \frac{\Lambda_{UV}}{\mu} (R_4 R_2)_{I_1 I_2 I_3 I_4} \right) + \cdots$$

Can we choose $a, b \sim \log \frac{\Lambda_{UV}}{\mu}$ such that

$$g_{IJ} = \delta_{IJ} \mapsto \tilde{g}_{IJ} = \delta_{IJ} + a_{IJ}$$

$$\xi^I \mapsto \tilde{\xi}^I = \xi^I + b^I_J \xi^J$$

Answer: YES! $a_{IJ} = \frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu} R_{IJ}$

$$b^I_J = -\frac{1}{6\pi} \log \frac{\Lambda_{UV}}{\mu} R^I_J$$

2021.11.8.

Caculus on Feymann integral — Feymann diagram R_G

divergence issue $\longrightarrow$ renormalization.

$$\begin{cases} g_{0\,IJ} = g_{IJ} + \dfrac{1}{2\pi}\log\dfrac{\Lambda_{UV}}{\mu}. \\ \xi_0^I = \xi^I - \dfrac{1}{6\pi}\log\dfrac{\Lambda_{UV}}{\mu} R_{IJ}\xi^J. \end{cases}$$

cut-off : $0 < \mu \leq |\underline{k}| \leq \Lambda_{UV}$

vector

$\Rightarrow$ 1-loop ok!

General picture (Ken Wilson)

$S(\phi, g)$, ϕ : collection of fields

g : "coupling constants"

Fields at cut-off scale $\Lambda = \Lambda_{UV}$

$$\phi_0(x) = \int_{|k|\leq\Lambda_{UV}} \frac{dk}{(2\pi)^2} e^{ikx}\hat{\phi}(k).$$

$$Z = \int D\phi_0\, e^{-S(\phi_0, \underline{g_0})}$$

corresponding "g" in cut-off

: UV-divergence disappears!

$$\phi_0(x) = \underbrace{\phi_L(x)}_{\int_{0\leq|k|\leq\mu}} + \underbrace{\phi_H(x)}_{\int_{\mu\leq|k|\leq\Lambda_{UV}}}$$

$$e^{-S_{eff}(\phi_L, g_0)} = \int D\phi_H\, e^{-S(\phi_L+\phi_H, g_0)}.$$

Goal: Change the discription at low energy scale μ to make the effective action regular under $\Lambda_{UV}/\mu \longrightarrow 0$.

In many cases, it has the form $g_0 = g_0(g, \frac{\Lambda}{\mu})$

$\phi_0(x) = Z(g, \frac{\Lambda}{\mu})\,\phi(x) + \phi_H(x)$.

Hence, we view $g, \frac{\Lambda}{\mu}$ as new variables.

Definition The beta function for coupling constant g is

$$\beta(g) = \mu \frac{d}{d\mu} g\left(g_1, \frac{\mu}{\mu_1}\right)\Bigg|_{\substack{g_1 = g \\ \mu_1 = \mu}} .$$

For non-linear σ-model, at 1-loop level,

g : metric tensor

$$\beta_{IJ} = -\mu \frac{d}{d\mu} g_{0IJ} = \frac{1}{2\pi} R_{IJ} .$$

- $R_{ij} > 0$, called asymptotic freedom as $\Lambda \longrightarrow \infty$.

  $\left(\text{since} \quad g_{IJ} + \frac{1}{2\pi} \log \frac{\Lambda}{\mu} \underset{0}{\underset{\vee}{R_{IJ}}} \nearrow \infty \right)$

  i.e. the purterbation theory "becomes better." (since $R_{ijkl} \sim 0$).
- $R_{ij} = 0$, get scale invarance at 1-loop level and also 2-loop level since it involves only up to ∇R_{IJ}. (hence $= 0$)

  But $\beta \neq 0$ for 4-loops if (M, g) is NOT flat.
- R_{ij} is general, e.g. < 0.

  $\rightsquigarrow$ Ultraviolet singularity.

  σ-model is not well-defined.

Remark RG flow also works for supersymmetry σ-model.

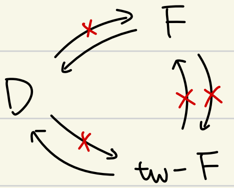
e.g. 1-loop β function is not modified by Fermion (easy to see.)

$*$ F-term non-renormalization theorem $\quad W_{eff} = W$, $\widetilde{W}_{eff} = \widetilde{W}$.

idea:
- promote parameters to fields.
- consider S_t : 1-parameter family (deformation) s.t. $t \longrightarrow 0^+$ gives the original theory.

  " $\frac{1}{t} \Delta S_t$ "
- choose ΔS_t so that it decouples (say in effective action).

  among fields for all $\varepsilon > 0$.

$\Rightarrow$ 

X : not affect !

In practice, let $S = \int dx^2 d\theta^4 \, K(\Phi_i, \bar{\Phi}_i, \tilde{\Phi}_{\tilde{i}}, \bar{\tilde{\Phi}}_{\tilde{i}}, \gamma_b) + \int d^2\theta \, W(\Phi_i, \lambda_a) + c.c$

parameters

$+ \int d^2\tilde{\theta} \, \tilde{W}(\tilde{\Phi}_{\tilde{i}}, \tilde{\lambda}_{\tilde{a}}) + c.c.$

Φ_i : Chiral superfields

$\tilde{\Phi}_i$: twisted-Chiral superfields

Consider $\Delta S_\varepsilon = \frac{1}{\varepsilon} \int d^2x \, d^4\theta \left(\sum_b \pm |\Gamma_b|^2 + \sum_a |\Lambda_a|^2 - \sum_{\tilde{a}} |\tilde{\Lambda}_{\tilde{a}}|^2 \right)$

$\begin{cases} + \text{ for } \Gamma_b \text{ c.s.f.} \\ - \text{ for } \Gamma_b \text{ tw-c.s.f.} \end{cases}$ the sign is required so that it is correct for component fields.

(i) No matter how we get W_{eff}, $\tilde{W}_{eff}$, we know $\tilde{\Lambda}_{\tilde{a}} \not\rightarrow W_{eff}$, $\Lambda_a \not\rightarrow \tilde{W}_{eff}$

since supersymmetry is preserved for all $\varepsilon > 0$.

(ii) Let $\varepsilon \to 0^+$, the D-term is large

$\Rightarrow$ any non-trivial rotation of Λ_a, $\tilde{\Lambda}_{\tilde{a}}$, Γ_b over (Σ, h) gives very large action.

$\Rightarrow$ these fields $\longrightarrow$ constant

i.e. $\begin{cases} \text{scalar components} \longrightarrow \text{constant} \\ \text{other components} \longrightarrow 0 \end{cases}$

i.e. the construction of W_{eff} goes back to the original W

$\Rightarrow$ D term parameters do not affect W.

(iii) Consider another deformation

$$\Delta S_\varepsilon = \frac{1}{\varepsilon} \int d^2x \, d^4\theta \left(\sum |\Phi_i|^2 - \sum |\tilde{\Phi}_{\tilde{i}}|^2 \right)$$

in $\varepsilon \to 0^+$, get only constant parameters hence is absorbed into the action.

(no effect in $\varepsilon \to 0^+$)

(ii) + (iii) $\Rightarrow$ D does not affect W, $\tilde{W}$ in effective theory.

Remark There is another proof in the book for $W = m\Phi^2 + \lambda\Phi^3$.

$\Rightarrow$ Homework read it and do the multivariable case.

<u>Remark</u> In our $1+1$ dimension QFT case, get ∞-dimensional group of symmetries.

Virasoro operators: $L_n = z^n \cdot z \frac{d}{dz}$ $\quad (\Sigma, h) \xrightarrow{\phi} (M, g)$

(generators) $\quad$ z: coordinate on Σ.

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0}$$

In the free field theory, we had seen $L_n = \frac{1}{2} : \sum_m \alpha_m \alpha_{n-m} :$

<u>Conjectures</u> (1) CY of dimension D : Ricci flat $\xrightarrow[\text{1-loop}]{\text{RG-flow}}$ CFT $=:$ fixed point of RG flow.

with $c = 3D$

(not necessary Ricci flat.)

(2) For LG to be CFT $\Rightarrow$ W is quasi-homogeneous with suitable D-term

$\downarrow$ RG-flow

$\exists!$ CFT

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Chiral rings

Let $Z=\tilde{Z}=0$ (central charges)

$\{\bar{Q}_+, \bar{Q}_-\} = Z = 0$, $\{\bar{Q}_+, \bar{Q}_-\} = \tilde{Z} = 0$.

$$Q = \begin{cases} Q_B := \bar{Q}_+ + \bar{Q}_- \\ Q_A := \bar{Q}_+ + Q_- \end{cases} \Rightarrow Q^2 = 0 \Rightarrow Q\text{-cohomology}.$$

e.g. let $\Sigma = \mathbb{R}\times S^1$: Q-cohomology of states $\simeq$ supersymmetry ground states in Kähler σ-model M $\simeq$ harmonic forms $H^*(M)$.

Definition $\mathcal{O}$: Chiral operators $[Q_B, \mathcal{O}]=0$.

twisted-Chiral operators $[Q_A, \mathcal{O}]=0$.

i.e. Q-closed operator.

e.g. For Chiral superfield $\Phi = \phi + \theta^\alpha \psi_\alpha + \theta^+\theta^- F$ at $y^\pm$

Chiral multiplet (16 components) at $x^\pm$.

Fact $[\bar{Q}_\pm, \phi]=0$

$$[\bar{Q}_\pm, \phi] = \left(-\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\partial_+\right)\left(\phi - i\theta^\pm\bar{\theta}^\pm\partial_\pm\phi - \theta^4\partial_+\partial_-\phi\right).$$

$\rightsquigarrow Q = Q_B$, $[Q, \phi]=0$.

ψ_α, F are determined by $\psi_\pm := [iQ_\pm, \phi]$, $F := \{Q_+, [Q_-, \phi]\}$ such that $[Q, \Phi]=0$.

If Q commute with $\mathcal{O}_1, \mathcal{O}_2$, then so does $\mathcal{O}_1\cdot\mathcal{O}_2$.

Definition (twisted-) Chiral ring $C(Q) :=$ Q-cohomology ring of (twisted-) Chiral operators.

e.g: $i\partial_-\mathcal{O} = \frac{i}{2}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\right)\mathcal{O} = [H-P, \mathcal{O}] = [\{Q_-, \bar{Q}_-\}, \mathcal{O}]$

worldsheet translation

$= -\{[\bar{Q}_-, \mathcal{O}], Q_-\} - \{[\mathcal{O}, Q_-], \bar{Q}_-\} = \{Q_B, [Q_-, \mathcal{O}]\}$

Q-closed

leads to Q-boundary.

$$\{[\bar{Q}_-, \mathcal{O}], Q_-\} = \{[\bar{Q}_+, \mathcal{O}], Q_-\} = -\{[\mathcal{O}, Q_-], \bar{Q}_+\} - [\underbrace{\{Q_-, \bar{Q}_+\}}_{=-\tilde{Z}=0}, \mathcal{O}]$$

Another way for Fact:

$$[\bar{Q}_\pm, \phi] = \bar{Q}_\pm\left(\mathcal{F}\big|_{\theta^\pm=\bar{\theta}^\pm=0}\right) = \left(\bar{D}_\pm + 2i\theta^\pm\partial_\pm\right)\mathcal{F}\big|_{\theta^\pm=\bar{\theta}^\pm=0} = 0.$$

<u>Twisting</u> Q: How to proceed if Σ is not flat?

First, we use $z = x^1 - x^0 = x^1 + ix^2$ (Wick rotation)

$\Sigma = T^2$

$\mathbb{C}|_\Lambda$

Lorentz $SO(1.1) \longrightarrow SO(2)_E = U(1)_E$, then try to work on Σ globally.

Recall supersymmetry action:

$$\delta S = \int_\Sigma \left[(\nabla_\mu \epsilon_+) G_-^\mu - (\nabla_\mu \bar{\epsilon}_+) \bar{G}_-^\mu - (\nabla_\mu \epsilon_-) G_+^\mu + (\nabla_\mu \bar{\epsilon}_-) \bar{G}_+^\mu \right] \sqrt{h}\, d^2x$$

$\epsilon_+, \bar{\epsilon}_+ \in \Gamma(S_+)$ $\quad S_+ = \bar{K}^{1/2}$ $\quad T^*\Sigma$ holomorphic.

$\epsilon_-, \bar{\epsilon}_- \in \Gamma(S_-)$, $\quad S_- = K^{1/2}$ $\quad$ $\parallel$ K: canonical bundle

If Σ is not flat, then $\nexists$ flat sections $\Rightarrow \delta S \neq 0$.

(covariant constant) $\Rightarrow$ no supersymmetry!!

If one of $U(1)_A$ or $U(1)_V$ exists,

		$U(1)_V$	$U(1)_A$	$U(1)_E$	$\mathcal{L}$	A-twist by F_V (→ twisted Chiral ring) $U(1)'_E$	$\mathcal{L}$	B-twist by F_A (→ Chiral ring) $U(1)'_E$	$\mathcal{L}$
ϕ		0	0	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$
ψ_-	Q_-	-1	1	1	$K^{1/2}$	$\boxed{0}$	$\mathbb{C}$	2	K
$\bar{\psi}_+$	$\bar{Q}_+$	1	1	-1	$\bar{K}^{1/2}$	$\boxed{0}$	$\mathbb{C}$	$\boxed{0}$	$\mathbb{C}$
$\bar{\psi}_-$	$\bar{Q}_-$	1	-1	1	$K^{1/2}$	2	K	$\boxed{0}$	$\mathbb{C}$
ψ_+	Q_+	-1	-1	-1	$\bar{K}^{1/2}$	-2	$\bar{K}$	-2	$\bar{K}$
		definition		(16.15) check!		$Q = \bar{Q}_+ + Q_-$		$Q = \bar{Q}_+ + \bar{Q}_-$	

We replace $U(1)_E$ by the diagonal $U(1)'_E$ in $U(1)_E \times U(1)_{\underline{R}}$

V or A

$(\Sigma, h) \longrightarrow (M, g)$ σ-model.

Kähler

A-twisted $\Rightarrow$ Gromov-Witten theory, Quantum cohomology.

Why called "Topological twist"?

① Independent of h in (Σ, h).

② Invariant of deformation of parameters in D-term.

③ Holomorphic dependence on Chiral parameters. (for B-twist)

Q-closed

$$①: \underbrace{\delta_h}_{d^2x\sqrt{h}} \langle \mathcal{O} \rangle = \langle \mathcal{O} \frac{1}{4\pi} \int_\Sigma d^2x \sqrt{h}\, \delta h^{\mu\nu} \underbrace{T^{tw}_{\mu\nu}}_{\text{energy momentum tensor.}} \rangle = 0$$

<u>Fact</u> In all the geometric models we discuss, we have $T^{tw}_{\mu\nu} = \{\underbrace{Q}_{\text{exact}}, G_{\mu\nu}\}$.

②: For B-twist, $\delta\langle\cdots\rangle$ means $\langle \cdots \int d^4\theta\, \Delta K \rangle$

Homework check it!

$$= \{ \bar{Q}_+ [\bar{Q}_-, \int d\theta^+ d\theta^- \Delta K|_{\bar{\theta}^\pm=0}] \}$$

$$= \{ \underbrace{\bar{Q}}_{\bar{Q}_+ + \bar{Q}_-} [\bar{Q}_-, \int d\theta^+ d\theta^- \Delta K|_{\bar{\theta}^\pm=0}] \} \quad : d\text{-exact}$$

$(\bar{Q}_- : \text{nilpotent.})$

ψ^i_-	scalar	χ^i
$\bar{\psi}^{\bar{i}}_+$		$\bar{\chi}^{\bar{i}}$
ψ^i_+	$\bar{K}$	$\rho^i_{\bar{z}}\, d\bar{z}$
$\bar{\psi}^{\bar{i}}_-$	K	$\rho^{\bar{i}}_z\, dz$

$$\Rightarrow S = \int d^2z \left(g_{i\bar{j}} h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \bar{\phi}^{\bar{j}} \sqrt{h} - i g_{i\bar{j}} \rho^{\bar{j}}_z D_{\bar{z}} \chi^i + i g_{i\bar{j}} \rho^i_{\bar{z}} D_z \chi^{\bar{j}} - \frac{1}{2} R_{i\bar{k}j\bar{\ell}} \rho^i_{\bar{z}} \chi^j \rho^{\bar{k}}_z \chi^{\bar{\ell}} \right)$$

$\delta = \bar{\epsilon}_- \bar{Q}_+ + \epsilon_+ Q_-$

$Q_A = Q = \bar{Q}_+ + Q_-$.

Set $\bar{\epsilon}_- = \epsilon_+ = \epsilon$

$\xRightarrow{\text{in components}}$

$$\begin{cases} \delta\phi = \epsilon\chi \\ \delta\rho^i_{\bar{z}} = 2i\bar{\epsilon}_- \partial_{\bar{z}} \phi^i + \epsilon_+ \Gamma^i_{jk} \rho^j_{\bar{z}} \chi^k \\ \delta\chi^i = 0 \end{cases}$$

$\rightsquigarrow$ localization to Q-fixed point : $\chi = 0$, $\partial_{\bar{z}}\phi^j = 0$ $\Rightarrow \Sigma \xrightarrow{\phi} M$: holomorphic.

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$V \longrightarrow$ A-twisted so that $\Sigma \longrightarrow (M, g)$ has supersymmetry.

$A \longrightarrow$ B-twisted $\qquad \mathbb{R}^1 \times S^1$

"Topological nature" of these twisted theories.

Basic descent relation for $\begin{cases} \text{Chiral superfield (B-twisted)} \\ \text{twisted Chiral superfield (A-twisted)} \end{cases}$ $\qquad Q = \begin{cases} Q_B = \bar{Q}_+ + \bar{Q}_- \\ Q_A = \bar{Q}_+ + Q_- \end{cases}$

(i) Chiral superfield $\Phi(\phi, \psi_\pm, F)(y^\pm)$.

lowest ϕ satisfies $[\bar{Q}_\pm, \phi] = 0$, then can determines

$\psi_\pm = [iQ_\pm, \phi]$, $F = \{Q_+, [Q_-, \phi]\}$ such that $\bar{Q}_\pm \Phi = 0$ (by Jacobi-identity).

(ii) Similar pattern for Q-closed operator $\mathcal{O}^{(0)} = \mathcal{O}$.

$0 = [Q, \mathcal{O}^{(0)}] \Rightarrow d\mathcal{O}^{(0)} = \{Q, \underbrace{\mathcal{O}^{(1)}}_{=\, i dz [Q_-, \mathcal{O}] - i d\bar{z} [Q_+, \mathcal{O}]}\}$

globally well-defined. (1-form operator)
for B-twisted

$\Rightarrow d\mathcal{O}^{(1)} = [Q, \underbrace{\mathcal{O}^{(2)}}_{=\, dz d\bar{z} \{Q_+, [Q_-, \mathcal{O}]\}}]$ 2-form operator (same proof by Jacobi.)

- Dependence on parameters : → give some div-term in each special cases to be discussed.

① S_h do not change the twisted-theory

metric on Σ

② D-term (for B-twist) : variation on D-term gives $\int d^4\theta \text{ "}\Delta K\text{"}$

$\underset{\text{正比於}}{\propto} \{\bar{Q}_+, [\bar{Q}_-, \int d\theta^+ d\theta^- \Delta K \big|_{\bar{\theta}^\pm = 0}]\}$

$Q_B = \bar{Q}_+ + \bar{Q}_-$ viewed as anti-Chiral on $\bar{\theta}^\pm$, apply (ii).

③ Independence of deformations on twisted-Chiral, anti-twisted Chiral, anti-Chiral.

Say for twisted-Chiral $\tilde{\Phi}$: $\int \underbrace{d^2\tilde{\theta}}_{= d\bar{\theta}^- d\theta^+} \Delta \tilde{W}(\tilde{\Phi}) \overset{(i)}{\propto} \int d^2\theta \{Q_+, [\bar{Q}_-, \Delta\tilde{W}(\tilde{\Phi})]\}$

killed by $\bar{Q}_+$ and Q_- $\qquad \bar{Q}_+ + \bar{Q}_- = Q_B$.

$- \{Q_B, [Q_+, \Delta\tilde{W}(\tilde{\Phi})]\} \pm [\{Q_B, Q_+\}, \Delta\tilde{W}(\tilde{\Phi})]$

$\bar{Q}_+ + \bar{Q}_- \qquad$ holomorphic.

$\{\bar{Q}_+, Q_+\} = H + P$. total derivative $= \partial_z$

④ It can depend on parameters in Chiral superfield holomorphically!

$$\int \sqrt{h}\, d^2x \int d^2\theta\, \Delta W(\Phi) \propto \int \sqrt{h}\, d^2x \left\{ Q_+, [Q_-, \Delta W(\Phi)] \right\} \propto \int \Delta W(\Phi)^{(2)}$$

Chiral ring & Twisted Chiral ring

from $g=0$, 3-point functions : $\Sigma = S^2 \longrightarrow M$.

$C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle_{\underset{\text{genus}}{0}}$, ϕ_i's : physical operators (i.e. Q-closed operators)

$\quad \| $

$\langle \phi_i(x_i)\, \phi_j(x_j)\, \phi_k(x_k) \rangle$

we omit all points x_i since it is a topological field theory

$\eta_{ij} = \langle \phi_i \phi_j 1 \rangle = C_{ij0} = \langle \phi_i \phi_j \rangle_0$ $\quad$ ($\eta_{ij} = (\phi_i \phi_j)_0 = \langle \phi_i | \phi_j \rangle$: Poincaré pairing)

$g=0$ enters have : Assumption $\underbrace{(\eta_{ij})}_{\text{topological metric}}$ is invertible! (verified for all example)

$$\phi_j \phi_k = \sum C_{jk}^{\ell} \phi_\ell \quad \Rightarrow \quad C_{ijk} = \eta_{i\ell} C_{jk}^{\ell}.$$

$\rightsquigarrow$ Chiral ring is determined by 3-point functions.

		A-twist by F_V		B-twist by F_A			
		$U(1)'_E$	$\mathcal{L}$	$U(1)'_E$	$\mathcal{L}$		
ψ_-	Q_-	$\boxed{0}$	$\mathbb{C}$	2	K	CY σ	V & A
$\bar\psi_+$	$\bar Q_+$	$\boxed{0}$	$\mathbb{C}$	$\boxed{0}$	$\mathbb{C}$	(M, g)	V
$\bar\psi_-$	$\bar Q_-$	2	K	$\boxed{0}$	$\mathbb{C}$	L-G, W : holomorphic	A
ψ_+	Q_+	-2	$\bar K$	-2	$\bar K$	L-G, W : quasi-homogeneous	V & A

$c_1 \geq 0$

A-twist for σ-model (M, g) $\qquad$ B-twist for LG, W : general holomorphic

① A & ③ B-twist for CY $\qquad$ A & ② B-twist for LG, W : quasi-homogeneous

Gromov-Witten $\qquad$ Kodaira-Spencer $\qquad$ (later...) JD!! $\qquad$ LG as before.

① A-twist for non-linear σ-model $(W=0)$

$\phi : \Sigma \longrightarrow X$ (Kähler) , $\chi^i = \psi_-^i$, $S = \int \cdots$ in $\chi^i, \bar{\chi}^i, \rho_z^i, \rho_{\bar{z}}^i$.

$\bar{\chi}^i = \bar{\psi}_+^i$

$\rho_{\bar{z}}^i = \psi_+^i \in \bar{K}$

$\rho_z^i = \bar{\psi}_- \in K$

$\delta = \bar{\epsilon}_- \bar{Q}_+ + \epsilon_+ Q_-$, set $\bar{\epsilon}_- = \epsilon_+ = \epsilon$

Q-variation $\delta\phi = \epsilon\chi$

$\delta\chi = 0$ $\quad + c.c.$

$\delta\rho_{\bar{z}}^i = 2i\bar{\epsilon} - \underbrace{\partial_{\bar{z}}\phi^i}_{=0} + \epsilon_+ \Gamma^i_{jk}\rho_{\bar{z}}^j \underbrace{\chi^k}_{=0}.$

Q-fixed points :

$\Rightarrow$ ϕ holomorphic.

Only have to consider operators made up by ϕ, χ , not ρ. $\rightarrow$ deformation

$\phi^i = z^i$, $Q_- = \partial$

$\chi^i = dz^i$, $\bar{Q}_+ = \bar{\partial}$, $Q = \partial + \bar{\partial} = d$

$\bar{\chi}^i = d\bar{z}^i$

$\{\text{physical operators}\} \simeq H^*_{dR}(M)$ in group level.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X\, \mathbb{Z})} \langle \prod_i \mathcal{O}_i \rangle_\beta = \sum_\beta \int_{\phi_*[\Sigma]=\beta} D\phi\, D\chi\, D\rho\; e^{-S}\, \mathcal{O}_1 \cdots \mathcal{O}_s .$$

$\mathcal{O}_i \longleftrightarrow \omega_i \in H^{p_i, q_i}(X).$

$\langle \prod_i \mathcal{O}_i \rangle = 0$ only if $q_V = -p_i + q_i$ symmetry fixed i.e. $\sum p_i = \sum q_i$.

$q_A = p_i + q_i$ (broken)

$$\sum (p_i + q_i) = 2 \underbrace{\operatorname{ind} \bar{\partial}}_{=} = 2k = \left(c_1(X).\beta + \dim X \cdot (1-g) \right)$$

$\begin{pmatrix} \chi & \text{0-mode} \\ -\rho & \text{0-mode} \end{pmatrix}$ $\qquad \sum p_i = \sum q_i = k$

$$\langle \prod_i \mathcal{O}_i \rangle_\beta = e^{-(\omega - iB)\beta} \int_{M_\Sigma(X,\beta)} \overbrace{e(V)}^{\text{Euler class}} \prod ev_i^* \omega_i \qquad D_i = \underbrace{PD}_{\text{Poincaré dual}}(\overbrace{\omega_i}^{\text{cycle}}) \quad , \quad \omega_i = \delta_{D_i}$$

- For non-zero mode, the Boson & Fermion determinant cancels out by supersymmetry. (Homework)

  V comes from the 0-mode of ρ. ← In general, it is an object in derived category control deformations. (vitural fundamental class)

If $V = 0 \longrightarrow$ G-W counting.

If $V \neq 0$, vector bundle $\Rightarrow Pf(F_V) = e(V)$ by 0-dimension QFT.

② B-twist of L-G model.

M: non-compact CY, $W: M \longrightarrow \mathbb{C}$ holomorphic.

Change spin: scalar $\begin{cases} \psi^{\bar{i}} = \bar{\psi}^i_- \\ \bar{\psi}^{\bar{i}} = \bar{\psi}^i_+ \end{cases}$ $\qquad S = \int \cdots$.

one-form $\begin{cases} \rho^i_z = \psi^i_- \in K \\ \rho^i_{\bar{z}} = \psi^i_+ \in \bar{K} \end{cases}$

under $\delta = \bar{\epsilon}_- \bar{Q}_+ - \bar{\epsilon}_+ \bar{Q}_-$, set $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$, $Q_B = \bar{Q}_+ + \bar{Q}_-$.

For simplicity, assume M = flat (e.g. $\mathbb{C}^N$)

$$\delta \phi^i = 0 \qquad\qquad \delta \bar{\phi}^{\bar{i}} = -\bar{\epsilon}(\psi^{\bar{i}} + \bar{\psi}^{\bar{i}})$$

$$Q: \quad \delta(\psi^{\bar{i}} - \bar{\psi}^{\bar{i}}) = \bar{\epsilon}\, g^{\bar{i}j} \underline{\partial_j W} \qquad\qquad \delta(\psi^{\bar{i}} + \bar{\psi}^{\bar{i}}) = 0$$

$$\delta \rho^i_z = 2i\bar{\epsilon}\, \underline{\partial_z \phi^i} \qquad\qquad \delta \rho^i_{\bar{z}} = -2i\bar{\epsilon}\, \underline{\partial_{\bar{z}} \phi^i} .$$

localize at Q-fixed points $\Rightarrow$ ϕ: constant map, $\partial_j W = 0$ for all j.
to $\mathrm{Critial}(W) \subseteq M$.

Assume isolated & non-degenerate critical points: $y_1, \dots, y_N$.

Correlation: operator = holomorphic function in ϕ^i, i.e. in M $\quad f \mapsto \mathcal{O}_f$

$$\langle \prod_i \mathcal{O}_{f_i} \rangle = \int D\phi D\psi D\rho\, e^{-S} \prod \mathcal{O}_{f_i} = \sum_{i=1}^N \langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle \Big|_{y_i}$$

At y_i, constant map kills the kinetic term. Boson - Fermion non-zero determinant $\longrightarrow 1$.

For constant mode:

$$\int d^{2n}\phi\, e^{-\frac{1}{4}g_{i\bar{j}}\partial_i W \partial_{\bar{j}}\overline{W}} \qquad \int d^n\psi\, d^n\bar{\psi}\, e^{-\frac{1}{2}W_{\bar{i}\bar{j}}\psi^{\bar{i}}\bar{\psi}^{\bar{j}}} \quad \cdot \int d^{ng}\rho\, d^{ng}\bar{\rho}\, e^{-\frac{1}{2}W_{\bar{i}\bar{j}}\rho^{\bar{i}}_{\bar{z}}\rho^{j}_{z}}$$

change of variable $u_i = \partial_i W$

$$\det(W_{ij})^{-2}(y_i) \qquad \overline{\det(W_{\bar{i}\bar{j}})}(y_i) \qquad \left(\det(W_{ij})\right)^{g}(y_i)$$

i.e. $\langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle_g = \sum_{i=1}^{N} f_1(y_i)\cdots f_s(y_i)\left(\det W_{ij}(y_i)\right)^{g-1}$ If $g=0 \Rightarrow$ ring structure.

③ B-twist of CY σ-model : $(W=0)$.

$$\eta^{\bar{i}} = (\psi^{\bar{i}} + \bar{\psi}^{\bar{i}}), \quad g^{\bar{i}j}\theta_j = \psi^{\bar{i}} - \bar{\psi}^{\bar{i}}$$

Q_B - variation
$$\delta\phi^i = 0$$
$$\delta\bar{\phi}^{\bar{i}} = \bar{\epsilon}\eta^{\bar{i}}$$
$$\delta\theta_i = 0$$
$$\delta\bar{\eta}^{i} = 0$$
$$\delta\rho^i_\mu = \pm 2i\bar{\epsilon}\partial_\mu\phi^i.$$

$\rightsquigarrow$ $\eta^{\bar{i}} \longleftrightarrow d\bar{z}^{\bar{i}}$ $\qquad Q = Q_B = \bar{\partial} \rightsquigarrow$ get Dobeault complex.

$\theta_i \longleftrightarrow \frac{\partial}{\partial z^i}$

$$\Omega^{0,g}(M, \wedge^p T) \xrightarrow{\bar{\partial}} \Omega^{0,g+1}(M, \wedge^p T)$$

Correlation. localize at Q-fixed points, $\partial_\mu\phi = 0$ for all μ $\Rightarrow$ constant map.

$$\Rightarrow \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int_M \mu_1^i \wedge \mu_2^j \wedge \mu_3^k \wedge \Omega_{ijk} \wedge \Omega$$

$\mathcal{O}_i \longleftrightarrow \mu_i \in H^1(T_M) = H^{0,1}(T_M)$ $\quad (k=3)$ holomorphic n-form, $\neq 0$.

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Gauged linear σ-model (e.g. $M = \mathbb{C}^n$ or $\mathbb{P}^n$)

e.g. $\bigoplus_{i=1}^{N} \mathcal{O}(Q_i) \to \mathbb{CP}^n \supseteq X$, $Q_i \in \mathbb{Z}$

$Q_i > 0$: complete intersection

$Q_i < 0$: e.g. $\mathcal{O}(-3) \to \mathbb{P}^2$ "bundle space" as a local part of some compact space.

Gauge Theory / Yang-Mills theory

Let $E \to (M, g)$ Riemann, G-bundle, $rk = N$

$(\, , \,) = \int_M \underbrace{< \, , \, >}_{\text{inner product on } \Lambda^2(\mathfrak{g}_E)} d\mu_g$ bundle

$\mathcal{A}$:= the space of G-connections has the form $\underbrace{A}_{\text{any fixed one}} + a$, $a \in \underbrace{\Omega^1(\mathfrak{g}_E)}_{\text{section}}$ i.e. affine.

Yang-Mills functional : $\mathcal{YM}(A) := \|F_A\|^2 = \int_M |F_A|^2 d\mu_g$

curvature $F_A =$ "dA" $+ A \wedge A$

$$F_{A+ta} = d(A+ta) + (A+ta) \wedge (A+ta)$$
$$= F_A + t\underbrace{(da + A \wedge a + a \wedge A)}_{= da + [A, a] = d_A a} + t^2 a \wedge a$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(A+ta) = 2\int_M < d_A a, F_A > d\mu_g = 2\int_M < a, d_A^* F_A > d\mu_g$$

A is a critical point $\Leftrightarrow d_A^* F_A = 0$: 2^{nd}-order non-linear PDE in A.

Notice that $d_A F_A = 0$ (Bianchi identity).

If $\dim M = 4$, oriented, compact $\rightsquigarrow \Lambda^2 \mathfrak{g}_E = \underbrace{\Lambda_+^2 \mathfrak{g}_E}_{\text{eigenvalue}=1} \oplus \underbrace{\Lambda_-^2 \mathfrak{g}_E}_{\text{eigenvalue}=-1}$ since $*^2 = 1$

If further $N = 2$, $G = SU(2)$, i.e. E is a rank 2, $SU(2)$ bundle.

$c_1(E) = \frac{\sqrt{-1}}{2\pi}[\mathrm{tr} F_A] = 0$, $\left(\frac{\sqrt{-1}}{2\pi}\right)^2 [\mathrm{tr} F_A^2] = c_1^2 - 2c_2 = -2c_2(E)$.

Let $\underset{\mathbb{Z}}{k} = c_2(E)[M] = \frac{1}{8\pi^2}\int_M \mathrm{tr} F_A^2 = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F_A^+)^2 + \mathrm{tr}(F_A^-)^2 = \frac{-1}{8\pi^2}\left(\|F_A^+\|^2 - \|F_A^-\|^2\right)$

su(2) : inner product $A \cdot B = \mathrm{tr}(\bar{A}^t B) = -\mathrm{tr}(AB)$

$F_A^+ \wedge F_A^+ = * F_A^+$ $\quad$ $F_A^- \wedge F_A^- = -* F_A^-$

But $\mathcal{YM}(A) = \|F_A\|^2 = \|F_A^+\|^2 + \|F_A^-\|^2 = 8\pi^2 k + 2\|F_A^+\|^2$

If $k>0$, then the minimal is achieved where $F_A^+ = 0$, i.e. $*F_A = -F_A$. ASD YM connection

1-st order non-linear PDE in A.

ADHM construction : ASD on S^4 for $k=1$.

A-S index theorem $\Rightarrow$ $\dim M_k^{ASD} = \text{rank} - 3(1 + b_1(X) + b_+^2(X))$

If $k=1$, $\pi_1(M)=0$, get $5 - 3b_+^2(X)$.

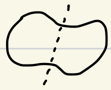
Donaldson (1983): M^4 : simply connected with negative integral form g_M on $H^2(M.\mathbb{Z})$

$\Rightarrow$ g_M is standard $\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ over $\mathbb{Z}$. (no E_8 etc.)

M. Freidman (1983): Any integral unimodular quadratic form over $\mathbb{Z}$ is the g_M for some topological 4 manifold M, $\pi_1(M)=0$.

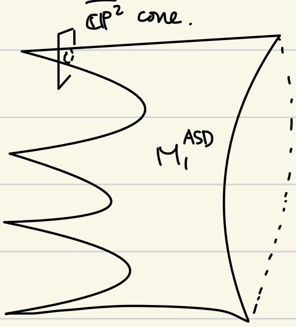
combine!

$\Longrightarrow$ $\exists$ 4-dim topological manifold with no C^∞-structure.

Later (1984) $\mathbb{R}^4$ has exotic differentiable structure.  K3

idea : $\dim M_1^{ASD} = 5$ has the shape :

$\overline{\mathbb{CP}}^2$ cone.



$\partial M_1^{ASD} \simeq M$

$$\underset{\circlearrowleft}{E} = L_1 \oplus L_2 .$$

Supersymmetric Guage Theory (baby version)

Classically : scalar field case $L = -\sum_{i=1}^N |\partial_\mu \phi_i(x)|^2 - U$

$U(\phi) = \frac{e^2}{2}\left(\sum_{i=1}^N |\phi_i|^2 - r\right)^2$ $r \geq 0$.

It is invariant under $\phi_i(x) \mapsto e^{i\alpha}\phi_i(x)$.

But if $\alpha = \alpha(x)$, then need $D_\mu\phi_i = \partial_\mu\phi_i + i v_\mu \phi_i$, v_μ (Gauge field) real

such that " $v_\mu \mapsto v_\mu - \partial_\mu\alpha$ "

$\Rightarrow$ $L = -\sum |D_\mu\phi|^2 - U$ is invariant since $\partial_\mu\phi_i \mapsto e^{i\alpha}(\partial_\mu + i\partial_\mu\alpha)\phi_i$.

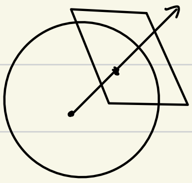
<u>Homework</u> The massless mode is $S^{2N-1}/U(1) \simeq \mathbb{CP}^{N-1}$ with FS metric.

Show $\partial_\mu = \frac{i}{2}\,\frac{\sum_{i=1}^{N} \bar\phi_i \partial_\mu \phi_i - (\partial_\mu \bar\phi_i)\phi_i}{\sum_{i=1}^{N} |\phi_i|^2}$. Q: What is the bundle?

Minimize $L \Rightarrow S^{2N-1}_{\sqrt{r}} = M_{Vac}$

massive mode

eigenvalues of $\partial_i \partial_j U(\phi) =: \text{mass}^2$



massless mode.

Apply this idea to Chiral superfield Φ.

$L = \int d^4\theta\, \bar\Phi \Phi$, under $\Phi \longmapsto e^{iA}\Phi$ is still Chiral.

A is also Chiral superfield

$\longmapsto \bar\Phi\, e^{-i\bar A + iA}\, \Phi$

Consider a real superfield $V(x^\mu, \theta^\pm, \bar\theta^\pm)$ such that $V \longmapsto V + i(\bar A - A)$.

<u>Homework</u> (Wess-Zumino)

Under a suitable gauge,

$$V = \theta^\mp \bar\theta^\mp (v_0 \mp v_1) - \theta^- \bar\theta^+ \sigma + i\theta^-\theta^+ \left(\bar\theta^- \lambda_1^- + \bar\theta^+ \overline{\lambda_+}\right) + \theta^-\theta^+\bar\theta^+\bar\theta^- D$$

real one form complex scalar field complex Dirac fermion real

$+\,c.c.$

and $L = \int d^4\theta\, \bar\Phi e^V \Phi$ is invariant.

The supersymmetry $\delta = \pm\epsilon_\pm Q_\mp \mp \bar\epsilon_\pm \bar Q_\mp$ on component fields of Φ and V is determined.

<u>Remark</u> residual guage := those guage fixes V : $v_\mu \mapsto v_\mu - \partial_\mu \alpha$.

The superfield strength of V is $\Sigma := \bar D_+ D_- V$ which is twisted-Chiral.
(curvature)

$\Rightarrow \Sigma = \sigma(\tilde y) \pm i\theta^\pm \bar\chi_\pm(\tilde y) + \theta^+\bar\theta^-\left(D(\tilde y) - i v_{01}(\tilde y)\right)$, $v_{01} = \partial_0 v_1 - \partial_1 v_0$ (i.e. curvature of v)

$\tilde y^\pm = x^\pm \mp i\theta^\pm \bar\theta^\pm$

Now, the supersymmetric Guage-invariant Lagrangian is

$$L = \int d^4\theta \Big(\underbrace{\bar{\Phi} e^V \Phi}_{L_{kin}} - \frac{1}{2e^2} \underbrace{\bar{\Sigma}\Sigma}_{L_{guage}} \Big) + \frac{-t}{2} \underbrace{\int d^2\tilde{\theta}\, \Sigma}_{\text{twisted-Chiral F-term}} + c.c.$$

e : coupling constant , $t = r - i\theta$ theta angle

Féjer-Iliopoulos parameter (or FI parameter)

Under (0.2) change to Σ, get $U(1)_V \times U(1)_A$ symmetric for classical system.

As before, eliminating F and D from equation of motion, get

$$\begin{aligned}
(15.40)\quad L = & - D^\mu\bar{\phi}D_\mu\phi + i\bar{\psi}_-(D_0+D_1)\psi_- + i\bar{\psi}_+(D_0-D_1)\psi_+ \\
& - \frac{e^2}{2}\left(|\phi|^2 - r\right)^2 - |\sigma|^2|\phi|^2 - \bar{\psi}_-\sigma\psi_+ - \bar{\psi}_+\bar{\sigma}\psi_- \\
& - i\bar{\phi}\lambda_-\psi_+ + i\bar{\phi}\lambda_+\psi_- + i\bar{\psi}_+\bar{\lambda}_-\phi - i\bar{\psi}_-\bar{\lambda}_+\phi \\
& + \frac{1}{2e^2}\left(-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- + i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ + v_{01}^2\right) \\
& + \theta v_{01}.
\end{aligned}$$

In general, for $\Phi_1 \dots \Phi_N$, under $U(1)^k = \prod_{a=1}^{k} U(1)_a$: $\Phi_i \longmapsto e^{i\sum_{a=1}^{k} Q_{ia}A_a}\,\Phi_i$.

Get

$$\begin{aligned}
(15.42)\quad L = & \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i\, e^{Q_{ia}V_a}\Phi_i - \sum_{a,b=1}^{k} \frac{1}{2e^2_{a,b}}\bar{\Sigma}_a\Sigma_b \right) \\
& + \frac{1}{2}\left(\int d^2\tilde{\theta} \sum_{a=1}^{k}(-t_a\Sigma_a) \ + c.c. \right),
\end{aligned}$$

If $\exists\, W(\Phi_i)$ polynomial, guage-invariant, then we can add the F-term

$$L_W = \int d^2\theta\, W(\Phi_i) + c.c.$$

Eliminating D_a, F_a, get

$$\begin{aligned}
U \ = & \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2\,|\phi_i|^2 + \sum_{a,b=1}^{k} \frac{(e^{a,b})^2}{2}\left(Q_{ia}|\phi_i|^2 - r_a\right)\left(Q_{jb}|\phi_j|^2 - r_b\right) \\
(15.44)\quad & + \sum_{i=1}^{k} \left|\frac{\partial W}{\partial\phi_i}\right|^2,
\end{aligned}$$

where $(e^{a,b})^2$ is the inverse matrix of $1/e^2_{a,b}$ and the summations over a and i, j are implicit.

$b_a := \sum_{i=1}^{N} Q_{ia}$: coefficient of $\log \frac{\Lambda_{UV}}{\mu}$

$= 0 \Rightarrow$ anomaly free !

2021.11.22

Classical + supersymmetry $\rightarrow$ Quantum theorem

"vacuum"

$<$ path integral via spectum decomposition
lattice model

Physics : "renormalization"

Mathematics : Consider twisted model so that it is well-defined.

Gauged linear σ-model.

$\Phi_1, \dots, \Phi_N$: Chiral superfield $U(1)^k$: $\Phi_i \longmapsto e^{i\sum_{i=1}^{k} Q_{ia}A_a} \Phi_i$

$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{Q_{ia}V_a} \Phi_i - \sum_{a,b=1}^{k} \frac{1}{2e_{ab}^2} \bar{\Sigma}_a \Sigma_b \right) + \frac{1}{2}\underbrace{\left(\int d^2\tilde{\theta} \sum_{a=1}^{k} -t_a \Sigma_a + c.c. \right)}_{\text{twisted F-term}} + \int d^2\theta\, \underline{W(\Phi_i)} + c.c.$$

if $\exists$ gauge-invariant polynomial W

V, $\Sigma := \bar{D}_+ D_- V$: twisted-Chiral superfield

$t_a := \underline{r_a} - i\,\underline{\theta_a}$. theta angle
FI-parameter

Elimating D_a and F_i, get potential term

$$U = \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2 |\phi_i|^2 + \sum_{a,b=1}^{k} \frac{(e^{a,b})^2}{2} (Q_{ia}|\phi_i|^2 - r_a)(Q_{jb}|\phi_j|^2 - r_b)$$

$$(15.44) \qquad + \sum_{i=1}^{k} \left| \frac{\partial W}{\partial \phi_i} \right|^2,$$

where $(e^{a,b})^2$ is the inverse matrix of $1/e^2_{a,b}$ and the summations over a and i, j are implicit.

Quantum theory : Consider special case $k=1$, $N=1$, charge $Q_{ia}=1$.
effective theory at scale μ, i.e. integral over $\mu \le |k| \le \Lambda_{UV}$.

The part on L related to D field before substitude equation of motion

$$\frac{1}{2e^2} D^2 + D(\underline{|\phi|^2} - \underline{r_0})$$

replace $|\phi|^2$ by $\langle |\phi|^2 \rangle$.

FI-parameter

$$\int D\phi \longmapsto \frac{1}{2e^2} D^2 + D\left(\underbrace{\log \frac{\Lambda_{UV}}{\mu} - r_0}_{\overset{!!}{-r}} \right)$$

For Λ_{UV}, r_0 rearranged suitably, can get $r(\mu) = \log \frac{\mu}{\Lambda}$

Anomaly of $U(1)_A$: the system is broken due to

$$-2i\overline{\psi}_- D_z \psi_- + 2i\,\overline{\psi}_+ D_{\bar{z}} \psi_+$$

$$k := \text{"}\psi_-\ \text{zero"} - \text{"}\overline{\psi}_-\ \text{zero"} = c_1(E) \neq 0 .$$

$$\left.\begin{array}{l} \text{So } \ D\psi D\overline{\psi} \longmapsto e^{-2ki\alpha}\, D\psi D\overline{\psi} . \\ \dfrac{i}{2\pi}\displaystyle\int \theta\, v_{12}\, dx^1 dx^2 = ik\theta \end{array}\right\} \Rightarrow \text{equivalent to } \ \theta \longmapsto \theta - 2\alpha$$

General case : $b_a = \sum_{i=1}^{N} Q_{ia}$, $r_a(\mu) = b_a \log \dfrac{\mu}{\Lambda} + \tilde{r}_a$.

$$\theta_a \longmapsto \theta_a - 2b_a\alpha$$

If $b_a = 0$ for all a, then $U(1)_A$ is anomaly-free.

& $t_a = r_a - i\theta_a$ and (FI-theta) parameters of the quantum theory.

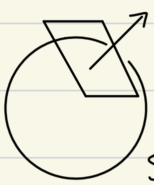
<u>Non-linear σ-model from Gauge linear σ-model</u>

(1) $\mathbb{CP}^{N-1}$. This is the case $U(1)^{k=1}$, $N = N$, no W.

$$U = \sum_{i=1}^{N} |\sigma|^2 |\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N} |\phi_i|^2 - r\right)^2 .$$

Only for $r > 0$, we have classical supersymmetry vacuum $\simeq S^{2N-1}/U(1) = \mathbb{CP}^{N-1}$.

$\sigma = 0$, ϕ_i : constant.



$$\text{transverse mass}^2 = \frac{1}{2}\frac{\partial^2}{\partial \rho^2} U(\rho)\Big|_{\rho = \sqrt{r}} = e^2 \cdot 2r .$$

$$\text{mass} = e\sqrt{2r}$$

S^{2N-1} Tangent of vacuum manifold is massless.

<u>Fact</u> Gauge fields also has mass $e\sqrt{2r}$.

v_μ (Higgs mechanism)

If $\psi_\pm^i, \overline{\psi}_\pm^i$ satisfy $\sum_{i=1}^N \overline{\phi}^i \psi_\pm^i = 0 = \sum_{i=1}^N \overline{\psi}_\pm^i \phi_i$ (i.e. tangent of $\mathbb{CP}^{N-1}$ at (ϕ_i))

then it has mass $= 0$.

Other modes and $\lambda_\pm, \overline{\lambda}_\pm$ (Fermion in V) has also mass $e \cdot \sqrt{2r}$.

Now let $e \longrightarrow \infty$, the system decouple. Classical theory is reduced to massless mode only

Claim: this is the non-linear σ-model to $\mathbb{CP}^{N-1}$.

(1) Classical: a direct check on L e.g. $ds^2 = \frac{r}{2\pi} g^{FS}$ with $B = \frac{Q}{2\pi} g_{FS}$.

need $r > 0$.

(2) Quantum level: The effective theory of massless mode is by integrate out all massive mode $M = e\sqrt{2r}$ & massless mode $\mu < |k| < \Lambda_{UV}$

(if $\mu << e\sqrt{2r}$.)

From $r(\mu) = (\underbrace{\sum_{i=1}^N \underbrace{Q_i}_{=1}}_{=N}) \log \frac{\mu}{\Lambda} + 0 \Rightarrow$ " $r = r' + N \log \frac{\mu}{\mu'}$ " $(*)$

This is from Gauge linear σ-model.

Recall the RG flow for metric in NLSM $\tilde{g}_{ij} = g_{ij} + \frac{1}{2\pi} \log \frac{\Lambda_{UV}}{\mu} R_{ij}$

apply this to $(r, \mu), (r', \mu')$.

Now, R_{ij} for $\frac{r}{2\pi} g^{FS}$ is independent of r, $= N g_{ij}^{FS}$.

So $\underbrace{g_{ij}'}_{\text{at scale } r'} = \frac{1}{2\pi} \left(r - N \log \frac{\mu}{\mu'} \right) g_{ij}^{FS}$, this agrees with $(*)$.

Also, $[\omega - iB] = \left[\frac{r - i\theta}{2\pi} \omega^{FS} \right] = \frac{t}{2\pi} [\omega^{FS}]$

i.e. the complexified Kähler class is a twisted-Chiral parameter t.

(2) Toric manifolds : $U(1)^k$, $\mathcal{N}=\mathcal{N}$, $e_{ab}^2 = \delta_{a,b}\, e_a^2$, no W.

The vacuum manifold $X_r = \left\{(\phi_1 \dots \phi_N) \,\middle|\, \underbrace{\sum_{i=1}^N Q_{ia}|\phi_i|^2 - r_a}_{\mu_a} = 0,\ a=1,2\dots k \right\} \Big/ U(1)^k$

As in (1), $e \longrightarrow \infty$ get NLSM on X_r.

$\omega_{\mathbb{C}^N}$ descends to a symplectic form on X_r, "ω".

complex structure : $X_r \simeq X_{\beta} = (\mathbb{C}^N \setminus \beta) \big/ (\mathbb{C}^\times)^k$: GIT quotient,

β = the set of points whose $(\mathbb{C}^\times)^k$ orbit has no solution in $\{\mu_a = 0\}$.

depends on $r = \{r_a\}$.

Recall in chapter 7, β. X_β can be constructed from a fan Σ.

Δ_Σ = convex hull of $\Sigma(1)$ $\rightsquigarrow$ $X_\Sigma = \bigcup_\sigma \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^N]$.

X_Σ is Fano ($c_1(X)>0$) $\Leftrightarrow$ Δ_Σ is reflective. (Thm 7.10.2 in textbook.)

e.g. $X = \mathbb{P}(Q_1,\dots,Q_N)$: weighted projective space.

$U(1)^{k=1}$, $Q_i \in \mathbb{N}$.

e.g. $\sum Q_i > 0$, but only $Q_1,\dots,Q_\ell > 0$

$$\Rightarrow X = \left[\bigoplus_{j=\ell+1}^N \mathcal{L}^{Q_j} \longrightarrow \mathbb{P}(Q_1,\dots,Q_\ell) \right]$$ for r large. ($r \ll 0$, reverse).

e.g. $\sum Q_i = 0$, the FI-parameter does not "run", all r are possible.

X is also a bundle space.

$\mathcal{O}(-N) \longrightarrow \mathbb{P}^{N-1}$ v.s. $\mathbb{C}^N/\mathbb{Z}_N$ (check it!) $\Phi_1,\dots,\Phi_N$, $Q_i = 1$, P, $Q_p = -N$.

$N|p|^2 = -r + \sum_{i=1}^N |\phi_i|^2$ $r \ll 0$

$\mathcal{O}(-1)\oplus\mathcal{O}(-1) \longrightarrow \mathbb{P}^1$ in two ways.

$\Phi_1, \Phi_2, \Phi_3, \Phi_4$, $Q_i = 1, 1, -1, -1$

M_{vac} : $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r$. $r > 0$

flop.

$r < 0$

$r=0 \longleftrightarrow xy = zw$

2021. 11. 25

Example ③ Hypersurfaces in $\mathbb{CP}^{N-1}$.

Let $W = \underbrace{G(\Phi_1, \dots, \Phi_N)}_{\text{homogeneous polynomial of } \deg = d} \cdot P$, where $\Phi_1, \dots, \Phi_N, P$ are Chiral superfield of charge $1, 1, \dots, 1, -d$.

This is Gauge-invariant

$$L = \int d^4\theta \left(\sum \bar{\Phi}_i e^V \Phi_i + \bar{P} e^{-dV} P - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) + \frac{1}{2} \int d^2\tilde{\theta} (-t\Sigma) + c.c.$$
$$+ \frac{1}{2} \int d^2\theta \, P G(\Phi_i) + c.c.$$

Potential term for scalar fields,

$$U = |\sigma|^2 \sum |\phi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2} \left(\sum |\phi_i|^2 - d|p| - r \right)^2 + \frac{1}{4} |G(\phi_i)|^2 + \frac{1}{4} \sum |p|^2 |\partial_i G|^2$$

Classical theory : 3 phases :

(i) $r > 0$: $U = 0 \Rightarrow \exists i$ s.t. $\phi_i \neq 0 \Rightarrow \sigma = 0 \Rightarrow p = 0$ (otherwise $G = 0 = \partial_i G$ $\Rightarrow \phi_i = 0$ for all i)

i.e. $\{\sum |\phi_i|^2 = r, \, G(\phi_i) = 0\} / U(1) \simeq M$, i.e. hypersurface defined by $G = 0$.

Some fields have mass $e\sqrt{r}$ or a_I (coefficient of W)

In a scaling s.t. $e, a_I \longrightarrow \infty$, then the system goes to non-linear σ-model on M. $[\omega - iB] = \frac{t}{2\pi} [\omega^{FS}] \Big|_M$.

(ii) $r < 0$: $U = 0 \Rightarrow p \neq 0 \Rightarrow \sigma = 0 \Rightarrow \phi_i = 0 \Rightarrow |p| = \sqrt{\frac{|r|}{d}}$, i.e. a circle.

$\Rightarrow$ vacuum = 1 point.

Let $\langle p \rangle := \sqrt{\frac{|r|}{d}}$ a vacuum value $\rightsquigarrow U(1)$ symmetry breaks to $\mathbb{Z}_d$.

$e \longrightarrow \infty$, we get LG theory with $W = \langle p \rangle G(\Phi_1, \dots, \Phi_N)$ with $\mathbb{Z}_d$ symmetry i.e. LG = orbifold.

(iii) $r = 0$: $U = 0 \Rightarrow \sum |\phi_i|^2 = d|p|^2 \Rightarrow p = 0$ (otherwise $\exists \phi_i \neq \phi_j$, contradict to $G = \partial_j G = 0$ for all $j \Rightarrow \phi_i = 0$ for all i)

$\Rightarrow$ complex σ-plane.

Quantum theory:

Renormalization of FI parameter r. "$N-d$"

- $d < N$: $r(\mu) = \underbrace{(N-d)}_{\sum Q_i > 0} \log \frac{\mu}{\Lambda}$

  As in $\mathbb{P}^{N-1}$ case, the system $\longmapsto$ non-linear σ-model on M.

  $c_1(M) = (N-d) H|_M$

- $d = N$: the theory is parameterized by $t = r - i\theta$, $c_1(M) = 0$.

  $r \gg 0 \longmapsto$ CY non-linear σ-model, t parameterized the complex Kähler class.

  $r \ll 0 \longmapsto e\sqrt{|r|} \to \infty \longmapsto$ LG orbifold by the massive as in (ii).

  $r = 0$

LG orbifold	$r=0$	CY σ-model
$r \ll 0$	?	$r \gg 0$

  "Low energy dynamics"

- $d > N$: M is of general type.

  Conjecture : LG orbifold for $\mathbb{Z}_{2(d-N)}$.

Witten, phases in $N=2$ theories.

Two more concepts from TFT:

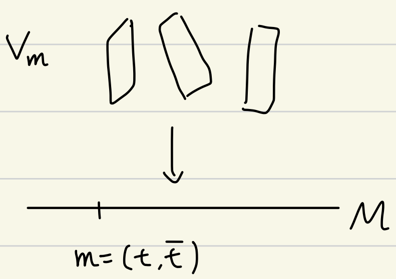
Variations of vacuum bundle & tt^* equations (Cecotti-Vafa 1992?)

Vacuum bundle vacuum $\equiv$ ground state (in QFT)

$\mathcal{H} \supseteq V_m = \ker Q \cap \ker Q^\dagger$ depends on chiral superfields holomorphically.

Fixed Hilbert space

$\longrightarrow$ moduli $\mathcal{M}$, $m \in \mathcal{M}$



Pick any Chiral superfield ϕ_i , $\underline{\phi_i|0\rangle} = |i\rangle$.

exist by axiom of QFT.

e.g. "1" in σ-model case

Pick a basis $\{\phi_i\}$ $\Rightarrow$ $\phi_i \underbrace{|j\rangle}_{\phi_j|0\rangle} = \underbrace{C_{ij}^k}_{\text{Chiral ring coefficient}} |k\rangle$

up to Q-deformation of operators and states.

<u>Exercise</u> See Textbook, the symbol is independent on choice of $|0\rangle$ or basis.

(Levi-Civita)

Connection A : Let $|j\rangle$: orthonormal basis of V_m.

$$\partial_i = \frac{\partial}{\partial t_i} \left(= \frac{\partial}{\partial m_i} \right)$$

$$(A_i)_j^k = \langle k| \partial_i |j\rangle , \quad \eta_{ij} = \langle i|j\rangle$$

Topological dependence $\Rightarrow$ <u>Fact</u>: $(A_{\bar{i}})_j^k = 0$, $\partial_{\bar{k}} \eta_{ij} = 0$, $\partial_{\bar{\ell}} C_{\bar{i}j}^k = 0$.

<u>Theorem</u> (tt^*-equations)

Let $D_i = \partial_i - A_i$. Then the improved connection (with parameter α)

$$\begin{cases} \nabla^\alpha = D + \alpha C \\ \bar{\nabla}^\alpha = \bar{D} + \alpha^{-1} C \end{cases}$$ is flat.

<u>Remark</u> Chiral ring is only for 3 point (small)

tt^* is "big" and even more, say it gives the "real structure".

<u>BPS soliton in LG theory</u>

W: isolated critical points $\phi_1, \dots, \phi_N$, $N = \dim \underbrace{R}_{\mathbb{C}[\Phi_i]/(\partial_i W)}$

$$\underset{(t,x')}{\Sigma = T \times \mathbb{R}} \xrightarrow{(\phi^i)} M = \mathbb{C}^n \xrightarrow{W} \mathbb{C}$$

<u>Definition</u> Soliton = time independant solution $(\phi^i(x'))_{i=1}^n$ connecting $\underset{x'=-\infty}{\underline{\phi_a}} \mp \underset{x'=\infty}{\underline{\phi_b}}$

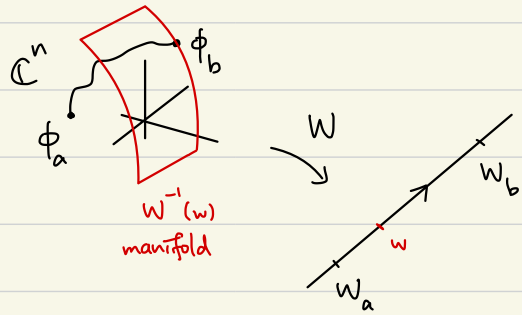
$$E_{ab} = \int_{-\infty}^{\infty} dx^1 \left(g_{i\bar{j}} \frac{d\phi^i}{dx^1} \frac{d\bar{\phi}^{\bar{j}}}{dx^1} + \frac{1}{4} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} \right)$$

$$= \int_{-\infty}^{\infty} dx^1 \left| \frac{d\phi^i}{dx^1} - \frac{\alpha}{2} g^{i\bar{j}} \partial_{\bar{j}} \bar{W} \right|^2 + \mathrm{Re}\, \bar{\alpha} \big(\underbrace{W(\phi_b)}_{W_b} - \underbrace{W(\phi_a)}_{W_a} \big) \quad \text{for any } |\alpha| = 1.$$

$$\geq |w(b) - w(a)|$$

BPS soliton := minimal , i.e. $\frac{d\phi^i}{dx^1} - \frac{\alpha}{2} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} = 0$, $\alpha = \frac{W_b - W_a}{|W_b - W_a|}$

$$\Rightarrow \partial_{x^1} W = \frac{\alpha}{2} \underbrace{g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W}}_{\in \mathbb{R}}$$



Q: Count # of BPS soliton.

# of soliton via invariant # of vanishing cycles Δ_a's.

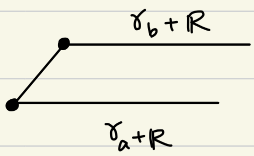
near ϕ_a, choose "Morse coordinate" such that $W(\phi) - W(\phi_a) = \sum_{i=1}^{n} (u_a^i)^2$

($\alpha = 1$) e.g. $\Rightarrow \begin{cases} \sum (\mathrm{Re}\, u_a^i)^2 = w - w_a \\ \mathrm{Im}\, u_a^i = 0. \end{cases}$ $\Delta_Q(w) = \Delta_a \circ \Delta_b$ at w. $= A_{ab} = \gamma_a \circ \gamma_b'$ in $\mathbb{C}^n$

We can do this for any homotopic curve:

Alternatively, use non-compact cycles γ_a

$W(\gamma_a) = I_a := W_a + \mathbb{R}^+$



then $\gamma_a \in H_n(\mathbb{C}^n, B)$ is a basis.

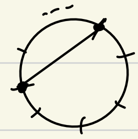
$B \subseteq \mathbb{C}^n$ s.t. $\mathrm{Re} W|_B \gg 0$

2021.11.29

tt^* equation, BPS soliton, D-branes

Example (1) $W(x) = \frac{1}{k+2} x^{k+2} - x$ ($N=2$ minimal model)

$W'(x) = x^{k+1} - 1 \rightsquigarrow$ critical points are $(k+1)$-root of unity.



Homework $\exists!$ soliton connecting each pair of critical points.

For general $W(x) = \frac{1}{k+2} x^{k+2} - x + $ (lower degree $\leq k+1$)

$\rightsquigarrow$ Q: How to determine the # of solitons?

Example (2) $M = \mathbb{CP}^{N-1}$ $\dashrightarrow$ some LG model

σ-model   mirror symmetry   over a non-compact CY.

(Later, we will see it is $(\mathbb{C}^\times)^{N-1}$: (non-compact CY)

with $W(x) = x_1 + \cdots + x_{N-1} + \frac{\lambda}{x_1 \cdots x_{N-1}}$ i.e. $x_1 \cdots x_N = \lambda$ in $\mathbb{C}^n$.)

For $N=2$, i.e. $\mathbb{C}^\times$, this is called sine-Gordon model.

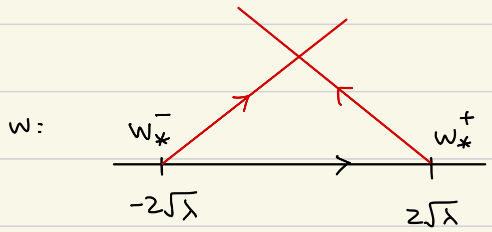
$W(x) = x + \frac{\lambda}{x}$, $W'(x) = 1 + \frac{\lambda}{x^2}$, $x_*^\pm = \pm\sqrt{\lambda}$, $w_*^\pm = \pm 2\sqrt{\lambda}$.

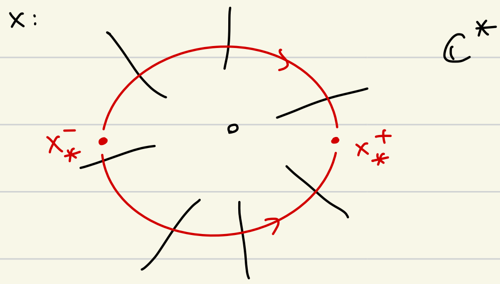
Need to solve $X(s) + \frac{\lambda}{X(s)} = 2\sqrt{\lambda}(2s-1)$. $s \in [0,1]$.

$\longrightarrow$ get 2 solutions : $X(s) = \sqrt{\lambda}\,\underline{(2s-1)} \pm 2\underline{i}\sqrt{\lambda}\,\underline{\sqrt{s(1-s)}} \in S^1_{\sqrt{\lambda}}$

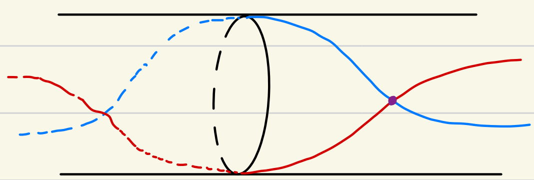
$(2s-1)^2 + 2^2 s(1-s) = 1$

$= \sqrt{\lambda}\, e^{\pm \tan^{-1} \frac{2\sqrt{s(1-s)}}{2s-1}}$





$\downarrow S$



For $N \geq 3$, similar calculations work, but need to guess an "ansatz solution of the soliton equation". (reading)

D-brane (Dirichlet membrane)

i.e. ∂-conditions

$$\Sigma = S^1 \times \mathbb{R} \xrightarrow{\phi} M = \mathbb{R}$$

Euclidean action $S = \int_{\text{part of } \Sigma} d^2x \, |d\phi|^2$.

$$\delta S = 0 \Rightarrow \Delta \phi = 0.$$

For general $(\Sigma, h) \xrightarrow{\phi} (M, g)$, get harmonic maps.

If $\partial\Sigma \neq \phi$, need boundary condition:

$$\delta\phi \cdot \underline{\partial_n \phi}\Big|_{\partial\Sigma} = 0.$$

normal derivative

Homework Do the general (M, g) case.

For $\Sigma = S^1 \times \mathbb{R} \ni (s, t)$ (check!)

Neumann condition: $\partial_n \phi \Big|_{\partial\Sigma} = 0. \longleftrightarrow *d\phi\Big|_{\partial\Sigma} = 0.$ $\quad *d = \partial - \bar\partial$

Dirichlet condition: $\delta\phi\Big|_{\partial\Sigma} = 0 \longleftrightarrow d\phi\Big|_{\partial\Sigma} = 0.$ $\quad d = \partial + \bar\partial$

In general, $N^p \hookrightarrow M$, get the notion of Dp-brane

with ∂-condition (N- or D-).

T-duality Let $M = S^1_R$. $R \xrightarrow{T} 1/R$. $\quad R\, d\varphi = \frac{i}{R} * B = i\left(\frac{1}{R}\right) * d\vartheta$

$\partial_t \varphi \longleftrightarrow \partial_s \vartheta$ exchanging winding #

$\partial_s \varphi \qquad\quad \partial_t \vartheta$ and momentum #.

We had seen $\partial_s \longleftrightarrow \partial_t$, $\partial = \frac{1}{2}(\partial_s - i\partial_t)$

$$\mapsto \frac{1}{2}(\partial_t - i\partial_s) = \frac{-i}{2}(\partial_s + i\partial_t) = -\bar\partial$$

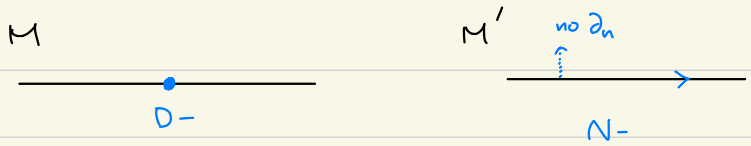
Similarly, $\bar\partial \longleftrightarrow \partial$.

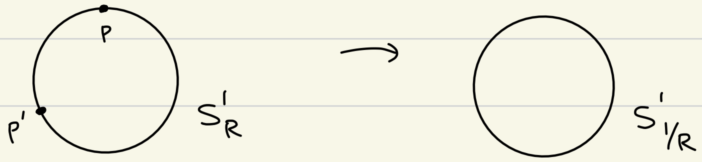
So $d \longmapsto *d$. $(*d \longrightarrow -d)$

i.e. N-, D- conditions are exchanged.

$\Rightarrow$ D0 brane $\xleftrightarrow{T}$ D1 brane.

(Fake example, odd dimension)

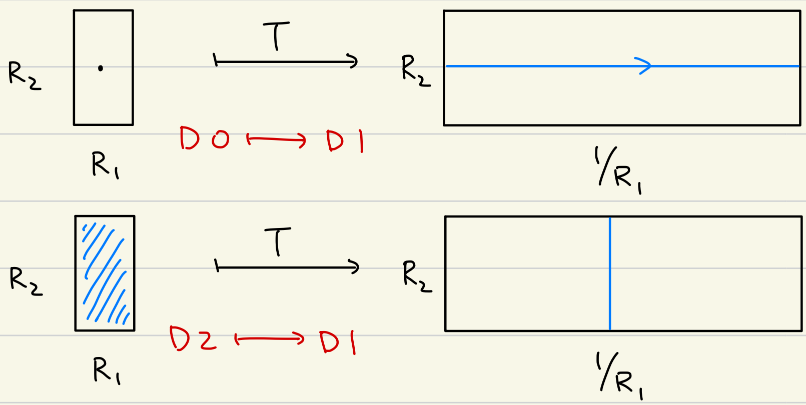




" + U(1) connection "

$H^1(S^1_{1/R}, \mathbb{R}/\mathbb{Z}) \simeq S^1$

first true
<u>Example</u> $M = S^1_{R_1} \times S^1_{R_2} \xrightarrow{T} M' = S^1_{1/R_1} \times S^1_{R_2}$.



but wrapped in $S^1_{R_1}$ instead

holomorphic submanifold $\longleftrightarrow$ symplectic structure $dx \wedge dy = 0 \Rightarrow x$ or y : constant.

<u>Homework</u> D-brane conditions preserve only half (2 of 4) supercharges.

$$Q_A = \overline{Q}_+ + Q_- \ , \quad \overline{Q}_A = \cdots$$

$$Q_B = \overline{Q}_+ + \overline{Q}_- \ , \quad \overline{Q}_B = \cdots$$

e.g. A-model $Q_A, \overline{Q}_A$ are preserved when $N \subseteq M$ is Lagrangian
B-model $Q_B, \overline{Q}_B$ holomorphic

($\Rightarrow$ other D-brane $\longleftrightarrow$ states $\longleftrightarrow$ deformations)

In fact, we had seen for CY σ-model. Mirror symmetry "predicts"

$$h^{p,q} \longleftrightarrow h^{d-p,q}$$

$$H^q(M, \Omega^p) \quad H^q(M, \wedge^p T_M).$$

<u>Supersymmetric cycles</u> submanifold + " U(1)-connection on it. "

<u>Remark</u> In $e^{2\pi i \int_\Sigma \phi^* B} \longmapsto \int_\Sigma \phi^*(B + d\Lambda) = \int_\Sigma \phi^* B + \underline{\int_{\partial\Sigma} \phi^* \Lambda}$

To remedy it, we add 1-form A connection on the D-brane.

$$S \curvearrowright S - 2\pi i \int_{\partial\Sigma} \phi^* A$$

$(B, A) \longmapsto (B + d\Lambda, A + \Lambda)$ is invariant.

(physics)

Hori-Vafa's proof of Mirror Symmetry (2000)

Step 1 T-duality on a charged field.

GLσM : $$L = \int d^4\theta \left(\bar{\Phi} e^{2QV} \Phi - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) + \frac{1}{2} \int d^2\tilde{\theta} \left((-t\Sigma) + c.c. \right)$$

$Q \in \mathbb{Z}$ $\Sigma = \bar{D}_+ D_- V$. $t = r - i\theta$

Vacuum manifold $\{\phi \in \mathbb{C} \mid |\phi|^2 - \underbrace{r}_{>0} = 0\} / U(1) = pt.$

Consider $$L_0 = \int d^4\theta \left(e^{2QV+B} - \frac{1}{2}(Y+\bar{Y})B \right)$$ s.t. $\operatorname{Im} Y$ is periodic in 2π.

B: Chiral superfield Y: twisted-Chiral superfield

If $\int DY \Rightarrow \delta Y$ takes the form $\bar{D}_+ D_- Z \Rightarrow \bar{D}_+ D_- B = 0 = D_+ \bar{D}_- B$

$\Downarrow$

$B = \Psi + \bar{\Psi}$ for some Chiral Ψ.

Get $L_1 = \int d^4\theta \, \bar{\Phi} e^{2QV} \Phi$, $\Phi = e^{\Psi}$.

If $\int DB \Rightarrow \int \left(e^{2QV+B} - \frac{1}{2}(Y+\bar{Y}) \right) \delta B = 0 \Rightarrow B = -2QV + \log \frac{Y+\bar{Y}}{2}$

$\int d^4\theta \underbrace{(Y+\bar{Y})}_{\text{real}} = 0$ get $$L_2 = \int d^4\theta \left(QV(Y+\bar{Y}) - \frac{Y+\bar{Y}}{2} \log(Y+\bar{Y}) \right)$$

But $\int d^4\theta \, VY = -\frac{1}{2} \int d\theta^+ d\theta^- \underbrace{\bar{D}_+ D_- V Y}_{\text{twisted-Chiral component}} = \frac{1}{2} \int d^2\tilde{\theta} \, \Sigma Y$

So L is T-dual to

$$\tilde{L} = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma}\Sigma - \frac{Y+\bar{Y}}{2} \log(Y+\bar{Y}) \right) + \frac{1}{2} \int d^2\tilde{\theta} \left(\Sigma(QY - t) + c.c. \right)$$

2021.12.2.

Hori-Vafa's physics proof of Mirror symmetry for those come from "GLσM"

Recall Step 1 T-duality on "a" charged field.

$$L = \int d^4\theta \left(\bar{\Phi} e^{2QV} \Phi - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) + \frac{1}{2} \int d^2\tilde{\theta} \left((-t\Sigma) + c.c. \right) \quad , \text{ where}$$

D-term F-term

$Q \in \mathbb{Z}$, $t = r - i\theta = -i(\theta + ir)$, $\Sigma = \bar{D}_+ D_- V$, V: real superfield.

r: Kähler, θ: B-field

Consider $L_0 = \int d^4\theta \left(e^{2QV+B} - \frac{1}{2}(Y+\bar{Y})B \right) \xrightarrow{\int DY}$ get $L_1 = \int d^4\theta\, \bar{\Phi} e^{2QV} \Phi$

$\xrightarrow{\int DB}$ get $L_2 = \int d^4\theta \left(QV(Y+\bar{Y}) - \frac{Y+\bar{Y}}{2} \log(Y+\bar{Y}) \right)$

Quantum theory: For $\delta = \epsilon_+ Q_- + \bar{\epsilon}_- \bar{Q}_+$, from B, we get

$$Y + \bar{Y} = 2\bar{\Phi} e^{2QV} \Phi$$

$y = \varrho - i\vartheta$, $\bar{X}_+$, X_-: Fermion

V: v, σ, λ, D (A_μ in the Hori-Vafa's paper)

Relations on component fields:

$$\begin{cases} \varrho = \rho^2 \\ \partial_\pm \vartheta = \pm 2(-\rho^2 \partial_\pm \varphi + Q A_\pm) + \bar{\psi}_\pm \psi_\pm \\ X_+ = 2\bar{\psi}_+ \phi \\ \bar{X}_- = -2\phi^\dagger \psi_- \end{cases}$$

($A_\pm$: v)

$d\varphi = * d\vartheta$.

Fermion variation $= 0 \Rightarrow \sigma = 0$, $D_{\bar{z}}\phi = 0$, $F_{12} = e^2(|\phi|^2 - r_0)$

Vortices (= instanton) $k := \frac{1}{2\pi} \int F_{12} d^2x \in \mathbb{Z}$ (topological number.)

Caution: metric on Y variable is given by $\frac{1}{4} \cdot \frac{|dy|^2}{\varrho}$

$\left(\Rightarrow \varrho > 0 \right.$, actually $\varrho = 0 \Rightarrow |\Phi| = 0$ i.e. $\rho = 0$, i.e. NO T-duality$\left.\right)$

RG $\Rightarrow Y_0 = \log \frac{\Lambda_{UV}}{\mu} + Y$.

$\mathrm{Re}\, y_0 \geq 0$ $\quad \Lambda_{UV} \to \infty$ $\quad \Rightarrow$ no constraint

$U(1)_A : e^{i\alpha}$ acts on this topological sector by $e^{2ik\alpha}$ $\Rightarrow$ only $k=1$ contributes!

twisted-potential has axial charge 2

$\chi_+ \bar{\chi}_-$

$\Rightarrow$ For $k=1$, $0 \neq \langle \chi_+(x) \bar{\chi}_-(y) \rangle$.

(heavy calculation! 10 pp.)

This can only come from $\int d^2\tilde{\theta}\, e^{-Y}$.

Reason: Since $\Delta \tilde{W}$ is holomorphic in t, periodic in theta angle ϑ, R-symmetry with suitable asymatotic behavior.

Dynamical generation of twisted superpotential via vertices.

$\Rightarrow$ So $\tilde{W} = \Sigma(QY - t) + e^{-Y}$.

#

<u>Step 2</u> The mirror of toric (Fano) varieties

$c_1 > 0$.

We do weighted projective spaces:

$$\begin{array}{c} \mathbb{C}^n \\ \downarrow \\ S^1 \times \mathbb{R} \xrightarrow{\phi} M = \mathbb{P}(Q_1, \ldots, Q_n) \end{array}$$

For $U(1)^{k=n}$, $N = n$, $\Phi_1, \ldots, \Phi_N$,

get dual effective superpotential $\tilde{W} = \sum_{i=1}^{n} \left(Q_i Y_i - t_i \right) \Sigma_i + e^{-Y_i}$.

Now, keep only the diagonal action, set $\frac{1}{e_{ab}^2} = 0$ in D-term variations.

This does not effect F-term, reduce to $U(1)^{k=1}$, $\Sigma_i = \Sigma$ for all i.

$t = \Sigma t_i$.

$$\tilde{W} = \left(\sum_{i=1}^{n} Q_i Y_i - t \right) \Sigma + \sum_{i=1}^{n} e^{-Y_i}.$$

Now, $\int D\Sigma$: i.e. $\partial_\Sigma \tilde{W} = 0$. $\Rightarrow \Sigma Q_i Y_i = t$ with potential $\tilde{W} = \sum_{i=1}^{n} e^{-Y_i}$.

The low energy limit is NLσM on $P(Q_1, \ldots, Q_n) \xleftrightarrow{\text{T-duality}}$ LG on variables Y_i.

Example $\mathbb{CP}^{n-1}$: $Q_i = 1$ for all i. Let $x_i = e^{-Y_i}$.

Then, $\widetilde{W} = x_1 + \cdots + x_n$ on $\prod_{i=1}^{n} x_i = e^{-t}$.

Recall t = FI-theta parameter = Kähler moduli of $\mathbb{CP}^{n-1}$.

Equivalently, $W(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-1} + \dfrac{e^{-t}}{x_1 \cdots x_{n-1}}$ on $\underbrace{(\mathbb{C}^\times)^{n-1}}_{CY}$.

$$\nabla \widetilde{W} = 0 \iff 1 - \frac{e^{-t}}{x_i (x_1 \cdots x_{n-1})} = 0 \quad \text{for all } i.$$

$$\iff x_i = \underbrace{\omega}_{\omega^n = 1} \cdot e^{-t/n} \quad \text{for all } i$$

$\#$ critical points $= n$

$\longleftrightarrow$ cohomology basis of $H^*(\mathbb{CP}^{n-1})$

identity $\quad H \in H^2(\mathbb{CP}^{n-1}) \longleftrightarrow -\partial_t \widetilde{W} = \dfrac{e^{-t}}{x_1 \cdots x_{n-1}}$

$QH(\mathbb{CP}^{n-1}) \simeq$ twisted-Chiral ring

$H^n \longleftrightarrow \underbrace{\left(\dfrac{e^{-t}}{x_1 \cdots x_{n-1}}\right)^n}_{\hat{H}^n =} = e^{-t}$.

Step 3 The hypersurface (or complete intersection) case.

(*) Consider GLσM on non-compact toric variety.

$n+2$ Chiral superfield $(p, \Phi_1, \ldots, \Phi_{n+1})$ charge $= (-d, 1, 1, \ldots, 1)$.

low energy limit $\longmapsto$ NLσM on $\left[\begin{array}{c} \mathcal{O}(-d) \\ \downarrow \\ \mathbb{CP}^n \end{array}\right]$

As before, we set $x_0 = e^{-P}$ $\qquad \widetilde{W} = x_0 + \cdots + x_{n+1}$

$x_i = e^{-Y_i}$ $(i = 1, 2, \ldots, n+1)$ $\qquad$ with $x_0^{-d} x_1 \cdots x_{n+1} = e^{-t}$.

Redefine $\tilde{x}_i = x_i^{1/d}$, $1 \le i \le n+1$ $\Rightarrow x_0 = e^{t/d} \tilde{x}_1 \cdots \tilde{x}_{n+1}$. (on $\mathbb{C}^{n+1}$)

So $\widetilde{W} = \tilde{x}_1^d + \cdots + \tilde{x}_{n+1}^d + e^{t/d} \tilde{x}_1 \cdots \tilde{x}_{n+1}$ with orbifold structure by $\mathbb{Z}_d^n \hookrightarrow \mathbb{Z}_d^{n+1}$

preserving $\tilde{x}_1, \ldots, \tilde{x}_{n+1}$.

Example For $d = n+1$, $\begin{bmatrix} \mathcal{O}(-(n+1)) \\ \downarrow \\ \mathbb{CP}^n \end{bmatrix} \xleftrightarrow{\text{T-duality}}$ LG with homogeneous $\widetilde{W}$

is considered as a non-compact CY
(local CY)

(Later ...)

To get compact hypersurface, we need to polentical

$W = p \cdot G_d(\Phi_i)$ with the same d viewed as a perterbation term of $(*)$.
d

Key point: For A-twisted theory (Q-coh), $Q_A = \overline{Q}_+ + Q_-$ dependent only on twisted Chiral ϕ but not on the W-variations.

(twisted F-term)

But the low energy limit in the NLσM on M. $M = \{G_d = 0\}$
$C_1(M) = n+1-d \geq 0$, i.e. $n+1 \geq d$.

$\rightsquigarrow \begin{matrix} \mathcal{O}(-d) \\ \downarrow \\ \mathbb{CP}^n \end{matrix} \underset{\text{for A-model}}{\simeq} M = (d) \subseteq \mathbb{CP}^n.$

$n+1 > d \xleftrightarrow{\text{T-duality}} \widetilde{W}$ inhomogeneous.

Greene - Plasser (1986) (Orbifold construction)

For $d = n+1$, i.e. $M =$ CY hypersurface $\subseteq \mathbb{CP}^n$.
(any) A-model

$\xleftrightarrow{\text{T-dual}}$ homogeneous $\widetilde{W} = X_1^{n+1} + \cdots + X_{n+1}^{n+1} + e^{-t/d}\, \widetilde{X}_1 \cdots \widetilde{X}_{n+1}$ on $\mathbb{C}^{n+1}$.

LG orbifold
$\mathbb{Z}_{n+1}^n \hookrightarrow \mathbb{Z}_{n+1}^{n+1}$

B-model

But it is clear $\widetilde{W} = 0$ defines a compact CY hypersurface in $\mathbb{P}^n$ as well!
W'

Consider $W'' = W'/G$ but this has quotient singularity
The real W is the CY resolution of W''.
mirror of M (Crepant)

$\begin{matrix} W \quad \text{CY?} \\ \downarrow \\ W'' = W'/G \end{matrix}$

LG - periods (BPS mass) on the mirror side (T-dual)

LG B-model : $\int_{\underset{=}{\gamma}} d\tilde{Y}_1 \cdots d\tilde{Y}_{n+1}\, e^{-\tilde{W}} = \int_\gamma \frac{d\tilde{x}_1}{\tilde{x}_1} \cdots \frac{d\tilde{x}_{n+1}}{\tilde{x}_{n+1}}\, e^{-\tilde{W}}$

non-compact cycle $H_{n+1}(\mathbb{C}^{n+1}, \underset{=}{B})$

$\mathrm{Re}\, W \to \infty$

$\tilde{W} = \tilde{x}_1^d + \cdots + \tilde{x}_{n+1}^d + e^{-t/d}\, \tilde{x}_1 \cdots \tilde{x}_{n+1}$

$$\Big\downarrow -\partial_t$$

$$\int_\gamma \frac{e^{td}}{d}\, d\tilde{x}_1 \cdots d\tilde{x}_{n+1}\, e^{\underline{-\tilde{W}}}$$ ≈ 0 for $d = n+1$

$$\rightsquigarrow \int_{\gamma_c} \Omega .$$

2021.12.6.

Reference: Fulton - Pandharipande: Notes on Stable maps.

Week 1: Gromov - Witten theory / Q-cohomology

Week 2: Virtual Localization

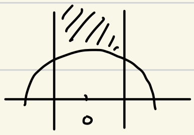
Week 3: Proof of classical mirror symmetry.

Moduli M_g: Riemann surface of geneus $g \geq 2$.

$\dim M_g = 3g-3$

$M_0 = pt$ (only $\mathbb{P}^1$)

$M_{1,1}$: (1 = genus, 1 = # of marked point)



$\overline{M}_{g,n}$: Dilgne - Mumford's moduli of stable curves

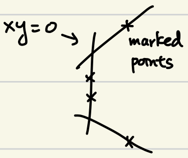
$(C, p_1 \cdots p_n)$ C = nodal curves / $\mathbb{C}$ s.t. $|\mathrm{Aut}\, C| < \infty$

$= \bigcup C_i$

$p_1 \cdots p_n \in C_{sm}$ (smooth points)

$g(C_i) = 0 \Rightarrow$ contains ≥ 3 special points ($=$ nodes $\cup$ marked points)

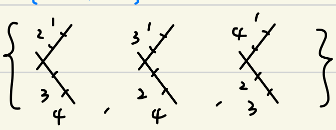
$g(C_i) = 1 \Rightarrow 1$ special point.



Examples $M_{0,3}$ = point

$M_{0,4} = \mathbb{P}^1 \setminus \underbrace{3 \text{ points}}_{= \{0,1,\infty\}}$

$\Rightarrow \overline{M}_{0,4} = \mathbb{P}^1 \cup \{$



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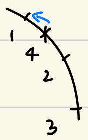
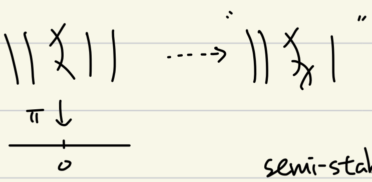


blow up at point 1

In general, $\overline{M}_{g,n}$ is compact (proper) due to Mumford's semi-stable reduction theorem.

$D = \pi^{-1}(0) = \sum n_i C_i$



How to make all $n_i = 1$?

answer: base-change & blow up

semi-stable $\longrightarrow$ stable

$|\mathrm{Aut}\, D| < \infty \Leftrightarrow \pi$ - ample (always OK by MMP for π.)

$(\overline{M}_g)$, $\overline{M}_{g,n}$ is not a scheme in general, it is only a stack.

D-M stack ←·····→ orbifold

(algebraic geometry) (top differential geometry)

i.e. locally a finite quotient of schemes.

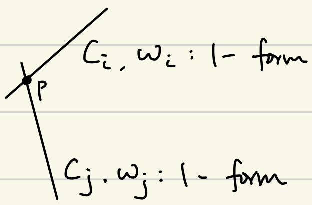
X/G, we say that M is smooth if X is non-singular on all charts.

Construction ⊛ $\mathcal{L}$: ample ⇔ stable curve

$(C, \omega_C(p_1 + \cdots + p_n))$ stable curve

dualizing sheaf

ω_C

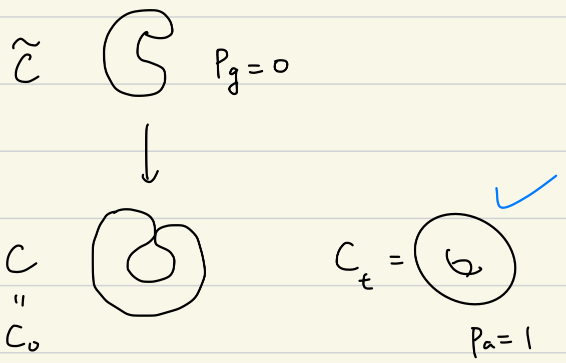


ω_C : sheaf of meromorphic 1 - form with simple pole at p s.t. $\mathrm{Res}_p\, \omega_i + \mathrm{Res}_p\, \omega_j = 0$.

$\exists\, f \in \mathbb{N}$ s.t. $\mathcal{L}^{\otimes f}$ is very ample, $h^1 = 0$.

$$h^0(C, \mathcal{L}^{\otimes f}) = \chi(\mathcal{L}^{\otimes f}) = f((2p_a(C) - 2) + n) + (1 - p_a(C)) \qquad g := p_a(C).$$
$$= (2f-1)(g-1) + fn.$$

Here, $g = p_a(C)$ is the arithmetic genus



$\rightsquigarrow$ embedding $\Phi : C \hookrightarrow \mathbb{P}(\underbrace{H^0(\mathcal{L}^{\otimes f})}_{\cong \mathbb{C}^N =: W}) = \mathbb{P}^{N-1}$

Let $P(m) = mf(2g-2+n) + (1-g)$ be the Hilbert polynomial.

The case without marked points, we have $H = \mathrm{Hilb}_{\mathbb{P}^{N-1}}(P)$. (Grothendieck : fine moduli space.)

parameterize "all" schemes ↗ $\circlearrowleft$ $PGL(N)$

not only "stabe" or "semi-stable". linear algebraic group. $\mathrm{Hilb}(P)$ is projective

$\rightsquigarrow$ take a quasi-projective $H^\circ \subseteq H$.

$\rightsquigarrow$ Naively: $H^\circ / G \overset{?}{=} (H^\circ / \tilde{G}) / \text{finite}$

Actually: Need Mumford's GIT. (Geometric invariant theory)

The case with marked points, each p_i determines a point in $\mathbb{P}(W)$.

So we look at $Z_1 \hookrightarrow Z \hookrightarrow H \times \mathbb{P}(W)^n$ incidence subscheme.

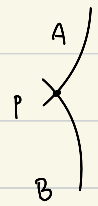
$\Rightarrow$ has good property, e.g. locally closed.

Finally, $\overline{M}_{g,n} = Z_1 /\!/ G$. (In fact, it is projective.)

course moduli space $\equiv$ forget all the finite stabilizer.

$$\mathcal{U}_{g,n} \to \overline{M}_{g,n}$$

∂-strata? this is the most important source of subvarieties



$\{1,2,3,\dots,n\} = A \amalg B$

$D(g_1, A \mid g_2, B)$ ∂-divisor

$\overline{M}_{g_1, |A|+1} \times \overline{M}_{g_2, |B|+1}$.

$\overline{M}_{g,n}(X, \beta)$ Kontsevich's space of stable maps. i.e. $|\mathrm{Aut}\, \mu| < \infty$.

$\quad \Psi$ $\qquad X$: projective manifold $/\mathbb{C}$, $\beta \in H^2(X, \mathbb{Z})$

$(C, \{p_i\}, \mu)$ $\qquad NE(X)$

$p_a(C) = g$ $\qquad \mu : C \longrightarrow X$ If $\mu(C_i) = pt$, then $\begin{cases} g(C_i) = 0 \Rightarrow 3 \text{ special point} \\ g(C_i) = 1 \Rightarrow 1 \text{ special point} \end{cases}$

$C = \bigcup C_i$ nodal $\qquad$ s.t. $\mu_*[C] = \beta$.

If $\mu(C_i)$ is not just a point, then no conditions.

stability $\Leftrightarrow$ $|\mathrm{Aut}| < \infty$

$\Leftrightarrow$ $\mathcal{L} := \omega_C(\Sigma p_i) \otimes \mu^* \mathcal{O}_{\mathbb{P}^r}(3)$ is ample.

if $X = \mathbb{P}^r$, $\beta = d$.

e.g. $r=0$, $\overline{M}_{g,n}(\mathbb{P}^0, d) = \overline{M}_{g,n}$

$\uparrow$ no effect.

$d=0$, $\overline{M}_{g,n}(\mathbb{P}^r, 0) = \overline{M}_{g,n} \times \mathbb{P}^r$

$\overline{M}_{0,0}(\mathbb{P}^1, 1) = \text{point}$

We will construct $\overline{M}_{g,n}(\mathbb{P}^r, d)$ for all other (g,n,r,d).

Theorem 1 $\exists$ coarse moduli space $\overline{M}_{g,n}(X,\beta)$ as a scheme.

Theorem 2 If X smooth projective convex (i.e. $\forall\ \underset{\mathbb{P}^1}{\overset{\|}{C}} \xrightarrow{f} X$, $H^1(C, f^*T_X)=0$),

then $\overline{M}_{0,n}(X,\beta)$ is normal of pure dimension $= c_1(X).\beta + \dim X\cdot(1-g) + n-3$.

called the expected, or virtual dim.

Moreover, $\overline{M}^*_{0,n}(X,\beta) =$ auto-free stable maps is a fine moduli space.

(with universal family)

Theorem 3 If X is smooth projective convex, then the boundary of $\overline{M}_{0,n}(X,\beta)$ is a NCD up to a finite quotient

Coarse moduli space is also useful (good enough) e.g. evaluation maps:

$\theta_i : \underline{\mathcal{M}_{g,n}}(X,\beta) \longrightarrow \mathrm{Hom}(*, X)$: natural transformation.

moduli functor

$$T \quad \left\{\begin{array}{c} S \\ \downarrow \\ T \end{array}\right\} \rightsquigarrow \begin{tikzcd} p^*\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ T \arrow[r, "p"'] & \overline{M} \end{tikzcd}$$

induces a unique morphism $ev_i : \overline{M}_{g,n}(X,\beta) \longrightarrow X$.

$$(C, \{p_i\}, \mu) \longmapsto \mu(C)$$

idea of proof of thm 1: (rigidification)

(construction) Let $\mathbb{P}^r = \mathbb{P}(V)$, $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = V^*$, $\bar{t} = (t_1, \dots, t_r)$ basis of V^*.

A $\bar{t}$-rigid stable family of (g, d, n) stables maps to $\mathbb{P}^r$ is a collection of data:

$$\left(\underset{A}{\begin{array}{c} C \\ \pi\downarrow \\ S \end{array}}, \underset{B}{\{p_i\}_{i=1}^{n}}, \underset{C}{\{q_{ij}\}_{i=0,\ j=1}^{r\quad d}}, \underset{D}{\mu} \right) \text{ s.t.}$$

(i) $A+B+D$ is a stable family of maps to $\mathbb{P}^r$

(ii) $A+B+C$ is a family of Deligne-Mumford stable curves with $n+d(r+1)$ points.

$\{p_i\}$, $\{q_{ij}\}$ are marked sections. $(\Rightarrow q_{ij} \neq p_i)$

(iii) For $0 \le i \le r$, $\mu^* t_i = q_{i1} + \cdots + q_{id}$ as Cartier divisor on C.

The new sub functor $\overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ is simpler.

($g=0 \Rightarrow$ representable (with universal family) as a quasi-projective variety)

Reason:

$$\begin{array}{ccc} C & \Rightarrow & \overline{U}_{g,m} \qquad m = n+d(r+1) \\ \downarrow & & \pi\downarrow \quad \text{universal family} \\ S & & S \longrightarrow \overline{M}_{g,m} \qquad \mathcal{H}_i = \mathcal{O}_{\overline{U}_{0,m}}(q_{i1} + \cdots + q_{id}) \end{array}$$

Let image $\subseteq B$: locally closed subscheme (Zariski open for $g=0$).

$(S \to B, \mu^* t_i)_{i=0\dots r}$

Apply "Theorem of the cube II" on equality of $\mathcal{H}_i$'s relative to π

$\Rightarrow$ the map μ is determined only by $\mathrm{div}(\mu^* t_i)$ for all i.

The $\bar{t}$-rigid moduli space = total space of these $\underset{(r+1)-1}{r}$-distinct $\mathbb{C}^*$ bundles over B.

When $g=0$, $\overline{M}_{0,m}$ is fine and B is Zariski open.

For $g>0$, B can be distinctized by universal conditions, called $\mathcal{H}$-balanced condition.

$\overline{M}_{g,n}(\mathbb{P}^r, d)$ is obtained by gluing $\overline{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ for varies $\bar{t}$.

$\overline{M}_{g,n}(X, \beta) \hookrightarrow \overline{M}_{g,n}(\mathbb{P}^r, d)$ if $X \hookrightarrow \mathbb{P}^r$, $i_* \beta = d\ell$.

2021. 12. 9.

Hilbert scheme $\rightsquigarrow \overline{M}_{g,n} \rightsquigarrow \overline{M}_{g,n}(\mathbb{P}^r, d) \overset{?}{\rightsquigarrow} \overline{M}_{g,n}(X,\beta)$

??? OK for $X \hookrightarrow \mathbb{P}^r$
$\overset{\shortparallel}{(d)}$: degree d hypersurface.

<u>Definition</u> X : convex if $H^1(C, f^*T_X) = 0$ for all genus 0 stable maps (C, f).

<u>Tangent - Obstruction complex (exact sequence)</u>

$\underset{\text{smooth}}{C} \xrightarrow{f} X \quad \rightsquigarrow 0 \to T_C \to f^*T_X \to N_f \to 0$

$\vec{p} = (p_1 \dots p_n) \quad \Rightarrow 0 \to \mathrm{Aut}(C, \vec{p}) \to \mathrm{Def}_{(C,\vec{p})}(f) \to \mathrm{Def}(C, \vec{p}, f)$ — $H^0(C, \cdot)$

$\to H^1(C, T_C(-\vec{p})) \to \mathrm{Ob}_{(C,\vec{p})}(f) \to \mathrm{Ob}(C, \vec{p}, f) \to 0$ — $H^1(C, \cdot)$

easy! we know how to compute. we want!

$(C, \vec{p}, f) \in \overline{M}_{g,n}(X, \beta)$

Zariski-tangent space = ? looking at $\mathrm{Def}(C, \vec{p}, f)$. Singularity = ? looking at $\mathrm{Ob}(C, \vec{p}, f)$

In general,

REMARK 24.4.1. The terms of the deformation long exact sequence can be defined cohomologically as follows:

$$\begin{aligned}
\mathrm{Aut}(\Sigma, p_1, \dots, p_n) &= \mathrm{Hom}(\Omega_C(p_1 + \dots + p_m), \mathcal{O}_C), \\
\mathrm{Def}(\Sigma, p_1, \dots, p_n) &= \mathrm{Ext}^1(\Omega_C(p_1 + \dots + p_m), \mathcal{O}_C), \\
\mathrm{Aut}(\Sigma, p_1, \dots, p_n, f) &= \mathbb{H}\mathrm{om}(f^*\Omega_X \to \Omega_C(p_1 + \dots + p_n), \mathcal{O}_C), \\
\mathrm{Def}(\Sigma, p_1, \dots, p_n, f) &= \mathbb{E}\mathrm{xt}^1(f^*\Omega_X \to \Omega_C(p_1 + \dots + p_n), \mathcal{O}_C), \\
\mathrm{Ob}(\Sigma, p_1, \dots, p_n, f) &= \mathbb{E}\mathrm{xt}^2(f^*\Omega_X \to \Omega_C(p_1 + \dots + p_n), \mathcal{O}_C),
\end{aligned}$$

where Ω is the sheaf of algebraic differentials, and $\mathbb{H}\mathrm{om}$ and $\mathbb{E}\mathrm{xt}$ are hypercohomology functors. We will avoid using these facts in our calculations.

<u>Big Theorem</u> $\exists$ functorial virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{virt}} \subseteq A_{\underbrace{D^{\mathrm{virt}}}_{\text{virtual dimension}}}(\overline{M}_{g,n}(X,\beta))$

$$\begin{aligned}
D^{\mathrm{virt}} &= \dim \mathrm{Def}(C, \vec{p}, f) - \dim \mathrm{Ob}(C, \vec{p}, f) \\
&= c_1(X).\beta + (\dim X - 3)(1-g) + n
\end{aligned}$$

above long exact sequation

- If $\overline{M}_{g,n}(X,\beta)$ is $\underbrace{\text{smooth}}_{\mathrm{Ob} \equiv 0}$, then $[\quad]^{\mathrm{virt}} = \overline{M}_{g,n}(X,\beta)$.

e.g. if $g = 0$, X is convex $\Rightarrow \mathrm{Ob} \equiv 0$.

- If Ob is of constant dimension for all $(C, \vec{p}, f)$, then get a vector bundle $\begin{array}{c} Ob \\ \downarrow \\ \overline{M}_{g,n}(X,\beta) \end{array}$

$\rightsquigarrow [\ \ ]^{vir} = e(Ob)$ (Euler class)

e.g. $(d) \subseteq \mathbb{P}^r$ can compute the virtual class as some Euler class.

Idea of proof of theorem 3:

∂ of $\overline{M}_{0,n}$ is clearly. $\overline{M}_{0,A\cup\{0\}} \times \overline{M}_{0,B\cup\{0\}}$

$A \amalg B = \{1,2,\dots n\}$

For $\overline{M}_{0,n}(X,\beta)$, we similarly have $D(A,\beta_1 | B,\beta_2) = \overline{M}_{0,A\cup\{0\}}(X,\beta_1) \underset{X}{\times} \overline{M}_{0,B\cup\{0\}}(X,\beta_2)$

(also for $g=g$, $g_1+g_2=g$, $\beta_1+\beta_2=\beta$) $= (e_A \times e_B)^{-1}(\Delta)$

e_A, e_B : evaluation $C \xrightarrow{f} X$, f at the last point.

Key point: $D(A,B)$'s form NCD of $\overline{M}_{0,n}$.

By the construction of $\overline{M}_{0,n}(X,\beta,\bar{t})$ which is locally Zariski open in $\overline{M}_{0,m} \times (\mathbb{C}^\times)^r$.

Hence, the ∂-divisor has the same behavior $\Rightarrow$ NCD up to finite quotient.

GW-invariant $g=0$ case

$M_{0,n}(X,\beta) \xrightarrow{e_i} X$, $\gamma_1,\dots,\gamma_n \in H^*(X)$, $\deg \gamma_i = 2i$, $\sum i =$ virtual dimension.

$\langle \gamma_1,\dots,\gamma_n \rangle_\beta := \int_{M_{0,n}(X,\beta)} \prod_{i=1}^n e_i^* \gamma_i$. (General g: $\langle \gamma_1 \dots \gamma_n \rangle_{g,\beta} = \int_{[M_{g,n}(X,\beta)]^{virt}} \prod_{i=1}^n e_i^* \gamma_i$.)

Fact: For X convex, $g=0$,

$\langle \gamma_1,\dots,\gamma_n \rangle_\beta = \#$ maps $f: \mathbb{P}^1 \to X$ s.t. $f(p_i) \in \underbrace{PD(\gamma_i)}_{\text{Poincaré dual}}$ for all i.

Facts: I. $\langle \gamma_1,\dots,\gamma_n \rangle_{\beta=0} = 0$ unless $n=3$

$\hookrightarrow$ since $M_{0,n}(X,0) = \underbrace{M_{0,n}}_{\text{dim} > 0 \text{ for } n>3.} \times X$

$\langle \gamma_1 \gamma_2 \gamma_3 \rangle_0 = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$.

II. $\langle 1, \gamma_2, \dots, \gamma_n \rangle_{\beta \neq 0} = \int_{M_{0,n}(X,\beta)} e_2^*\gamma_2 \cdots e_n^*\gamma_n \equiv 0$

$M_{0,n}(X,\beta) \searrow M_{0,n-1}(X,\beta)$ relative dim $= 1 > 0$

fundamental class axiom.

III. Divisor axiom : $\gamma_1 \in A^1(X)$

$$\langle \gamma_1, \dots, \gamma_n \rangle_\beta = \underline{(\gamma \cdot \beta)} \langle \gamma_2, \dots, \gamma_n \rangle_\beta$$

Poincaré pairing

proof: $\varphi : M_{0,n}(X,\beta) \xrightarrow[e_1]{} X \times M_{0,n-1}(X,\beta)$

$$\varphi_* M_{0,n}(X,\beta) = \beta_1 \times M_{0,n-1}(X,\beta) + \alpha \quad \Rightarrow \beta_1 = \beta.$$

Künneth

$$\int_{M_{0,n}(X,\beta)} e_1^*\gamma_1 \cup \cdots \cup e_n^*\gamma_n = \int_{\varphi_* M_{0,n}(X,\beta)} \gamma_1 \times (e_2^*\gamma_2 \cup \cdots \cup e_n^*\gamma_n)$$

#

Key idea : (Mumford)

cycle in $\overline{M}_{g,n} \Rightarrow$ relation of invariance

$\equiv$ differential equation (flow).

Algebraic cycle : $\begin{cases} \partial\text{-strata} & (\partial\text{-divisor}) \\ \psi \text{ class} \end{cases}$

$$\begin{array}{ccc} \mathcal{U} : \text{universal curve} & \equiv & \overline{M}_{g,n+1}(X,\beta) \\ \pi \downarrow & & \\ (C, p_1, \dots, p_n, f) \in & & \overline{M}_{g,n}(X,\beta) \end{array}$$

$\mathbb{L}_i := T^*_{p_i} C$ at p_i.

$\psi_i := c_1(T^*_{p_i} C)$. Alternatively, $\mathbb{L}_i := \underline{p_i^*} \, \underline{\omega_\pi}$

a section relative dualizing sheaf

The most important starting point in G-W theory is to understand relation between ψ_i and ∂-strata.

e.g. for $g=0$, all ψ classes are $\sim \partial$-divisors. $n \geq 3 \Rightarrow \psi_1 = D_{1|23}$ for any 1.2.3.

Comparison lemma $\psi_i - \pi^*\psi_i =$ some ∂-divisor.

Quantum Cohomology

$\underbrace{T_1, \ldots, T_p}_{} \in A^1(X) = H^2$

Basis of cohomology $T_0 = 1, T_1, \ldots, T_m \in A^*(X)$

$$g_{ij} = \int_X T_i \cup T_j \equiv (T_i, T_j).$$

$\rightsquigarrow$ dual basis $T^i := g^{ij} T_j$, $g^{ij} = (g_{ij})^{-1}$.

$\Rightarrow \Delta \subseteq X \times X$ has $[\Delta] = \sum_{e,f} g^{ef} T_e \otimes T_f = \sum T^i \otimes T_i$.

$\Rightarrow T_i \cup T_j = \sum_k \langle T_i, T_j, T_k \rangle_0 T^k = \sum_k \langle T_i, T_j, T^k \rangle T_k$

Then, we get small quantum product $T_i \overset{s}{*} T_j = \sum_{\substack{\beta \in A_1(X) \\ NE(X) \\ H_2(X,\mathbb{Z})}} \sum_k \langle T_i, T_j, T_k \rangle_\beta \, q^\beta \, T^k$.

small

q^β : formal variable, Novikov variable. (Semi-group ring over $NE(X)$.)

In general, consider pre-potential

$$\Phi(\gamma) := \sum_n \sum_\beta \frac{\langle \gamma^n \rangle_\beta}{n!} q^\beta, \quad \gamma = \sum_{i=0}^m t^i T_i \in A(X) = H^{even}(X)$$

$\langle \gamma^n \rangle_\beta = \langle \underbrace{\gamma, \gamma, \ldots, \gamma}_{n} \rangle_\beta$

It is formal power series in $t_0, \ldots, t_m, q^\beta$.

(Big-) Quantum product

$$T_i \underset{\gamma}{*} T_j := \sum_k \Phi_{ijk} T^k, \quad \Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t^i \partial t^j \partial t^k} = \sum_{n=0}^\infty \sum_{\beta \in A_1(X)} \frac{1}{n!} \langle T_i, T_j, T_k, \gamma^n \rangle_\beta \, q^\beta$$

($\gamma = 0 \rightsquigarrow$ small Q-product) i.e. a family of product over $\gamma \in A(X)$.

$\Rightarrow T_0 = \mathrm{id}$ for $*$ since $\underset{1}{\underset{\shortparallel}{T_0}} * T_j = \langle \underset{1}{\underset{\shortparallel}{T_0}}, T_j, T_k \rangle_0 T^k = g_{jk} T^k = T_j$.

Theorem $QH(X)$ is a commutative ring.

As a group, it is $H^{even}(X) \otimes \mathbb{Q}[[t, q^\beta]]$.

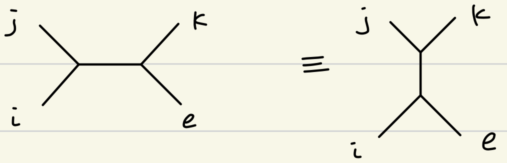
(if we define on cohomology $H^*(X)$, then it is a $\mathbb{Z}_2$-graded commutative ring.)

proof: $(T_i * T_j) * T_k \overset{?}{=} T_i * (T_j * T_k)$

$$(T_i * T_j) * T_k = \Phi_{ije}\, g^{ef}\, T_f * T_k = \Phi_{ije}\, g^{ef}\, \Phi_{fkc}\, T^c$$

$$T_i * (T_j * T_k) = T_i * \Phi_{jke}\, g^{ef}\, T_f = \Phi_{jke}\, g^{ef}\, \Phi_{ifc}\, T^c$$

$\to$ WDVV equation



why?

Consider $D(A, \beta_1 \mid B, \beta_2) = \mathcal{M}_{0,A\cdot}(X, \beta_1) \underset{X}{\times} \mathcal{M}_{0,B\cdot}(X, \beta_2) \xhookrightarrow{\iota} \mathcal{M}_{0,A\cdot}(X, \beta_1) \times \mathcal{M}_{0,B\cdot}(X, \beta_2)$

$A \amalg B = \{1, \dots, n\}$

$\beta_1 + \beta_2 = \beta$

$$\downarrow \alpha$$

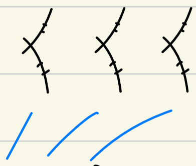
$$\mathcal{M}_{0,n}(X, \beta)$$

lemma Splitting axiom:

$$\iota_* \alpha^* (e_1^* \gamma_1 \cup \dots \cup e_n^* \gamma_n) = \sum_{ef} g^{ef} \Big(\prod_{a \in A} e_a^* \gamma_a \cup e_\cdot^* T_e \Big) \times \Big(\prod_{b \in B} e_b^* \gamma_b \cup e_\cdot^* T_f \Big)$$

proof: DIY.

Try to use the linear equivalence :



$$\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1 = (\mathbb{P}^1 \setminus \{0, 1, \infty\}) \cup \{0, 1, \infty\}$$

$$\sum \Phi_{ije}\, g^{ef}\, \Phi_{fke} = \sum \frac{1}{n_1!\, n_2!} \langle T_i, T_j, T_e, \gamma^{n_1} \rangle_{\beta_1}\, g^{ef} \langle T_k, T_e, T_f\, \gamma^{n_2} \rangle_{\beta_2}$$

Let $G(qr \mid st) := \sum g^{ef} \langle \prod_{\substack{a \in A \\ \cup\!\shortmid \\ \{q, r\}}} \gamma_a, T_e \rangle_{\beta_1} \langle \prod_{\substack{b \in B \\ \cup\!\shortmid \\ \{s, t\}}} \gamma_b, T_f \rangle_{\beta_2} = \sum \int_{D(A, \beta_1 \mid B, \beta_2)} \prod e_i^* \gamma_i$

// Claim

$G(rs \mid qt)$

$$\overline{M}_{0,n}(X,\beta) \longrightarrow \overline{M}_{0,n} \longrightarrow \overline{M}_{0,4} \simeq \mathbb{P}^1$$

$$\{q,r,s,t\} = \{1,2,3,4\} \qquad \overset{\shortparallel}{\{i,j,k,l\}}$$

$$G(qr|st) = (n-4)! \sum_{e,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl}$$

$$D(ij|kl) = \sum_{\substack{i,j\in A\\ k,l\in B}} D(A,\beta_1|B,\beta_2) = \sum \Phi_{ije}\, g^{ef}\, \Phi_{fkl}$$

#

On $X = \mathbb{P}^2$, $N_d = ?$

$T_0 = 1$, $T_1 = \text{line}$, $T_2 = pt$

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0$$

$$\text{WDVV} \Rightarrow \Phi_{222} = \Phi_{112}^2 - \Phi_{111}\Phi_{122}$$

$$\Phi(\gamma) = \Phi(t_1 l + t_2 pt) = \sum_{n\geq 0}\sum_{d\geq 0} \frac{\langle (t_1 l + t_2 pt)^{\otimes n} \rangle_{dl}}{n!}$$

$$= \sum_{\substack{n\geq 0\\ n_1+n_2=n}} \sum_{d\geq 0} \frac{1}{n_1!\, n_2!} t_1^{n_1} d_1^{n_1} t_2^{n_2} d_2^{n_2} \langle pt^{\otimes n_2} \rangle_{dl} = \sum_{d\geq 0} e^{dt_1} \underline{N_d} \frac{t_2^{3d-1}}{(3d-1)!}$$

$$N_d = \langle \underbrace{pt, pt, \dots, pt}_{3d-1} \rangle_{dl}$$

$$\text{Plug into WDVV} \Rightarrow N_d = \sum_{\substack{d_1+d_2=d\\ d_i>0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

2021.12.13.

Special case of virtual cycle $X=(\ell)\subseteq \mathbb{P}^m$, defined by $s=0$, $\deg s=\ell$.

for $g=0$.

$i:\overline{M}_{0,n}(X,\underline{d}) \hookrightarrow \overline{M}_{0,n}(\mathbb{P}^m,d)$

$\underline{d}$: collect all $\beta\in H_2(X,\mathbb{Z})$ s.t. $[\beta]\sim d\cdot$ line.

virtual dimension $=\underbrace{(m+1-\ell)}_{c_1(X)}d+\underbrace{(m-1)}_{\dim X}-3+n=\underbrace{\dim \text{ of RHS}}_{(\text{virtual dimension on } \mathbb{P}^m)}-(d\ell+1)$

Plan: realize $[\overline{M}_{0,n}(X,d)]^{vir}$ as an Euler class for a vector bundle of rank $d\ell+1$ on $\overline{M}_{0,n}(\mathbb{P}^m,d)$.

$$\begin{tikzcd} & & \mathcal{O}(\ell) \arrow[d] \\ \overline{M}_{0,n+1}(\mathbb{P}^m,d) \arrow[rr, "e_{n+1}"] \arrow[d, "\pi"] & & \mathbb{P}^m \arrow[u, bend left, dashed, "s"] \\ \overline{M}_{0,n}(\mathbb{P}^m,d) & & \end{tikzcd}$$

$R\pi_* e_{n+1}^*\mathcal{O}(\ell)$, here $R^1=0$, we just take $R^0\pi_*=\pi_*$.

$\rightsquigarrow E_d:=\pi_*e_{n+1}^*\mathcal{O}(\ell)\longrightarrow \overline{M}_{0,n}(\mathbb{P}^m,d)$

is a vector bundle of rank $d\ell+1$.

Since $\dim H^0(\Sigma,f^*\mathcal{O}_{\mathbb{P}^m}(\ell))=d\ell+1$.

$H^1=0$

Homework $\pi_*e_{n+1}^*(s)$ is a section of E_d, vanishes on $i(\overline{M}_{0,n}(X,d))$ exactly.

$\Rightarrow$ Theorem $\iota_*[\overline{M}_{0,n}(X,d)]^{vir}=e(E_d)\cap \overline{M}_{0,n}(\mathbb{P}^m,d)$.

$$\int_{[\overline{M}_{0,n}(X,\underline{\beta})]^{vir}}\phi=\int_{\overline{M}_{0,n}(\mathbb{P}^m,\beta)}e(E_d)\cdot\phi$$

($\underline{\beta}$ = d)

How to do localization?

$T=(\mathbb{C}^\times)^{m+1}$ acts on $V=\mathbb{C}^{m+1}$ diagonally.

$(t_0,t_1,\dots,t_{m+1})$ $\qquad (x_0,\dots,x_m)$

$H_T^*\equiv H_T^*(pt):=H^*(BT)=H^*((\mathbb{CP}^\infty)^{m+1})=\mathbb{Q}[\alpha_1,\dots\alpha_m]$

where $\alpha_i=c_1(\mathcal{L}_i)$, $\mathcal{L}_i=$ the i-th $\mathcal{O}_{\mathbb{CP}^\infty}(1)$.

$$\begin{array}{cl} EG & \text{principal } G\text{-bundle, contractible.} \\ \downarrow & \\ BG & \text{classifying space} \end{array}$$

e.g. $G=\mathbb{C}^\times$

$$\begin{array}{c} \mathbb{C}^{n+1}\setminus\{0\} \\ \downarrow \mathbb{C}^\times \\ \mathbb{CP}^n \end{array}$$

$\leftarrow \pi_i=0$ for $i<n-1$

$\rightsquigarrow$ Take $n\to\infty$.

If T acts on X, then $H^*_T(X) := H^*(X \times_T BT)$ is a H^*_T-module.

e.g. $X = \mathbb{P}(V) \simeq \mathbb{P}^m$, $H^*_T(X) = \mathbb{Q}[\alpha_0 \dots \alpha_m][H] \Big/ \prod_{i=0}^{m} (H-\alpha_i)$.

By Leray-Hirsch: $X \times_T BT \to BT$, fiber $= X = \mathbb{P}^m$, $H^*(X) = \mathbb{Q}[H]$.

We may take different action (linearization) on $\mathcal{O}_{\mathbb{P}^m}(-1)$.

e.g. $(x_0, \dots, x_m) \longmapsto t_0^{-1}(t_0 x_0, t_1 x_1, \dots, t_m x_m)$ get $\mathcal{O}(-H+\alpha_0)$

Let $P_0, \dots, P_m$ be T-fixes point of $\mathbb{P}^m$.

$\phi_i := H^{2m}_T(\mathbb{P}^m)$ be the equivariant class of $P_i \xhookrightarrow{\iota_i} \mathbb{P}^m$

$\rightsquigarrow$ pairing $a, b \in H^*_T(\mathbb{P}^m)$, $(a,b) = \int_{\mathbb{P}^m} a \cup b \in H^*_T$.

integrate out H terms

__Facts__ (1) $T_{\mathbb{P}^m}\big|_{P_i}$ has weight $\alpha_i - \alpha_j$ for all $j \neq i$ by definition of Hom.

(2) $H^0(\mathbb{P}^1, T\mathbb{P}^1)$ has weights $\alpha_0 - \alpha_1$, 0, $\alpha_1 - \alpha_0$.

By exact sequence $0 \to \mathcal{O} \to \underbrace{\mathcal{O}(1) \otimes V}_{\text{Global section } x_0, x_1} \to T\mathbb{P}^1 \to 0$.

weight $= 0$

$\nearrow \otimes S^{-1} = \mathcal{O}(1)$

$V = \mathcal{O}(-1)$, $\pi \downarrow$, $\mathbb{P}^m$

$0 \to S \to \pi^* V \to Q \to 0$

weight $= \{\alpha_0, \alpha_1\} \times \{-\alpha_0, -\alpha_1\}$

$\rightsquigarrow 0,\ \alpha_0 - \alpha_1,\ \alpha_1 - \alpha_0,\ 0$

$\rightsquigarrow$ Take H^0 $(H^1(\mathcal{O}) = 0)$ $\Rightarrow$ weight $= \alpha_0 - \alpha_1, 0, \alpha_1 - \alpha_0$.

(3) $(f(H,\alpha), \phi_i) = \iota_i^* f(H,\alpha) = f(\alpha_i, \alpha)$.

(4) $\phi_i = \prod_{j \neq i} (H - \alpha_j)$

(5) $a = b \Leftrightarrow (a, \phi_i) = (b, \phi_i)$ for all i. §4.3

Atiyah - Bott localization §4.3.4.4

For any $\phi \in H_T^*(X)$, $\phi = \sum_F \frac{\iota_{F*}\iota_F^*\phi}{e_T(N_{F/X})}$

F fixed submanifolds $\iota_F : F \hookrightarrow X$.

$$\rightsquigarrow \int_X \phi = \sum_F \int_F \frac{\iota^*\phi}{e(N_{F/X})}$$

$\Rightarrow$ Bott - residue formula:

$$X = \mathbb{P}^m, \quad \int_{\mathbb{P}^m} f(H,\alpha) = \sum_{i=0}^{m} \operatorname*{Res}_{H=\alpha_i} \frac{f(H,\alpha_i)}{\prod_j (H-\alpha_j)} \quad \text{by (1).}$$

(6) For $f: \Sigma \longrightarrow \mathbb{P}^1$, $d:1$ cover, branched only at $0, \infty$.

$\Rightarrow \Sigma \simeq \mathbb{P}^1$ and the map is $(z_0, z_1) \longmapsto (z_0^d, z_1^d) = (x_0, x_1)$

(7) On $\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}(H)$, $H^0(\Sigma, f^*\mathcal{O}(1))$ has weights $\frac{1}{d}(i\alpha_0 + (d-i)\alpha_1)$, $i = 0 \dots d$.

Proposition For "$\mathcal{O}(1) \simeq \mathcal{O}(H-\alpha_1)$" in (6), (7)

this means we choose a different weight

as $t_1^{-1}(t_0 x_0, t_1 x_1)$

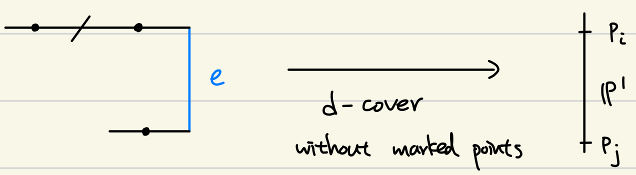
$\Rightarrow$ $H^0(\Sigma, f^*\mathcal{O}(1))$ has weights $\frac{i}{d}(\alpha_0 - \alpha_1)$ $\quad i = 0, 1, \dots, d$

$H^1(\Sigma, f^*\mathcal{O}(-1))$ $\quad \frac{i}{d}(\alpha_0 - \alpha_1)$ $\quad i = 1, 2, \dots, d-1$

$\hookrightarrow$ using R-R + Serre duality.

T-localization on $\overline{M}_{g,n}(\mathbb{P}^m, d)$

$f: (\Sigma, x_1, \dots, x_n) \longrightarrow \mathbb{P}^m$ is T-fixed $\Leftrightarrow$



Graph: edge $\leftrightarrow$ non- constant component with $\deg(e) = d_e$.
$(g=0)$

vertex $\leftrightarrow$ connected component of $f^{-1}\{p_1 \dots p_m\}$ with $p_v = f(v)$.

tails $\leftrightarrow$ marked points

$\mathrm{val}(v) = \#$ tails $+$ edges. $\quad \sum_e d_e = d$

Given $\Gamma \leftrightarrow$ T-fixed point.

$$\gamma : \overline{M}_\Gamma = \prod_v \overline{M}_{g(v),\, val(v)} \longrightarrow \overline{M}_{g,n}(\mathbb{P}^m, d)$$

$$A_\Gamma := \mathrm{Aut}(\overline{M}_\Gamma) : \quad 1 \longrightarrow \prod_e \mathbb{Z}/d_e \longrightarrow A_\Gamma \longrightarrow \mathrm{Aut}(\Gamma) \longrightarrow 1$$

$\cong$ as a semi-direct product.

<u>Definition</u> Flag $F = (e, v)$ i.e. $v \in e$.

$$w_F := \frac{\alpha_\mu(v) - \alpha_\mu(v')}{d_e} \quad : \text{weight on } T_{\cdot}\mathbb{P}^1$$

point P_F

For $g=0$, hence $g(v)=0$ for all v since $H^1(\Sigma, f^* T_{\mathbb{P}^m}) = 0$.

convex

$$0 \to \mathrm{Aut}(\Sigma, x_i) \longrightarrow \mathrm{Def}(f) \longrightarrow \underline{\mathrm{Def}(\Sigma, x_i, f)}$$

$$\to \mathrm{Def}(\Sigma, x_i) \longrightarrow 0$$

what we want in the "moving point" i.e. $wt \neq 0$.

$$e(N_\Gamma) = \frac{e(\mathrm{def}(f)^{mov})\,③ \cdot e(\mathrm{def}(\Sigma, x_i)^{mov})\,②}{e(\mathrm{aut}(\Sigma, x_i)^{mov})\,①}$$

$$① = \prod_{val(v)=1} w_F$$ e.g. the point P_F is not special.

② : boundary lemma : $\iota : \overline{M}_{g_1, A \cup p} \times \overline{M}_{g_2, B \cup q} \longrightarrow \overline{M}_{g,n}$

$$N_i = (L_p \boxtimes L_q)^*$$ i.e. $\otimes$ of T_p and T_q.

deformation of constant component $\longrightarrow$ weight 0, so we consider smoothing of nodes.

$$\prod_F (w_F - \psi_F) \cdot \prod_{val(v)=2} (w_{F v_1} + w_{F v_2})$$

③: On $\operatorname{def}(f) = H^0(\Sigma, f^* T\mathbb{P}^m)$,

Partial normalization $0 \to \mathcal{O}_\Sigma \to \bigoplus_v \mathcal{O}_{\Sigma_v} \oplus \bigoplus_e \mathcal{O}_{\Sigma_e} \longrightarrow \bigoplus_F \mathcal{O}_{p_F} \to 0$

as T- representation.

$h^1(\mathcal{O}_\Sigma) = 0 \Rightarrow H^0(\quad) = H^0(\quad) \oplus H^0(\quad)$.

<u>Theorem</u> $\dfrac{1}{e(N_\Gamma)} =$

$$\prod_{\text{flags}} \frac{1}{\omega_F - \psi_F} \prod_{\nu \neq \mu(F)} (\alpha_{\mu(F)} - \alpha_\nu)$$

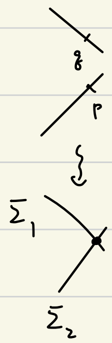
$$\prod_{\text{vertices}} \prod_{\nu \neq \mu(v)} \frac{1}{\alpha_{\mu(v)} - \alpha_\nu} \prod_{\text{val}(v)=2} \frac{1}{\omega_{F_{v,1}} + \omega_{F_{v,1}}} \prod_{\text{val}(v)=1} \omega_F$$

$$\prod_{\text{edges}} \frac{(-1)^{d(e)} d(e)^{2d(e)}}{(d(e)!)^2 (\alpha_i - \alpha_j)^{2d(e)}} \prod_{\substack{a+b=d(e) \\ k \neq i,j}} \frac{1}{\frac{a}{d(e)}\alpha_i + \frac{b}{d(e)}\alpha_j - \alpha_k}.$$

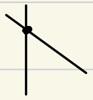
2021.12.16.

$e(N_\Gamma) = \ ?$

"smoothing of nodes" ∂-lemma.

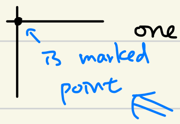


$N_i = (N_p \boxtimes N_q)^*$ i.e. $T_p\Sigma_1 \otimes T_q \Sigma_2$



two non-contracted components $\Rightarrow \omega_{F_{v,1}} + \omega_{F_{v,2}}$

$$\omega_F = \frac{\alpha_{\mu(v)} - \alpha_{\mu(v')}}{d(e)}$$



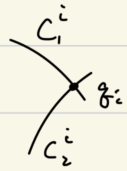
one $\underline{\text{contracted}}$, one non-contracted component $\Rightarrow \omega_F - \psi_F$.

stable map

$$\rightsquigarrow \prod_{\substack{\text{flags} \\ (\text{val} \geq 3)}} (\omega_F - \psi_F) \prod_{\text{val}(v)=2} (\omega_{F_{v,1}} + \omega_{F_{v,2}})$$

$\underline{\partial\text{-lemma}}$ (Lebnitz rule ?)

$$H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \simeq \bigoplus_{i \text{ nodes}} (T_{q_i} C_1^i \otimes T_{q_i} C_2^i)$$



first order deformation (smoothing) for A_1-singularity

proof: This is local, we may assume $\underset{(xy=0)}{C} \subseteq \underset{\text{chart at } (0,0) = q \subseteq \mathbb{C}^2}{S}$. $I = I_C = (xy)$: ideal sheaf.

$$0 \to I/I^2 \to \Omega_S^1|_C \longrightarrow \Omega_C^1 \longrightarrow 0$$
$$f \longmapsto df|_C$$

Kähler differential

$$\Rightarrow \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C) \simeq \operatorname{coker}(T_S|_C \xrightarrow{h} (I_C/I_C^2)^*)$$

$T_S|_C$ generated by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$(I_C/I_C^2)^* = \operatorname{Hom}\left(\frac{(xy)}{(xy)^2}, \frac{\mathbb{C}[x,y]}{(xy)}\right)$

$xy \quad\quad 1, x, y$

$$h(\tfrac{\partial}{\partial x}) : xy \longmapsto y$$
$$h(\tfrac{\partial}{\partial y}) : xy \longmapsto x$$

Claim $T_q C_1 \otimes T_q C_2 \xrightarrow{\sim} \mathcal{E}xt^1(\Omega^1_C \otimes \mathcal{O}_C)$ extend to vector field on S.

$v \otimes w \longmapsto$ the map in I/I^2 by $f \mapsto \tilde{v}\tilde{w}(f)\big|_C$.

$\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}(xy) = 1$. Done!

#

Next, on $\mathrm{Def}(f) = H^0(\Sigma, f^* T\mathbb{P}^m)$

$\Big(\mathcal{O}b(f) = H^1(\Sigma, \underset{\substack{\uparrow \\ \text{if } g(\Sigma)=0}}{f^* T\mathbb{P}^m}) = 0 \Big)$

Normalize sequence : $0 \to \mathcal{O}_\Sigma \to \bigoplus_v \mathcal{O}_{\Sigma_v} \oplus \bigoplus_e \mathcal{O}_{\Sigma_e} \to \bigoplus_F \mathcal{O}_{P_F} \to 0$

$$0 \longrightarrow H^0(\Sigma, f^* T\mathbb{P}^m) \longrightarrow \bigoplus_v H^0(\Sigma_v, f^* T\mathbb{P}^m) \oplus \bigoplus_e H^0(\Sigma_e, f^* T\mathbb{P}^m) \longrightarrow \bigoplus_F T_{P_{\mu(F)}}\mathbb{P}^m$$

$$\longrightarrow \underbrace{H^1(\Sigma, f^* T\mathbb{P}^m)}_{\text{this is } \neq 0 \text{ if } g(\Sigma) \geq 1.} \longrightarrow \bigoplus_v \underbrace{H^1(\Sigma_v, f^* T\mathbb{P}^m)}_{*} \oplus \underline{0} \longrightarrow 0$$

$H^0 - H^1$ defines "virtural" part of N_Γ

e. non-contract $\Rightarrow \Sigma_e = \mathbb{P}^1 \Rightarrow H^1 = 0$

The additional term $*$ (for $g(\Sigma) \geq 1$):

$$H^1(\Sigma_v, \underset{=}{f^* T\mathbb{P}^m}) = \underline{H^1(\Sigma_v, \mathcal{O}_{\Sigma_v})} \otimes T\mathbb{P}^m$$

contract $\Rightarrow$ constant map

$\cong$ $H^0(\Sigma_v, \omega_{\Sigma_v})^* =: \mathbb{E}^\vee$, rank $= g(\Sigma_v)$.

$\rightsquigarrow$ get additional weight $\prod_{\nu \neq \mu(v)} \Big(C_{g(v)}(\mathbb{E}_v) + C_{g(v)-1}(\alpha_{\mu(v)} - \alpha_\nu) + \cdots + (\alpha_{\mu(v)} - \alpha_\nu)^{g(v)} \Big)$

$$= \prod_{\nu \neq \mu(v)} c(\mathbb{E}^\vee)\Big(\frac{1}{\alpha_{\mu(v)} - \alpha_\nu} \Big) \cdot (\alpha_{\mu(v)} - \alpha_\nu)^{g(v)}$$

The other (original easier) part :

$$H^0(\Sigma, f^* T\mathbb{P}^m) = \bigoplus_v T_{P_v}\mathbb{P}^m + \bigoplus_e (\underset{\substack{\cong \\ \mathbb{P}^1}}{\underline{\Sigma_e}}, f^* T\mathbb{P}^m) - \bigoplus_F T_{P_F}\mathbb{P}^m$$

$\downarrow$ is a trivial bundle

$\overline{M}_\Gamma$ since $\Sigma_e = \mathbb{P}^1$ has no deformation, or rigid.

$$0 \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_{\mathbb{P}^m}(1) \otimes V \to T\mathbb{P}^m \to 0$$

$$\overset{f^*}{\rightsquigarrow} \; 0 \to \underline{\mathbb{C}} \to H^0(\Sigma_e, \mathcal{O}_{\Sigma_e}(d_e)) \otimes V \to H^0(\Sigma_e, f^* T\mathbb{P}^m) \to 0$$

$wt = 0$ $\qquad wt = \frac{1}{d_e}(a\alpha_i + b\alpha_j) - \alpha_k$ $\qquad i, j, k = 0 \cdots m$, $a + b = d_e$

The case $k=i$ or j / $k\neq i,j$

$\rightsquigarrow$ $a=0$ or $b=0$ gives 2 weights 0. Another weight 0 to be cancelled is
$k=j$ $\quad$ $k=i$ $\qquad$ in $\mathrm{Aut}(\Sigma)$.

Theorem $\dfrac{1}{e(N_\Gamma)} =$

$$\prod_{\text{flags}} \frac{1}{\omega_F - \psi_F} \prod_{\nu \neq \mu(F)} (\alpha_{\mu(F)} - \alpha_\nu)$$
$$\prod_{\text{vertices}} \boxed{\prod_{\nu\neq\mu(v)} \frac{1}{\alpha_{\mu(v)} - \alpha_\nu}} \prod_{\text{val}(v)=2} \frac{1}{\omega_{F_{v,1}} + \omega_{F_{v,1}}} \prod_{\text{val}(v)=1} \omega_F$$
$$\prod_{\text{edges}} \frac{(-1)^{d(e)} d(e)^{2d(e)}}{(d(e)!)^2 (\alpha_i - \alpha_j)^{2d(e)}} \prod_{\substack{a+b=d(e)\\ k\neq i,j}} \frac{1}{\frac{a}{d(e)}\alpha_i + \frac{b}{d(e)}\alpha_j - \alpha_k}.$$

only term to be corrected in $e(N_\Gamma^{vir})$
(i.e. $g(v)\geq 2$)

Corollary For $\bar{\Gamma} = \overset{d}{\underset{P_0 \quad P_1}{\bullet\!\!-\!\!-\!\!\bullet}}$, $\dfrac{1}{e(N_\Gamma)} = \dfrac{(-1)^{d-1} d^{2d-2}}{(d!)^2 (\alpha_0-\alpha_1)^{2d-2}}$

Aspinwall - Morrison's multiple cover formula for $g=0$:

$$\begin{array}{c} \mathcal{O}(-1)^{\oplus 2} \\ \downarrow \quad \subseteq X^3 \\ \Sigma \longrightarrow \mathbb{P}^1 = C \end{array}$$

degree d cover has contribution $\dfrac{1}{d^3}$ in $d[C] \in H_2(X,\mathbb{Z})$

proof: We will show $\displaystyle\int_{\bar{M}_0(\mathbb{P}^1,d)} e\left(H^1(\Sigma, f^*\mathcal{O}(-1))^{\oplus 2}\right) = \frac{1}{d^3}$

$\hookrightarrow R^1\pi_* f^*\mathcal{O}(-1)$

Key: Choose action weight on the bundle directions by $-H+\alpha_0$, $-H+\alpha_1$.

Claim: If $\Gamma \neq \bar{\Gamma}$, then the contribution $=0$.

another graph

subpf: If $\Gamma\neq\bar{\Gamma}$, it has node b_i. $\tilde{\Sigma} = \cup \Sigma_i \longrightarrow \Sigma \longrightarrow C = \mathbb{P}^1$.

$\rightsquigarrow 0 \longrightarrow f^*\mathcal{O}(-1) \longrightarrow \bigoplus_i f|_{\Sigma_i}^* \mathcal{O}(-1) \longrightarrow f|_{b_i}^*\mathcal{O}(-1) \longrightarrow 0$

$\rightsquigarrow 0 \longrightarrow \bigoplus_i \underbrace{H^0(b_i, f^*\mathcal{O}(-1))}_{=\,\mathbb{C}} \hookrightarrow H^1(\Sigma, f^*\mathcal{O}(-1))$

$\mathcal{O}(-1)$ has no H^0.

If $f(b_i) = p_0$, get subline bundle of trivial weight on $\mathcal{O}(-1) \simeq \mathcal{O}(-H+\alpha_0)$ at p_0.

#

$$C(g,d) = \int_{[\bar{M}_g(\mathbb{P}^1,d)]^{vir}} C_{top}(R^1\pi_* \mu^* \underset{\overset{\shortparallel}{\mathcal{O}(-1)^{\oplus 2}}}{N})$$

$$C(1,d) = \frac{1}{12d}, \quad C(g,d) = \frac{|B_{2g}|}{2g(2g-2)!} d^{2g-3} = |\chi(M_g)| \frac{d^{2g-3}}{(2g-3)!}.$$

$$\sum_{g\geq 0} C(g,1) t^{2g} = \left(\frac{t/2}{\sin(t/2)} \right)^2.$$

GW for $g=0 \iff$ QH, $\quad \Phi_{ijk}(t) = \langle\langle T_i, T_j, T_k \rangle\rangle$

$$= \sum \langle T_i, T_j, T_k, t^n \rangle / n!$$

Q: Where is tt^*? Dubrovin conjecture: How to put real structure on QH?

$(t, \bar{t})$ $\quad$ (e.g. Γ-conjecture)

$=$???

Dubrovin connection: $\nabla_{\partial t^i} := \frac{\partial}{\partial t^i} - \frac{1}{\hbar} T_i *_t$. $\quad (\nabla^\hbar := \hbar \nabla.)$

$t = \sum t^i T_i$

∇ is flat for all $\hbar \iff$ WDVV

integrable connection

<u>Definition</u> • F is a flat sections $:= \hbar \frac{\partial F}{\partial t^i} - T_i * F$.

• J-function: ($g=0$.) $J(t) = 1 + \frac{t}{\hbar} + \sum_{(n,\beta)\neq(0,0)} \frac{1}{n!} \left\langle \frac{T_b}{\hbar(\hbar-\psi)}, t^n \right\rangle_{\beta, n+1} T^b$.

(Givental's J-)

$= \sum_b J_b T^b$

$$\langle \tau_{a_1} T_{j_1} \cdots \tau_{a_n} T_{j_n} \rangle_{g,\beta} = \int_{[\bar{M}_{g,n}(X,\beta)]^{vir}} \prod ev_i^*(T_{j_i}) \cdot \prod \psi_i^{a_i}.$$ desendant GW.

Consider $\Phi_{ab} = \hbar \frac{\partial}{\partial t^a} J_b = g_{ab} + \sum_{n,\beta} \sum_{k\geq 0} \frac{\hbar^{-(k+1)}}{n!} \cdot \langle T_a, \tau_k T_b, t^n \rangle_\beta$

<u>Proposition</u> Φ_{ab} is a fundamental solution matrix for QDE, i.e.

$$\hbar \frac{\partial}{\partial t^i} (\Phi_{ab} T^a) = T_i * (\Phi_{ab} T^a).$$

proof: TRR: topological recursion relation.

SE, DE, TRR.

"div E"

$\psi_1 = D(1|23)$ for any $\{1.2.3\} \subseteq \{1.2.3\ldots n\}$.

$(n \geq 3)$

$\overline{M}_{0,n}(X,\beta) \xrightarrow{\pi} \underset{\underset{\text{point}}{\shortparallel}}{\overline{M}_{0,3}}$, $\quad L_1 = \underset{\underset{0}{\shortparallel}}{\underline{\pi^* L_1'}} + D(1|23)$.

$$LHS = \sum_{k\geq 0} \frac{\hbar^{-k}}{n!} \langle T_a, \tau_k T_b, T_i, t^n \rangle_\beta T^a$$

$\shortparallel$ TRR!

$$RHS = T_i * T_b + \sum_{\substack{k\geq 0 \\ n_i, \beta_i}} \frac{\hbar^{-(k+1)}}{n_1!\, n_2!} \langle \underline{T_a}, \tau_k T_b, t^{n_1} \rangle_{\beta_1} \underline{g^{aj}} \langle T_i, \underline{T_j}, T_s, t^{n_2} \rangle_{\beta_2} T^s.$$

#

2021.12.20.

$X = (\ell) \hookrightarrow \mathbb{P}^m$, "determine all genus 0 GW-invariant on X."

degree ℓ hypersurface

$$\begin{array}{c} \mathcal{O}(\ell) \\ \downarrow \\ \mathbb{P}^m \end{array} \circlearrowleft T$$

graph sum

$\exists$ algorithm to do it! (OK!)

The mirror symmetry predicts that "$GW_{g=0}$" is determined by certain linear differential equation.

comes from Gauss-Manin connection or VHS.

QDE: $\hbar \partial_i \partial_j J(t) = \sum_k \underbrace{C_{ij}^k(t)}_{= (T_i *_t)_j^k} \partial_k J(t)$ ← What we want is not this since we do not know $C_{ij}^k(t)$ yet

Recall: $\frac{\partial}{\partial t} \vec{x}(t) = A(t) \vec{x}(t) \longrightarrow$ scalar higher order ODE?

GW $g=0$ $\Leftrightarrow$ the cyclic $D^\hbar$-module generated by J. $(\hbar\partial_i)(\hbar\partial_j) J = \sum C_{ij}^k(t) \hbar \partial_k J$.

$\hookrightarrow \hbar\partial_i$

It has a "good" (easier) D-module theory for 1-dimensional case.

In our case, $H^2(X) = H^2(\mathbb{P}^m) = \mathbb{Z}$. $t = \underset{\substack{\uparrow \\ H^0}}{t_0} + \underset{\substack{\uparrow \\ H^1}}{t_1} + \underset{\substack{\uparrow \\ H^2}}{t_2}$

$(\mathrm{Pic}(X) = \mathbb{Z})$

$t_0 \to$ SE, $t_1 \to$ Div E

$$\rightsquigarrow \sum_n \frac{1}{n!} \langle T_i, T_j, T_k, t^{\otimes n} \rangle_\beta \, q^\beta = q^\beta e^{\int_\beta t_1} \sum_n \frac{1}{n!} \langle T_i, T_j, T_k, t_2^{\otimes n} \rangle_\beta$$

Set $t_0 = 0 \Rightarrow$ OK!

Set $t_2 = 0 \Rightarrow$ small QH. This is OK if $H^*(X)$ is generated by $H^2(X)$.

Manin-Kontsevich's reconstruction theorem

$\rightsquigarrow$ WDVV $\Rightarrow$ need only divisor insertions for $n \geq 3$ point functions.

$\langle \underset{\text{divisor}}{\underline{D}}, T_i, T_j, T_k \rangle_\beta$ "=" $\langle T_i, D, T_j, T_k \rangle_\beta$ up to lower "β".

ODE $\longleftrightarrow$ recursive relations $\left\langle \begin{array}{l} \text{linear} \to \text{OK.} \\ \text{quadratic} \to ?? \end{array} \right.$

$c_1(X) = (m+1-\ell) H$

X: Fano if $\ell \leq m$

X is CY if $\ell = m+1$

$$S(t,\hbar) = \sum_{d\geq 0} e^{(\frac{H}{\hbar}+d)t}\, ev_{2*}\left(\frac{e(E_d)}{\hbar-\psi_2}\right) \quad . \quad " < 1, \frac{e(E_d)}{\hbar-\psi_2} >_{n=2} "$$

$$0 \longrightarrow E_{d'} \longrightarrow E_d \longrightarrow ev_2^* \mathcal{O}(\ell) \longrightarrow 0$$

$$ev_2 : \overline{M}_{0,2}(\mathbb{P}^m, d) \longrightarrow \mathbb{P}^m$$

$$< \frac{S}{\ell H}, \phi_i > = e^{\alpha_i \frac{t}{\hbar}} \overbrace{\sum_{d\geq 0} e^{dt} \int_{\overline{M}_{0,2}(\mathbb{P}^m, d)} \frac{e(E_d')}{\hbar-\psi_2} ev_2^*(\phi_i)}^{=: Z_i(e^t, \hbar)}$$

$\hookrightarrow \dim = (m+1)d + m(1-g) + 2 + 3g - 3 = (m+1)d + m - 1$

$\mathrm{rank}(E_d') = \ell d$, " $\deg \phi_i = m$ " $\Rightarrow$ ψ_2 $\deg \geq \underbrace{(m+1-\ell)d - 1}_{=\ \delta}$ to get non-trivial term. $\delta \geq 0$ if $\ell \leq m$.

$$\text{So } Z_i(e^t, \hbar) = 1 + \sum_{d>0} \left(\frac{e^t}{\hbar^{m+1-\ell}}\right)^d \int_{\overline{M}_{0,2}(\mathbb{P}^m, d)} \frac{\psi_2^\delta \cdot e(E_d')}{\underline{1-\psi_2/\hbar}} ev_2^*(\phi_i)$$

!?

For $i \in [0,m]$, localization has fixed loci:

$$G_d = G_d^{i*} \sqcup G_d^{i0} \sqcup G_d^{i1}$$

G_d^{i*}: $x_2 \not\mapsto p_i$

G_d^{i0}: $\Sigma_v \mapsto p_i$, $\Sigma_v \ni x_2$, i.e. $\mu(v) = i$, $\mathrm{val}(v) \geq 3$

G_d^{i1}: $\mu(v) = i$, $\mathrm{val}(v) = 2$, Σ_2 is not contracted, $\Sigma_2 \ni x_2$

Let $G^{i0} = \bigcup_{d>0} G_d^{i0}$, $G_d^i = G_d^{i0} \cup G_d^{i1}$

- $\Gamma \in G_d^{i*}$: clearly $ev_2^* \phi_i = 0$
- $\Gamma \in G_d^{i0}$: $\psi_2|_{\overline{M}_\Gamma}$ has trivial T-action $\Rightarrow \psi_2^{\overbrace{\dim(v)+1}^{\mathrm{val}(v)-3\ (\dim \text{ of } \overline{M}_{0,\mathrm{val}(v)})}} = 0$

  Γ has at most d edges, 2 tails, no loops $\Rightarrow \mathrm{val}(v) \leq d+2$

  So nilpotency of $\psi_2 \leq d$ (i.e. $\psi_2^d = 0$)
- $\Gamma \in G_d^{i1}$: Let $e = \overline{vv'}$, $j = \mu(v')$.

  If $d_e < d$, let $\Gamma_j =$ " Γ with e contracted ". Then, $\Gamma_j \in G_{d-d_e}^j$

  $|\mathrm{Aut}(\Gamma_j)| = |\mathrm{Aut}(\Gamma)|$.

Lemma Write $Z_i(e^t, \hbar) =: 1 + \sum_{d>0} e^{dt} \zeta_{id}(\alpha, \hbar)$, then ζ_{id} is a rational function and regular at $\hbar = \frac{\alpha_i - \alpha_j}{n}$ for all $j \neq i$, $n \geq 1$.

proof: $$\zeta_{id} = \sum_{\Gamma \in G_d^{io}} \sum_{k=0}^{d-1} \frac{P_{\Gamma,k}(\alpha)}{\hbar^{k+1}} + \sum_{\Gamma \in G_d^{i1}} \frac{P_\Gamma(\alpha)}{\hbar + \frac{\alpha_i - \alpha_j}{d_e}}$$ $\leftarrow$ Since $\psi_2\big|_{\overline{M}_\Gamma}$ is topological trivial of weight $= \frac{\alpha_j - \alpha_i}{d_e}$.

with $P_{\Gamma,k}(\alpha), P_\Gamma(\alpha) \in \mathbb{Q}(\alpha)$

#

- If $\ell \leq m$, denote $z_i(Q, \hbar) = Z_i(Q\hbar^{m+1-\ell}, \hbar)$.

Lemma If $\ell < m$, then the contribution of G_d^{io} to $z_i(Q, \hbar)$ is 0.

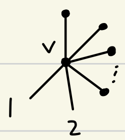
proof: $\begin{matrix} \ell < m \\ d > 0 \end{matrix} \Rightarrow S(d) = (m+1-\ell)d - 1 \geq d$.

The integrant of $\psi_2^{S(d)}$ now gives 0 on $\overline{M}_\Gamma$.

#

G_d^{io}:

- If $\ell = m$, only one case survives: $\displaystyle\int_{\overline{M}_\Gamma} \psi_2^{d-1} \frac{e(E_d') \, ev_2^*(\phi_i)}{e(N_\Gamma)}$.

$\Gamma =$  $\mathrm{val}(v) = d+2$, d edges with each $d_e = 1$.

Calculated by localization formula: $\displaystyle\int_{\overline{M}_{0,n}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} = \binom{n-3}{\beta_1 \cdots \beta_n}$.

$\rightsquigarrow$ get $$c_i(Q, \hbar) = -1 + \exp\left(-m!\, Q + \frac{(m\alpha_i)^m}{\prod_{j \neq i}(\alpha_i - \alpha_j)} Q\right)$$

G_d^{i1}: If $d_e = d$, get contribution $Q^d c_i^j(d, \hbar)$, where
$$c_i^j(d, \hbar) = \frac{1}{\frac{\alpha_i - \alpha_j}{\hbar} + d} \; \frac{\prod_{r=1}^{\ell d}\left(\frac{\ell d \alpha_i}{(\alpha_j - \alpha_i)/d} + r\right)}{\prod_{k=0}^{m} \prod_{r=1}^{d}\left(\frac{\alpha_i - \alpha_k}{(\alpha_j - \alpha_i)/d} + r\right)}$$ (check!)

Key lemma (#1)

If $d_e < d$, then the contribution $\Gamma \mapsto \Gamma_j$

$$\Rightarrow \mathrm{Cont}_\Gamma \, z_i(Q, \hbar) = Q^{d_e} c_i^j(Q, \hbar) \cdot \mathrm{Cont}_{\Gamma_j} \, z_j\Big(Q, \underbrace{\frac{\alpha_j - \alpha_i}{d_e}}_{\text{replace } \hbar}\Big)$$

proof: Flag (v', e) in $\Gamma \longmapsto$ node of Σ.

N_Γ has a line bundle quotient $\longleftrightarrow$ deformation of node.

In $N_{\Gamma'}$, this deformation disappear, but $\psi_2 \longmapsto \psi_2 - \frac{\alpha_j - \alpha_i}{d_e}$

#

<u>Proposition</u> (linear recursion & uniqueness)

$$(R) \quad z_i(Q, \hbar) = 1 + C_i(Q, \hbar) + \sum_{j \neq i} \sum_{d > 0} Q^d C_i^j(d, \hbar)\, z_j\left(Q, \frac{\alpha_j - \alpha_i}{d}\right).$$

Also, (R) determines z_i uniquely.

#

Now, consider $$Z_i^*(e^t, \hbar) = \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{r=1}^{\ell d} (\ell \alpha_i + r\hbar)}{\prod_{j=0}^{k} \prod_{r=1}^{d} (\alpha_i - \alpha_j + r\hbar)}$$

$$S^*(t, \hbar) := \sum_{d \geq 0} e^{(H/\hbar + d)t} \frac{\prod_{r=1}^{\ell d} (\ell H + r\hbar)}{\prod_{j=0}^{k} \prod_{r=1}^{d} (H - \alpha_j + r\hbar)}$$ .... Quantum Lefschitz hyperplane theorem.

$$\langle S^*(t, \hbar), \phi_i \rangle = e^{\alpha_i t/\hbar}\, \ell \alpha_i \cdot Z_i^*(e^t, \hbar).$$

For $\ell \leq m$, $z_i^*(e^t, \hbar) := Z_i^*(Q\hbar^{m+1-\ell}, \hbar)$. $\underset{(\ell < m)}{z_i^*}$ and $\underset{(\ell = m)}{e^{-m! Q} z_i^*}$ satisfy (R)!

$$\Rightarrow \underset{(\ell < m)}{S = S^*} \quad \text{or} \quad \underset{(\ell \leq m)}{S = e^{-m! \frac{e^t}{\hbar}} S^*}. \quad \text{Done!}$$

Set $\alpha_j = 0$ in S, we will get Hypergeometric function.

#

2021.12.23.

Generally setup for "mirror theorem" or actually QLHT

quantum Lefschetz hyperplane theorem

X : proj. $/\mathbb{C}$. If you know $J^X(t)$ already (or just $t \in H^0 \oplus H^2$: small.)

$t \in H^*(X)$: big

e.g. $X = \mathbb{P}^m$

E . vector bundle $E = \bigoplus_{i=1}^{n} L_i$, L_i : line bundle

$\downarrow$

X

$$\begin{cases} L : \text{convex} : f : C \longrightarrow X & H^1(C, f^*T_X) = 0 \\ L : \text{concave} : f : C \longrightarrow X & H^0(C, f^*T_X) = 0 . \end{cases}$$

Associated hypergeometric factor : $\beta \in NE(X)$, $f_*[C] = \beta$.

$$H^L_\beta := \prod_{k=0}^{c_1(L).\beta} (c_1(L) + k\hbar) \qquad \text{convex}$$

$$:= \prod_{\underbrace{k = c_1(L)\beta + 1}_{<0}}^{-1} (c_1(L) + k\hbar) \qquad \text{concave}$$

Define $I^E := e^{\frac{t_0 + \overbrace{\sum t^i D_i}^{= t_1}}{\hbar}} \sum_{\beta \in NE(X)} e^{(t_1, \beta)} q^\beta J^X_\beta(t) \underbrace{\prod_j H^{L_j}_\beta}_{\text{polynomial in } \hbar} \notin \mathbb{Q}[\![q, \hbar^{-1}]\!]$. hypergeometric modification

$$J_X(q, \hbar) := \underbrace{e^{\frac{t_0 + \sum t^i D_i}{\hbar}}}_{\text{comes from SE + Div E}} \sum_\beta e^{(t_1, \beta)} q^\beta J^X_\beta(\underset{=}{t_2})$$

$t = t_0 + \underset{\substack{\shortparallel \\ \sum t^i D_i \in H^2(X)}}{\underline{t_1}} + t_2$

small if $t_2 = 0$

$\cap$

$\mathbb{Q}[\![\underset{\substack{= \\ q^\beta}}{e^t}, \hbar^{-1}]\!]$

$$\Rightarrow q^\beta \left\langle \frac{T_\mu}{\hbar - \psi}, t^n \right\rangle_\beta T^\mu$$

$$1 + \frac{t_0 + \sum t^i D_i}{\hbar} + O(\hbar^{-2})$$

<u>Theorem</u> If $I^E(q, \hbar) \in \mathbb{Q}[\![q, \hbar^{-1}]\!]$ $\left(\Leftrightarrow c_1(T_X) - \sum_{L_j : \text{convex}} c_1(L_j) + \sum_{L_j : \text{concave}} c_1(L_j) \geq 0 \right)$

e.g. $X = (\ell) \subseteq \mathbb{P}^m$, $S^*_X = \sum e^{(\frac{H}{\hbar} + d)t} \dfrac{\prod_{r=0}^{\ell d} (\ell H + r\hbar)}{\prod_{j=0}^{m} \prod_{r=1}^{d} (H + r\hbar)}$

$$I^E(q, \hbar) = e^{\frac{t}{\hbar}} \left(I_0(t) + \frac{I_1(t)}{\hbar} + \cdots \right)$$

$$\frac{I^E}{I_0} = e^{\frac{t}{\hbar}} \left(1 + \frac{I_1(t)}{I_0(t)} \frac{1}{\hbar} + \cdots \right) = \left(1 + \frac{t}{\hbar} + \cdot \right)\left(1 + \frac{I_1(t)}{I_0(t)} \cdot \frac{1}{\hbar} + \cdots \right) = 1 + \left(t + \frac{I_1(t)}{I_0(t)} \right) \frac{1}{\hbar} + \cdots$$

$\rightsquigarrow \underbrace{= J^E(T)}_{\text{mirror theorem}}$

change variable, let $T = \underbrace{t + \frac{I_1(t)}{I_0(t)}}_{\text{classical mirror transform}}$

Today: Mirror conjecture for CY hypersurface $\ell = m+1$ in $\mathbb{P}^m$.

Write $$Z_i(e^t, \hbar) = 1 + \sum_{d>0} e^{dt} \sum_{k=0}^{d-1} \frac{1}{\hbar^{k+1}} \int_{\overline{M}_{0,2}(\mathbb{P}^m, d)} \psi_2^k \, e(E_d') \, ev_2^*(\phi_i)$$

$\mathbb{Q}[[e^t, \hbar^{-1}]]$

$$+ \sum_{d>0} \left(\frac{e^t}{\hbar}\right)^d \int_{\overline{M}_{0,2}(\mathbb{P}^m, d)} \frac{\psi_2^d}{\hbar - \psi_2} e(E_d') \, ev_2^*(\phi_i)$$

$$z_i(Q, \hbar) := Z_i(Q\hbar, \hbar)$$

$(*)$ $$\rightsquigarrow = 1 + \sum_{d>0} \frac{Q^d}{d!} \underline{Q_{id}} + \sum_{d>0} \sum_{j \neq i} Q^d \, \underline{C_i^j(d, \hbar)} \cdot z_j\left(Q, \frac{\alpha_j - \alpha_i}{d}\right)$$

$\sum_{j=0}^{d} R_{id}^j \hbar^{d-j}$: polynomial $\deg_\hbar \le d$. as in last time

$$\text{Now, } C_i^j(d, \hbar) = \frac{1}{\alpha_i - \alpha_j + d\underline{\hbar}} \cdot \frac{1}{d!} \cdot \frac{\prod_{r=1}^{(m+1)d} \left((m+1)\alpha_i + r \cdot \frac{\alpha_j - \alpha_i}{d}\right)}{\prod_{k \neq i} \prod_{\substack{r=1 \\ (k,r) \neq (j,d)}}^{d} \left(\alpha_i - \alpha_k + r \frac{\alpha_j - \alpha_i}{d}\right)}$$

the only polynomial with $\hbar$

View $\underline{R_{id}}$ as initial data. ($\Rightarrow$ determine everything as last time!)

How to control π?

Abstractly, given $Y_i(e^t, \hbar) \in \mathbb{Q}[[e^t, \hbar^{-1}]]$, $i \in [0, m]$ s.t.

P

I. Rationality of $Y_i \in \mathbb{Q}(\alpha, \hbar)[[e^t]]$, regular at $\hbar = \frac{\alpha_i - \alpha_j}{n}$, $n \geq 1$

II. $y_i(Q, \hbar) := Y_i(Q\hbar, \hbar)$ satisfies $(*)$ $\left(\leftrightarrow \underline{I_{id}} \in \mathbb{Q}(\alpha)[\hbar], \deg \hbar \le d\right.$ i.e. R_{id} for $\left.Z_i\right)$

III. Polynomial condition (GPC)

I. $$y_i(Q, \hbar) = \sum_{d \geq 0} \frac{Q^d}{d!} \cdot \frac{N_{id}}{\prod_{j \neq i} \prod_{r=1}^{d} (\alpha_i - \alpha_j + r\hbar)}$$ $\leftarrow \exists \, N_{id} \in \mathbb{Q}(\alpha)[\hbar]$, $N_{i0} = 1$, $\deg N_{id} \le d + md = (m+1)d$

<u>Definition</u> Y_i satisfy GPC if the unique polynomial $E_d^Y \in \mathbb{Q}(\alpha, \hbar)[p]$ s.t. of $\deg \le (m+1)(d+1) - 1$

$$E_d^Y(\underbrace{\alpha_i + r\hbar}_{Q}) = (m+1)\alpha_i \, N_{ir}(\hbar) \cdot N_{i(d-r)}(-\hbar) \quad \text{for all } r \in [0, d].$$

Indeed, has $E_d^Y \in \mathbb{Q}[\alpha, \hbar, p]$.

<u>Example</u> Hypergeometric correlator (on $(d) \subseteq \mathbb{P}^m$)

$$\left(Z_i^*(e^t, \hbar) = \sum_{d \geq 0} e^{dt} \frac{\prod_{r=1}^{(m+1)d} ((m+1)\alpha_i + r\hbar)}{\prod_{j=0}^{m} \prod_{r=1}^{d} (\alpha_i - \alpha_j + r\hbar)} \right)_{i=0}^{m} \in \mathcal{P}.$$

$j=i$ included $\rightsquigarrow \frac{1}{d!\,\hbar^d}$ appear

In fact, $E_d^{Z^*} = \prod_{r=0}^{(m+1)d} ((m+1)p - r\hbar)$.

To prove $(Z_i)_{i=0}^m \in \mathcal{P}$, i.e. satisfies II:GPC, need the "graph space" and "polynomial space" as an "linear σ-model".

$M_{0,2}(\mathbb{P}^1 \times \mathbb{P}^m, (1,d))$: graph space $\quad \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^m$

$\rightsquigarrow \mathbb{P}^1 \xrightarrow{(id, f)} \mathbb{P}^1 \times \mathbb{P}^m$.

$\underset{[x_0:x_1]}{\mathbb{P}^1} \xrightarrow[\deg = d]{\varphi} \mathbb{P}^m$ is given by $\left(\sum_i \varphi_{ri}\, x_0^i\, x_1^{d-i} \right)_{i=0}^m \rightsquigarrow$ dimension $(d+1)(m+1)$.

$\rightsquigarrow L_d' \simeq \mathbb{P}^{(m+1)(d+1)-1}$: polynomial space

$$M_{0,2}(\mathbb{P}^1 \times \mathbb{P}^m, (1,d)) \xrightarrow[\mathbb{C}^\times \times T \text{ equivalent.}]{\substack{(J, L_i) \\ \text{morphism}}} L_d' \simeq \mathbb{P}^{(m+1)(d+1)-1}, \quad P = c_1(\mathcal{O}_{\mathbb{C}^\times \times T}(1))$$

dimension : $2 + (m+1)d + (m+1) - 3 + \underbrace{2}_{=\text{mark point}}$ dimension different !?

$\mathbb{P}^1 = \mathbb{P}(V)$, $V \cong \mathbb{C}^2$ $\quad H_{\mathbb{C}^\times} \simeq \mathbb{Q}[\hbar] \Rightarrow$ tangent weight : $\hbar$, $-\hbar$

$\mathbb{C}^\times \curvearrowright V$ of weight $(0,-1)$ $\quad$ at y_1, y_2

$x_0 = 0 \quad y_1$ | $\mathbb{P}^1 \ni [x_0 : x_1]$

$\mathbb{C}^\times$

$x_1 = 0 \quad y_2$

Introduce $L_d \subseteq M_{0,2}(\mathbb{P}^1 \times \mathbb{P}^m, (1,d))$

$\quad = ev_1^{-1}(y_1 \times \mathbb{P}^m) \cap ev_2^{-1}(y_2 \times \mathbb{P}^m)$

$$\rightsquigarrow f : L_d = ev_1^{-1}(y_1 \times \mathbb{P}^m) \cap ev_2^{-1}(y_2 \times \mathbb{P}^m) \longrightarrow L_d'$$

<u>Question</u> $\mathbb{C}^\times \times T$ fix loci ?

$E_d \geq H^0(\Sigma, \varphi_2^* \mathcal{O}_{\mathbb{P}^m}(m+1))$ $\quad$ rank $E_d = (m+1)d + 1$.

$\mathbb{C}^\times \times T$ equivalent bundle

$\downarrow \quad$ | fiber at φ

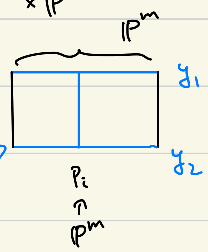
$L_d \ni \varphi = (\varphi_1, \varphi_2) : \Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^m$

(sketch) Compute $\Phi(z, e^t) = \sum_{d\geq 0} e^{dt} \int_{L_d} e^{z f^* p} \cdot e(E_d)$.

Fixed loci $\leftrightarrow$ $\Gamma = (i, \Gamma_1, \Gamma_2)$ $\qquad$ $\varphi : (\Sigma, \bar{x}_1, \bar{x}_2) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^m$

$i \in [0,m]$ $\qquad$ $\Sigma = \Sigma_1 \cup \Sigma_m \cup \Sigma_2$, $\quad \bar{x}_1 \in \Sigma_1$, $\quad \Sigma_2 \ni \bar{x}_2$

$\mathbb{P}^1 \times \mathbb{P}^m \ni p_i$ $\qquad$ $\Sigma_i \longmapsto y_i$



$\overline{M}_\Gamma \simeq \overline{M}_{\Gamma_1} \times \overline{M}_{\Gamma_2}$ $\qquad$ $f(\overline{M}_\Gamma) = [\Sigma_1 \otimes x_0^{d_2}, \Sigma_2 \otimes x_1^{d_1}]$

$d_1 + d_2 = d$. $\qquad$ $\rightsquigarrow f^* p$ is of pure weight $\alpha_i + d_1 \hbar$.

$$\mathrm{Cont}_\Gamma = \frac{(m+1)\alpha_i}{\prod_{j\neq i}(\alpha_j - \alpha_i)} \underbrace{e^{\alpha_i z} \cdot e^{z\hbar d_1}}_{(**)}$$

$$e^{d_1 t}\, \mathrm{Cont}_{\Gamma_1}\left(\int_{\overline{M}_{d_1}} \frac{e(E_d')}{\hbar - \psi_2} ev_2^* \phi_1 \right) \quad e^{d_2 t}\, \mathrm{Cont}_{\Gamma_2}\left(\int_{\overline{M}_{d_2}} \frac{e(E_d')}{-\hbar - \psi_2} ev_2^*(\phi_i) \right)$$

$e(E_d)\big|_{\overline{M}_\Gamma}$ is of pure weight and factors as $(m+1)\alpha_i \cdot e(E_d')\big|_{\overline{M}_{\Gamma_1}} \cdot e(E_d')\big|_{\overline{M}_{\Gamma_2}}$.

$$\Rightarrow \Phi(z, e^t) = \sum_{i=0}^{m} \frac{(m+1)\alpha_i}{\prod_{j\neq i}(\alpha_i - \alpha_j)} e^{\alpha_i z} Z_i(e^{t+z\hbar}, \hbar) \cdot Z_i(e^t, -\hbar)$$

remarkable fact : $\hbar \longleftrightarrow \hbar$: formal
equivalent parameter

$\in H^{(m+1)d+1}_{\mathbb{C}^* \times T}(L_d')$ has $(m+1)(d+1)$ fixed points.

But now $\Phi(z, e^t) = \sum_{d\geq 0} e^{dt} \int_{L_d'} e^{pz} \, \overbrace{f_* e(E_d)}$

L_d': projective space $\mathbb{P}^{(m+1)(d+1)-1}$

$\Rightarrow f_* e(E_d) = E_d^Z(\hbar, \alpha, p) \in \mathbb{Q}[\hbar, \alpha, p]$, homogeneous degree $= (m+1)d + 1$.

Bott residue formula $\Rightarrow$

$$= \frac{1}{2\pi i} \oint e^{pz} \sum_{d\geq 0} \frac{e^{dt} E_d^Z(\hbar, \alpha, p)}{\prod_{j=0}^{m} \prod_{r=0}^{d} (p - \alpha_i - r\hbar)}$$

#

<u>uniqueness lemma</u>: Let $\underset{i\in[0,m]}{Y_i, \overline{Y}_i} \in \mathcal{P}$.

If $Y_i \equiv \overline{Y}_i \pmod{\hbar^{-2}} \Rightarrow Y_i = \overline{Y}_i$ for all i.

proof: Let $I_{id}, \overline{I}_{id}$ be the initial data.

Then, $Y_i = \sum_{d\geq 0} e^{dt}\left(I_{id}^{0} + \frac{I_{id}^{1}}{\hbar}\right) \mod \hbar^{-2}$, so does $\overline{Y}_i$.

$\Rightarrow I_{id}^{0} = \overline{I}_{id}^{0}$, $I_{id}^{1} = \overline{I}_{id}^{1}$ for all i, d

$\Rightarrow I_{i1} = \overline{I}_{i1}$ for all i.

<u>Claim</u> $I_{id} = \overline{I}_{id}$ by induction

If ture for all $i \in [0,m]$, $k \leq d-1$, then $N_{ik} = \overline{N}_{ik}$ by II (recursion).

$\Rightarrow \delta E_d = E_d^{Y} - E_d^{\overline{Y}} = 0$ at $p = \alpha_i + r\hbar$ for all $\underset{r\in[1,d-1]}{i\in[0,m]}$ by III: GPC.

$\Rightarrow \prod_{j=0}^{m}\prod_{r=1}^{d-1}(p-\alpha_j - r\hbar) \mid \delta E_d \overset{\text{check!}}{\Longrightarrow} \delta E_d(\alpha_i + d\hbar)$

$= (m+1)\alpha_i \prod_{j\neq i}\prod_{r=1}^{d}(\alpha_i - \alpha_j + r\hbar)\cdot \delta I_{id}$.

As polynomial in $\hbar \Rightarrow \hbar^{d-1} \mid \delta I_{id}$

But $\delta I_{id} = \sum_{j=0}^{d} \delta I_{id}^{j}\hbar^{d-j}$ also $\delta I_i^{0} = 0 = \delta \overline{I}_i^{0}$

$\Rightarrow \delta I_d = 0$.