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Cantor. $Q_1 \supset Q_2 \supset Q_3 \supset \dots$ closed bounded $Q_i \neq \phi \ \forall i$

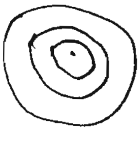
$\Rightarrow S := \bigcap_{i=1}^{\infty} Q_i \neq \phi$

let $x_i \in Q_i \subset Q_1 \Rightarrow \exists$ accum pt p

for any i, $p \in Q_i$: if not, $\exists \ B(p) \cap Q_i = \phi$

hence $B(p) \cap Q_j = \phi \ \forall j \geq i$

not accum pt. *

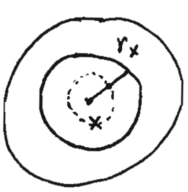


$\Rightarrow p \in S$.

Lindelöf. $A \subset \bigcup_{S \in F} S \Rightarrow A \subset$ countable subcover.

$G = \{A_1, A_2, \dots\}$ $A_k = B(\underset{\mathbb{Q}^n}{\underset{\cap}{x_k}}, \underset{\mathbb{Q}^+}{\underset{\cap}{r_k}})$

$x \in A \Rightarrow x \in S_\alpha$



$r_x \in \mathbb{Q}^+$; $\underset{\mathbb{Q}^n}{\underset{\cap}{y}} \in B(x, r_x/2) \Rightarrow x \in B(y, r_x/2) = A_{k_x}$

get subset in G

⇝ in F.

Heine-Borel. $\underset{\text{closed + bdd}}{A} \subset \overset{\text{open}}{\bigcup I_\alpha}$ in $\mathbb{R}^n \Rightarrow A \subset I_1 \cup \dots \cup I_m$.

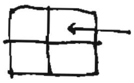
pf ① : $A \subset \bigcup_{k=1}^{\infty} I_k$ let $S_m = I_1 \cup \dots \cup I_m$

if $A \not\subset S_m$ then $\exists \ x_m \in A \setminus S_m =: Q_m$ closed bdd

↘ in m

Cantor $\Rightarrow \phi \neq \bigcap (A \setminus S_m) = A \setminus \bigcup S_m = \phi$ *

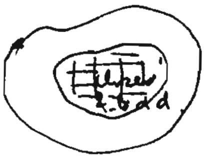
② : direct pf



$\exists$ a part without finite subcover

get accumulation pt $p \in I_\alpha$ some α.

Cor. distance of sets.



open

I_α

Thm: Cpt set in $\mathbb{R}^n$ (finite subcover property)

S cpt $\Leftrightarrow$ S closed + bdd $\Leftrightarrow$ infinite $x_i \in S$ has accum pt in S

$\xleftarrow{\text{H-B}}$ $\xrightarrow{\text{B-W}}$

$\longrightarrow$

S bdd : if not, $B(0,n)$ no cover

S closed : if not, $\exists \ p$ accum pt of S $\notin S$

then $\bigcup_{x \in S} B(x, \frac{|x-p|}{2})$ has no finite sub. *.

$\longleftarrow$: S bdd. if not $\exists \ |x_{k+1}| > |x_k| + 1$ *

S closed. if not, $\exists \ p$: accum of S but $p \notin S$ *

↘ hence accum of $\{x_i \in S\}$

Metric Spaces.

$d : S \times S \to \mathbb{R}^{\geq 0}$ St. $d(x,y) = 0 \iff x = y$

$d(x,y) = d(y,x)$

$d(x,y) + d(y,z) \geq d(x,z)$.

eg. $\mathbb{R}^n$ with $d(x,y) = \|x-y\|$ or $\max_i \{|x_i - y_i|\}$, or δ_{xy}, or

$B([a,b], \mathbb{R})$ with $d(f,g) = \sup|f-g|$ $\quad d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

Open ball $B_S(a,r) \rightsquigarrow$ open set $\rightsquigarrow$ closed set

interior pt $\quad$ adh pt. accum pt

subspace of metric space $S \subset M$: eg. $\mathbb{Z} \subset \mathbb{R}$, $\mathbb{Q} \subset \mathbb{R}$. open set = ? $\mathbb{R} \subset \mathbb{R}^2$?

fact : $X \subset S$ open (closed) $\iff$ $X = A \cap S$ for some $A \subset M$ open. (closed).

$\Leftarrow$: $p \in X$, A open $\Rightarrow$ $B_M(p,r) \subset A$ $\Rightarrow B_S(p,r) = B_M(p,r) \cap S \subset A \cap S = X$

$\Rightarrow$: $X = \bigcup_{x \in X} B_S(x, r_x) = \bigcup_{x \in X} (B_M(x, r_x) \cap S) = \left(\bigcup_{x \in X} B_M(x,r_x)\right) \cap S$.

for closedness : $X \subset S$ closed $\iff$ $S - X$ open $\iff$ $S - X = A \cap S$ $\quad A$ open

$X = B \cap S \iff S - B \cap S = (M-B) \cap S$ $\quad \overset{\|}{M-B}$ closed.

Thm' : $S \subset M$ metric space. TFAE

S closed $\iff$ $S \supset$ adh pt $\iff$ $S \supset$ acum pt $\iff$ $S = \bar{S}$

Now we consider those parts in $\mathbb{R}^n$ which really uses "$\mathbb{R}$".

Thm. $S \subset M$ metric space

S cpt $\xRightarrow{(1)}$ S closed + bdd

$\xRightarrow[(2)]{}$ Every ∞ subset of S has accum pt in S.

pf : (Same pt in $\mathbb{R}^n$) (1) S bdd : otherwise $\bigcup_{n=1}^{\infty} B(p,n)$ no finite sub.

S closed : if not, $\exists$ p accum to S, $p \notin S$

$S \subset \bigcup_{x \in S} B(p, \frac{1}{2} d(p,x)) = B(p, \frac{1}{2} d(p,x_1)) \cup \cdots \cup B(p, \frac{1}{2} d(p,x_n))$

let $r < \min\{\frac{1}{2} d(p,x_i)\}$ then $B(p,r) \cap S = \phi$ $\leftarrow\!\!\times$

New.

(2) Let $T = \{x_1, x_2, \cdots\} \subset S$. if no accum pt (in M, then must be in S : closed)

$\forall p \in M$, $\exists B(p, r_p)$ either $\cap T = \phi$

or $\cap T =$ single x_k (for $p = x_k$)

Then no finite sub cover. $\square$

Exercise : "$\Leftarrow$" of (2) holds !

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"$x_n \to p$" : $\forall \varepsilon > 0\ \exists N$ st. $n \geq N \Rightarrow d(x_n, p) < \varepsilon$

i.e. $d(x_n, p) \to 0$. Such p is unique.

$T = \{x_1, x_2, \dots\}$ is bounded & p is adh. to T (& conversely)

~~Cauchy sequence~~ : $\forall \varepsilon > 0\ \exists N$ st. $n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon$

$x_n \to p \Rightarrow \{x_n\}$ Cauchy

Def'n (S, d) <u>complete</u> if "$\Leftarrow$". eg. $[0,1), \mathbb{Q} \subset \mathbb{R}^1$ not complete; $\mathbb{R}^k$ is complete ($\Leftarrow$ B-W)

Thm: T cpt metric space $\Rightarrow$ complete.

let $\{x_n\}$ cs, $\Rightarrow \exists$ accum. pt. then same pf.

if not $\forall x \in T$, $\exists B(x)$ contains at most one x_n ~~x~~

Def'n : $f: \underset{\underset{S \ni p}{\cap}}{A} \to T$, $\lim\limits_{x \to p} f(x) = b \triangleq \forall \varepsilon > 0, \exists \delta$ st $0 < d_S(x, p) < \delta \Rightarrow d_T(f(x), b) < \varepsilon$.

Apostol requires: p acum. pt to A.

fact: $\Leftrightarrow \lim\limits_{n \to \infty} f(x_n) = b \quad \forall\ \underset{\nwarrow A \setminus \{p\}}{x_n} \to p$. ($\Leftarrow$ by pf by contrad.)

Eg. $T = \mathbb{R}, \mathbb{C}$ (4-rules), $T = \mathbb{R}^k$ (vector rules)

Def'n: f conti at $p \in A \triangleq p$ isolated or $\lim\limits_{x \to p} f(x) = f(p)$.

Thm: f conti $\Leftrightarrow f^{-1}(U)$ is open $\forall U$ open in T. $\Leftrightarrow$ "closed".

- V open $\not\Rightarrow f(V)$ open. eg $f(x) = x^2 : (-1, 1) \to [0, 1)$
  " closed $\not\Rightarrow$ " closed $\quad f(x) = \tan^{-1} x : [0, \infty) \to [0, \frac{\pi}{2})$.
- f, g conti $\Rightarrow f \circ g$ conti ($(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$ open)

Thm: V cpt $\Rightarrow f(V)$ cpt ($\Rightarrow f(V)$ closed + bdd)

Cor: $f: \underset{\searrow \text{cpt}}{A} \to \mathbb{R}$ conti $\Rightarrow \exists p, q \in A$, $f(p) = \inf\limits_A f$, $f(q) = \sup\limits_A f$.

Cor: $f: A \to T$ $\underset{\text{conti}}{\text{1-1.}}$ A cpt $\Rightarrow f^{-1}$ is conti on $f(A)$.
(need cpt $\Rightarrow$ closed in T. need T_2) ok for metric space

Eg. $[0, 1) \to \mathbb{C}$, $f(x) = e^{2\pi i x}$

9/27. Def'n : S <u>connected</u> $\triangleq S \neq A \amalg B \quad A, B \neq \phi$, open.

pre-Lindelöf $\Rightarrow \mathbb{R}, [a, b], (a, b)$ are connected. (Ex. $\Leftarrow$) $\Updownarrow$ $\forall\ S \xrightarrow{\text{conti}} \{0, 1\}$ is const.

Thm: $f: S \to T$ conti, $A \subset S$ conn $\Rightarrow f(A)$ conn.

In particular, for $T = \mathbb{R}$ get Intermediate Value Thm.

Fact: A_α conn. $\cap A_\alpha \neq \phi \Rightarrow \cup A_\alpha$ conn.

$\forall x$, $\exists\, U(x)$ the max conn subset $\ni x$

either $U(x) \cap U(y) = \phi$ or $U(x) = U(y)$ $>$ conn. components

$\Rightarrow S = \amalg\, U(x)$. Can we refine it?

Def'n: S is arcwise conn. if $\forall\, p, q \in S$ $\quad p \sim q$

$\exists$ conti function (curve) $f : [0,1] \to S$, $f(0) = p$, $f(1) = q$.

Fact: path conn $\Rightarrow$ conn:
$$\begin{tikzcd} & S \arrow[r, "g"] & \{0,1\} \\ {[0,1]} \arrow[ur] \arrow[urr, dashed] & & \end{tikzcd}$$
$\Rightarrow g(p) = g(q)$.

converse is not true:

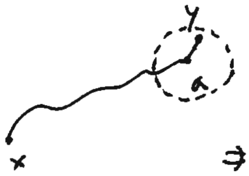
$y = \sin\frac{1}{x} \subset \mathbb{R}^2$

Thm: $S \subset \mathbb{R}^n$ open conn. $\Rightarrow$ path conn.

pf: (~~Method of Continuity~~)

let $x \in S$. $A := \{ y \in S \mid y \sim x \}$; $S = A \amalg B$

A is open: if $a \in A$



$\exists\, B(a) \subset S$

any $y \in B(a)$ has $y \sim a$

$\Rightarrow y \sim a \sim x$ ie. $y \in A$

B is open: if $b \in A$, $\exists\, B(b) \subset S$. if $\exists\, y \in B(b)$ st $y \sim x$

then $b \sim y \sim x$ * hence $B(b) \subset B$.

but S is conn. hence we must have $B = \phi$. done $\square$

Rmk: May replace $\mathbb{R}^n$ by metric space, even top space

st. S conn & ~~locally path conn~~.

for $\mathbb{R}^n$ may assume the path being piece wise-linear.

ie. ~~polygonal~~

Thm: Every $S \subset \mathbb{R}^n$ open, is a countable union

of (disjoint) open connected set.

pf: $S = \amalg\, U(x)$, S open $\Rightarrow \underset{T=}{U(x)}$ open : $y \in U(x) \Rightarrow U(x) = U(y)$

$\exists\, B(y) \subset S$, $B(y)$ connected $\Rightarrow B(y) \subset T$.

so this is a disjoint open cover $\xrightarrow[\text{Lindelöf}]{}$ countable $\square$.

Def'n: A region = conn. open set $\cup$ some boundary pts.

open region $\equiv$ domain $\equiv$ conn. open set.

9/29. $f : S \to T$ unif. conti on A if

$\forall \varepsilon > 0$, $\exists \delta$ st. $x, y \in A$, $d_S(x,y) < \delta \Rightarrow d_T(f(x), f(y)) < \varepsilon$.

Thm: If f conti on A & A cpt then $\Rightarrow$ unif. conti.

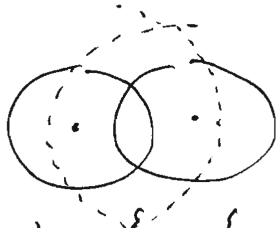
pf: Given $\varepsilon > 0$, $a \in A \Rightarrow \exists B_S(a, r_a)$, $d_T(f(x), f(a)) < \varepsilon/2$ for all $x \in B_S(a, r_a) \cap A$

$\Rightarrow A \subset \bigcup_{a \in A} B_S(a, \frac{r_a}{2}) \Rightarrow A \subset B_S(a_1, \frac{r_1}{2}) \cup \cdots B_S(a_n, \frac{r_n}{2})$

let $\delta = \min\{\frac{r_1}{2}, \cdots, \frac{r_n}{2}\}$.

for any $d_S(x,y) < \delta$, say $x \in B_S(a_k, \frac{r_k}{2})$

then $d_S(y, a_k) < \delta + \frac{r_k}{2} \le r_k$. i.e. y



$\Rightarrow d(f(x), f(y)) \le d(f(x), f(a_k)) + d(f(a_k), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ✱

Fixed pt thm: If (S, d) complete, $f : S \to S$ st.

$d(f(x), f(y)) \le \alpha\, d(x,y)$. $\alpha < 1$, Then $\exists!\, p$ st $f(p) = p$.

pf: Take $x_0 = a \in S$ any. $x_{n+1} = f(x_n)$

claim: $x_0, x_1, x_2 \ldots$ is Cauchy: $d(x_{n+1}, x_n) \le \alpha \cdot d(x_n, x_{n-1})$

$\le \cdots \le \alpha^n \cdot d(x_1, x_0)$

hence $d(x_m, x_0) \le \sum_{i=0}^{m-1} d(x_{i+1}, x_i) \le (\alpha^{m-1} + \cdots + 1)\, d(x_1, x_0)$

$d(x_{n+m}, x_n) \le \alpha^n \cdot d(x_m, x_0)$ $\qquad = \frac{1-\alpha^m}{1-\alpha} d(x_1, x_0) < \frac{d(x_1, x_0)}{1-\alpha}$

$\le \frac{\alpha^n}{1-\alpha} d(x_1, x_0) \longrightarrow 0$ as $n \to \infty$.

let $x_n \to p$. $\Rightarrow \lim_{n \to \infty} f(x_n) = f(p)$

$\qquad\qquad\qquad \overset{\shortparallel}{\lim_{n \to \infty}} x_{n+1} = p$ $\qquad$ i.e. $f(p) = p$.

Uniqueness: $f(p) = p$. $f(q) = q \Rightarrow d(f(p), f(q)) \le \alpha\, d(p, q)$

$\qquad\qquad\qquad \overset{\shortparallel}{d(p,q)} \Rightarrow d(p,q) = 0$ ✱

For functions on $\mathbb{R}$. Say $f : (a,b) \to \mathbb{R}$

define $f(c+) = \lim_{x \searrow c} f(x)$ if it exists.

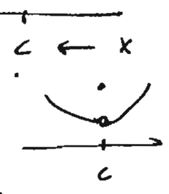
if $c \in (a,b)$ then f cont. at $c \Leftrightarrow f(c+) = f(c) = f(c-)$

if $f(c+) = f(c-) \neq f(c)$

then "removable ~~sing~~ discontinuity".

Also have:

monotone functions.



Apostol Ch5 Derivatives

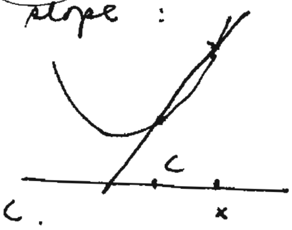
(*) $f'(c)$ exists at $c \in (a,b) \iff \exists f^*(x)$ conti at $x = c$ st $f(x) - f(c) = (x-c) f^*(x)$

and in fact $f^*(c) = f'(c)$.

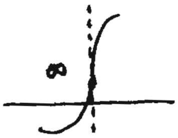
$f^* =$ slope :

$\Rightarrow$ 4 rules ; chain rule

$$g(f(x)) - g(f(c)) = (f(x) - f(c))\, g^*(f(x))$$
$$= g^*(f(x))\, f^*(x)\,(x-c) \quad , \text{ now let } x \to c.$$



1-side & ∞ derivative : $f'(c)$ exists $\iff f'_+(c) = f'_-(c) \in [-\infty, \infty]$.



fact : f has local max/min $\Rightarrow f'(c) = 0$.

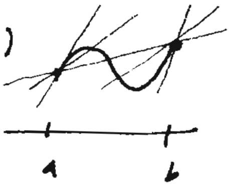
MVT : (1) Rolle : $f \in C([a,b])$, f' exists on (a,b); $\underline{\in [-\infty, \infty]}$
$f(a) = f(b) \Rightarrow \exists\, c \in (a,b).\ f'(c) = 0$.

(2) $f(b) - f(a) = f'(c)(b-a)$

(3) if f', g' not both infinite : $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$

Cor : $f' \in (0, \infty] \Rightarrow f \nearrow$, $f' \equiv 0 \Rightarrow f = \text{const}$.

~~Intermediate Value Thm for f'~~ : Any $m \in f'_+(a), f'_-(b)$ is achieved : $m = f'(c)$. ($\Leftarrow$ slope is conti.)



Cor : f' exists and monotone on $(a,b) \Rightarrow f'$ conti.

pf : if not conti at c, then f' omit the jump $f'(c-) \le f'(c) \le f'(c+)$

(4) Taylor : $f, g \in C^{n-1}([a,b])$, $f^{(n)}, g^{(n)}$ exist and finite on $\overset{*}{(a,b)}$

then
$$\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) g^{(n)}(\xi) \qquad \exists\, \xi \in (x,c).$$
$$= \left(g(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) f^{(n)}(\xi)$$

pf :
$$\frac{f(x) - T_{n-1}f(x)}{g(x) - T_{n-1}g(x)} = \frac{F(x)}{G(x)} = \frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(x_1)}{G'(x_1)} = \cdots = \frac{f^{(n)}(x_n)}{g^{(n)}(x_n)}$$

Partial derivatives : $f : \underset{\text{open}}{S \subseteq \mathbb{R}^n} \to \mathbb{R}$ $\quad D_k f(c) = f_k(c) = \frac{\partial f}{\partial x_k}(c)$

complex functions : $f : S \subset \mathbb{C} \to \mathbb{C}$ (*) still holds, 4 rule, chain.

Cauchy-Riemann eq^n : $f = u + iv$ $f'(c)$ exists $\Rightarrow \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$

Cor : f' exsist on $D \subset \mathbb{C}$, f is const if
u or v or $|f| = \text{const}$, or $f' \equiv 0$

eg. $|f|^2 = u^2 + v^2 = c \Rightarrow u u_x + v v_x = 0 \Rightarrow u v_y = v u_y$ (C.R.)
$u u_y + v v_y = 0$ ie. $(u^2+v^2) v_y = 0 \ldots$ #

Ch 6. BV functions. eg. f ↗ on $[a,b]$.

fact: f Lipschitz on $[a,b]$ (eg. $|f'| \le A$) $\Rightarrow$ $f \in BV([a,b])$.

Def'n: $V_f(a,b) = \sup_{p \in P[a,b]} \{ \sum |\Delta f_k| \}$

4 rules : $V_{f \pm g} \le V_f + V_g$, $V_{fg} \le \sup|f| \cdot V_g + \sup|g| \cdot V_f$

if $|f| \ge m > 0$ then $V_{1/f} \le V_f / m^2$.

Additivity : $a < c < b \Rightarrow V_f(a,b) = V_f(a,c) + V_f(c,b)$.

Thm: let $f \in BV[a,b]$, $V(x) = V_f(a,x)$, $x \in [a,b]$, then
V ↗ and $V - f$ ↗.

pf: $V(x+h) = V(x) + V_f(x, x+h) \ge V(x)$

Also $[V(x+h) - f(x+h)] - (V(x) - f(x)) = V_f(x, x+h) - (f(x+h) - f(x)) \ge 0$

Cor: $f \in BV[a,b] \Leftrightarrow f = f_1 - f_2$; f_i ↗. (could be strictly)

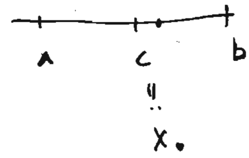
"⇒" $f = V - (V - f) =: f_1 - f_2 = (f_1 + x) - (f_2 + x)$.

Thm: let $f \in BV[a,b]$, f conti at x $\Leftrightarrow$ V conti at x.

pf: ⇐: $x < y \Rightarrow |f(y) - f(x)| \le V(y) - V(x)$. Let $y \to c+$, $x \to c-$.

⇒: Given $\varepsilon > 0$,

$\exists \delta$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon/2$



$\exists p \in P[c,b]$ $V_f(c,b) < \sum|\Delta f_k| + \varepsilon/2$ wrt p.

may assume $x_1 - c < \delta$.

$\Rightarrow$ $V_f(c,b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \sum_{k=2}^{n} |\Delta f_k| < \varepsilon + V_f(x_1, b)$

ie. $V(x_1) - V(c) = V_f(c, x_1) = V_f(c,b) - V_f(x_1, b) < \varepsilon$

ie. $V(c+) = V(c)$. Similarly, $= V(c-)$ *

Cor : $f \in C$ & BV on $[a,b]$ $\Leftrightarrow$ $f = f_1 - f_2$. f_i ↗ + conti.

Rectifiable paths (curve) & arc length :

$\vec{f} : [a,b] \to \mathbb{R}^n$, $\Lambda_f(p) := \sum_{k=1}^{n} \| \Delta \vec{f}_k \|$, $\Lambda_f(a,b) := \sup_{p \in P[a,b]} \Lambda_f(p)$.

conti \ image ⊂ cpt. connected curve.

Fact: $V_{f_i}(a,b) \le \Lambda_{\vec{f}}(a,b) \le \sum_{i=1}^{n} V_{f_i}(a,b)$, $\vec{f} = (f_1, \cdots, f_n)$.
$\forall i$.

Additivity of Λ_f $\Rightarrow$ arc length $s(x) := \Lambda_{\vec{f}}(a,x)$ ↗ and conti.

Change parameters: $\vec{f}, \vec{g}$ equiv. ie. $g(t) = f(u(t))$: u conti. str. monotone

thm. f, g 1-1, equiv. $\Longleftrightarrow$ f, g have the same curve (graph).

pf: f^{-1} exists & conti on image, let $u(t) = f^{-1}(g(t))$ *.

Def'n: $S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$ $\quad P : \underset{a}{\overset{\|}{x_0}} < x_1 < \cdots < \underset{b}{\overset{\|}{x_n}} \in \mathcal{P}[a,b]$.

$f \in R(\alpha)$ on $[a,b]$ if $\exists A$ st. $\forall \varepsilon > 0\ \exists P_\varepsilon$ st. $| S(P, f, \alpha) - A | < \varepsilon$ for any choice of $t_k \in [x_{k-1}, x_k]$ and $P \geq P_\varepsilon$.

A exists ⇒ unique $\equiv \int_a^b f\, d\alpha$.

This is very weak assumption.

Linearity: (1) $f \in R(\alpha)$, $g \in R(\alpha) \Rightarrow c_1 f + c_2 g \in R(\alpha)$

(2) $f \in R(\alpha)$ & $R(\beta) \Rightarrow f \in R(c_1\alpha + c_2\beta)$.

Pf of (2): Given $\varepsilon > 0$, $\exists P'_\varepsilon$ $\ P \geq P'_\varepsilon \Rightarrow | S(P, f, \alpha) - \int_a^b f\, d\alpha | < \varepsilon$

$\exists P''_\varepsilon$ $\ P \geq P''_\varepsilon \Rightarrow | S(P, f, \beta) - \int_a^b f\, d\beta | < \varepsilon$

for such $P \geq P_\varepsilon := P'_\varepsilon \cup P''_\varepsilon \Rightarrow | S(P, f, c_1\alpha + c_2\beta) - c_1\int f\, d\alpha - c_2 \int f\, d\beta |$ $< (|c_1| + |c_2|)\varepsilon$ ✻.

Thm: $a < c < b$. If 3 int in (1) has 2 exist the 3rd exists and $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$.

pf: Say I, III exists. refine partition P_I, P_{III} to produce P_{II}. ✻

Rmk: In fact only need to assume III exists. ↑ require Riemann's condi. $\leq U - L < \varepsilon$. for $\alpha \in BV$,

Def'n: $\int_b^a = -\int_a^b$ if $b \geq a$.

Thm (Integration by parts): $f \in R(\alpha) \Rightarrow \alpha \in R(f)$ on $[a,b]$ &

$$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a) =: A$$

Pf: Given $\varepsilon > 0$, $\exists P_\varepsilon$ st. $| S(P', f, \alpha) - \int f\, d\alpha | < \varepsilon$ $\ \forall P' \geq P_\varepsilon$

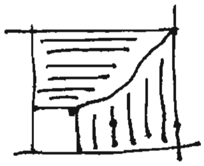
consider $S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\Delta f_k = \sum \alpha(t_k) f(x_k) - \sum \alpha(t_k) f(x_{k-1})$

write. $A = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1})$

$\Rightarrow A - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)(\alpha(x_k) - \alpha(t_k)) + f(x_{k-1})(\alpha(t_k) - \alpha(x_{k-1}))$

choose $P \geq P_\varepsilon$ and $P' = P \cup \{t_1, \cdots, t_n\} \geq P_\varepsilon$

the RHS is a case of $S(P', f, \alpha)$, hence



$| A - S(P, \alpha, f) - \int f\, d\alpha | < \varepsilon$ $\quad \forall P \geq P_\varepsilon$ and t_k arb.

i.e. $\int_a^b \alpha\, df$ exists and $= A - \int f\, d\alpha$. □

10/11, 2011

~~Fact (CVF)~~: g strictly ↑, conti $[a,b] \longleftrightarrow [c,d]$

$$f \in R(\alpha) \Rightarrow f\circ g \in R(\alpha\circ g) \ \& \ \int_a^b f\,d\alpha = \int_c^d f\circ g \, d(\alpha\circ g).$$

the pf is trivial by def. but interesting to compare:

Two fundamtal examples:

① <u>Thm:</u> let $\alpha \in C'([a,b])$, $f \in R(\alpha)$, then $f\alpha'$ Riem. int:ble

$$\int f\,d\alpha = \int f\alpha'\,dx.$$

pf: $|S(P,f,\alpha) - S(P,f\alpha')| = \left|\sum_{k=1}^{n} f(t_k)\cdot\left(\Delta\alpha_k - \alpha'(t_k)\Delta x_k\right)\right|$

use the same P, t_k in

$$\Delta\alpha_k = (\alpha'(t_k') - \alpha'(t_k)).\Delta x_k$$

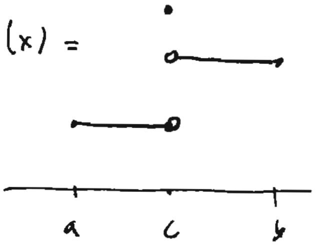
$< M\cdot(b-a)\cdot\varepsilon_1$ for any P, $\|P\| < \delta$

But $\exists P_\varepsilon$. $P \geq P_\varepsilon \Rightarrow \left|S(P,f,\alpha) - \int_a^b f\,d\alpha\right| < \varepsilon/2$

Pick $M(b-a)\cdot\varepsilon_1 < \varepsilon/2$ done ✱.

② ~~Another extreme: step functions~~ and sum.

let $\alpha(x) =$



<u>Thm.</u> Assume that one of f, α is right conti at c; also one left conti at c. Then $f \in R(\alpha)$ and

$$\int_a^b f\,d\alpha = f(c).(\alpha(c+) - \alpha(c-)).$$

pf: let $c \in P \in \mathcal{P}[a,b]$. Then $S(P,f,\alpha) = f(t_{k-1})(\alpha(c) - \alpha(c-)) + f(t_k)(\alpha(c+) - \alpha(c))$,

$$|\Delta| := |S(P,f,\alpha) - f(c)\cdot(\alpha(c+) - \alpha(c-))|$$
$$\leq |f(t_{k-1}) - f(c)|\cdot|\alpha(c) - \alpha(c-)| + |f(t_k) - f(c)|\cdot|\alpha(c+) - \alpha(c)|$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ st. $\forall P \ni \{c\}$ and $\|P\| < \delta \Rightarrow |\Delta| < \varepsilon$ ✱

~~Cor.~~ For a step function α wrt P (ie. const on (x_{k-1}, x_k), $\forall k$) st. f, α are not both disconti at c from L or R, then

$$\int_a^b f\,d\alpha = \sum_{k=0}^{n} f(x_k)\,\alpha_k, \quad \text{where } \alpha_k = \alpha(x_k+) - \alpha(x_k-) \quad k \neq 0, n$$

jump: $\alpha_0 = \alpha(x_0+) - \alpha(x_0)$

$\alpha_n = \alpha(x_n) - \alpha(x_n-)$.

eg. $\alpha(x) = [x]$, $f(x) = a_k$ for $(k-1, k]$, $f(0) = 0 \Rightarrow \sum_{k=1}^{n} a_k = \int_0^n f(x)\,d[x]$.

~~Thm (Euler summation formula)~~ let $\{x\} := x - [x]$. $f' \in C([a,b])$

③ then $$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\{x\}\,dx - f(x)\{x\}\Big|_a^b.$$

Riemann-Darboux criterion:

<u>Thm:</u> For α ↑ (i) $f \in R(\alpha) \Leftrightarrow$ (ii) $\forall \varepsilon > 0\ \exists P_\varepsilon$ st $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ $\forall P \geq P_\varepsilon$. $\Leftrightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$.

Thm: let $\alpha \in BV([a,b])$ then $f \in R(\alpha) \Rightarrow f \in R(V)$.

pf: Given $\varepsilon > 0$, will find P_ε st $P \geq P_\varepsilon \Rightarrow$

$U(P,f,V) - L(P,f,V) < \varepsilon$.

$$\overset{\shortparallel}{\sum_{k=1}^{n}} \left(M_k(f) - m_k(f)\right) . \big(\underbrace{\Delta V_k - |\Delta\alpha_k|}_{①} + \underbrace{|\Delta\alpha_k|}_{②} \big)$$

$$\overset{\wedge\shortparallel}{2M} \left(V(b) - \sum_{k=1}^{n} |\Delta\alpha_k| \right)$$

where $|f| \leq M$. $\overset{\wedge}{\varepsilon/2}$ ↘ cond. on P_ε

for ②, $f \in R(\alpha) \Rightarrow \left| \sum_{k=1}^{n} \left(f(t_k) - f(t_k')\right) \Delta\alpha_k \right| < \frac{\varepsilon}{4}$ $\forall t_k, t_k' \in [x_{k-1}, x_k]$

↖ compare both with $\int_a^b f \, d\alpha$.

if $\Delta\alpha_k \geqslant 0$, pick t_k, t_k' st. $M_k(f) - m_k(f) < f(t_k) - f(t_k') + h$

$\Delta\alpha_k < 0$ 〃 $M_k(f) - m_k(f) < f(t_k') - f(t_k) + h$

then ① $\leq \sum_{k=1}^{n} \left(f(t_k) - f(t_k')\right) \cdot \Delta\alpha_k + h \cdot \sum_{k=1}^{n} |\Delta\alpha_k| < \varepsilon/2$.

$\qquad\qquad\qquad \overset{\wedge}{\varepsilon/4} \qquad\qquad \overset{\wedge}{h \cdot V(b)}$ pick h st. $< \varepsilon/4$

~~Thm~~ **Cor.** $\alpha \in BV([a,b])$, $f \in R(\alpha)$ on $[a,b] \Rightarrow$ on $[c,d] \subset [a,b]$. ※

pf: $\alpha = V - (V - \alpha)$, $f \in R(\alpha) \Rightarrow f \in R(V) \Rightarrow f \in R(V-\alpha)$

enough to prove the thm for $\alpha \nearrow$. and for $[a,c] \subset [a,b]$

$\forall \varepsilon > 0$, $\exists P_\varepsilon \in \mathcal{P}[a,b]$, $P \geq P_\varepsilon \Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

may assume $c \in P_\varepsilon$ and let $P_\varepsilon' = P_\varepsilon \cap [a,c]$

then any $P' \geq P_\varepsilon' \Rightarrow P := P' \cup P_\varepsilon \geq P_\varepsilon$

$\Rightarrow U(P',f,\alpha) - L(P',f,\alpha) < U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

Basic existence criterion ‸ on $[a,c]$ ※.

Thm. $f \in C([a,b])$, $\alpha \in BV([a,b]) \Rightarrow f \in R(\alpha)$.

pf: May assume $\alpha \nearrow$. $\forall \varepsilon > 0$, $\exists \delta > 0$ st. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Pick P_ε with $\|P_\varepsilon\| < \delta$, then $\forall P \geq P_\varepsilon$: $M_k(f) - m_k(f) \leq \varepsilon$.

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) \leq \varepsilon \cdot \sum_{k=1}^{n} \Delta\alpha_k = \varepsilon \cdot (\alpha(b) - \alpha(a))$

i.e. Riemann condi holds ※

Cor. True for $f \in BV$, $\alpha \in C$.

MVT ① $\alpha \uparrow$, $f \in R(\alpha) \Rightarrow \int_a^b f\,d\alpha = c\int_a^b d\alpha$; $f \in C \Rightarrow c = f(x_0)$

$c \in [\inf f, \sup f]$

② $\alpha \in C$, $f \uparrow$ then $\exists\, x_0 \in [a,b]$ st.

$$\int_a^b f\,d\alpha = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$$. (using int. by parts)

Fund. Thm. of Calculus

① Let $\alpha \in BV([a,b])$, $f \in R(\alpha)$; Define $F(x) = \int_a^x f\,d\alpha$, $x \in [a,b]$. then $F \in BV$, same continuity as α. Moreover if $\alpha \uparrow$; α' exist + f conti at x $\Rightarrow$ $F'(x) = f\alpha'(x)$

pf: May assume $\alpha \uparrow$. Then $F(y) - F(x) = \int_x^y f\,d\alpha = c\,(\alpha(y) - \alpha(x))$ by MVT ①

now divide $y-x$ and $y \to x$ get $f(x)\,\alpha'(x)$ *

② Let g' exists & $\in R$ on $[a,b]$ $\Rightarrow$ $\int_a^b g'(x)\,dx = g(b) - g(a)$

pf: $g(b) - g(a) = \sum_{k=1}^n g(x_k) - g(x_{k-1}) = \sum_{k-1}^n g'(t_k)\,\Delta x_k = S(P, g')$ *

Application: $f \in R$, α conti & $\alpha' \in R$, then

$\int_a^b f\,d\alpha$, $\int_a^b f\alpha'$ both exist and equal.

This is Apostol : Thm 26 + Thm 33 + Thm 35. (**)

pf: $^{(*)}\int_a^b f\alpha'$ exists. Denote $g = \alpha'$ bonded by M.

$$\left| S(P,f,\alpha) - \int_a^b f\alpha' \right| = \left| \sum_{k=1}^n f(t_k)\int_{x_{k-1}}^{x_k} \alpha' - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t)\,\alpha'(t)\,dt \right|$$

$$= \left| \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(t_k) - f(t))\,\alpha'(t)\,dt \right| \le M \cdot \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (M_k(f) - m_k(f))\,dt$$

$f \in R \Rightarrow$ estimate * $= M(U(P,f) - L(P,f))$.

CVF: Let $g \in C^1([c,d])$, $f \in C(\underset{[a,b]}{g[c,d]})$. Then

$$\int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f(g(t))\,g'(t)\,dt.$$

pf: Define $F(u) = \int_{g(c)}^u f(x)\,dx$; $G(x) = \int_c^x f(g(t))\,g'(t)\,dt$

Then $F(g(x))' = G'(x)$ and $F(g(c)) = G(c) = 0$ *

by FTC ①, need conti of f & g'.

(*): Facts for $\alpha \uparrow$:

$f \in R(\alpha) \Rightarrow f^2 \in R(\alpha)$: $M_k(f^2) - m_k(f^2) \le 2M.(M_k(f) - m_k(f))$.

$f, g \in R(\alpha) \Rightarrow f \cdot g \in R(\alpha)$: $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$.

(**) Thm 7.26: $f, g \in R(\alpha)$, $\alpha \in BV([a,b])$ $\Rightarrow$ $\int_a^b fg\,d\alpha = \int_a^b f\,dG$; $G(x) := \int_a^x g\,d\alpha$

exists and

10/18. R-S int with parameter

Thm : let $f \in C(Q)$, $Q = [a,b] \times [c,d]$, $\alpha \in BV([a,b])$. Then

$$F(y) := \int_a^b f(x,y)\, d\alpha(x) , \quad y \in [c,d] \quad \text{is conti in } y .$$

Pf: May assume $\alpha \uparrow$. Then whenever $|y-y'| < \delta \Rightarrow$

$$|F(y) - F(y')| = \left| \int_a^b (f(x,y) - f(x,y'))\, d\alpha(x) \right| \le \epsilon (\alpha(b) - \alpha(a)) \quad ※$$

unif. conti of f on Q.

Thm′: True for $F(y) := \int_a^b f(x,y)\, g(x)\, dx$ where $g \in R$.

Pf : Write $F(y) = \int_a^b f(x,y)\, dG(x)$ with $G(x) = \int_a^x g(t)\, dt$ ※

Diff under integral sign :

Thm : If $D_2 f \in C(Q)$, then $F'(y)$ exists $= \int_a^b D_2 f(x,y)\, d\alpha(x)$.

Pf : $$\frac{F(y) - F(y_0)}{y - y_0} = \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0}\, d\alpha(x) = \int_a^b D_2 f(x, \bar{y})\, d\alpha(x)$$

now let $y \to y_0$ ※

changing order of integration :

Thm: Let $\alpha \in BV([a,b])$, $\beta \in BV([c,d])$; $f \in C(Q)$. Then

$$\int_a^b \left(\int_c^d f(x,y)\, d\beta(y) \right) d\alpha(x) = \int_c^d \left(\int_a^b f(x,y)\, d\alpha(x) \right) d\beta(y) .$$

Pf : May assume $\alpha, \beta \uparrow$. Use partition, mean value + unif. conti. ※
twice on x and y

~~Lebesgue criterion~~

~~Def~~: $S \subset \mathbb{R}$ has measure 0 if $\forall \varepsilon > 0$ $\exists$ countable cover $(a_k, b_k) = U_k$, $\bigcup_{k=1}^{\infty} U_k \supset S$ but $\sum |U_k| < \varepsilon$.

Fact: F_i measure 0 $\forall i = 1, 2, \dots \Rightarrow S = \bigcup_{i=1}^{\infty} F_i$ measure 0.

Ex 7.32. Cantor set is uncountable, with measure 0.

~~Def'n~~: Oscillation on T: $\Omega_f(T) = \sup(f(x) - f(y))$; $T \subset S$

at $x \in S$: $\omega_f(x) := \lim_{h \to 0+} \Omega_f(B(x,h) \cap S)$

Thm: Let $S = [a,b]$. If $\exists \varepsilon > 0$ st. $\omega_f(x) < \varepsilon$ $\forall x \in S$, Then $\exists \delta = \delta_\varepsilon > 0$ st. $\forall$ closed interval $T \subset S$ with $|T| < \delta$ we have $\Omega_f(T) < \varepsilon$.

Pf: $\forall x \in S$, $\exists B_x = B(x, \delta_x)$ st. $\Omega_f(B_x \cap S) < \varepsilon$.

$\bigcup B(x, \delta_x/2) \supset S$ (cpt) $\Rightarrow$ $S \subset B(x_1, \frac{\delta_1}{2}) \cup \dots \cup B(x_k, \frac{\delta_k}{2})$

let $\delta = \min\{\delta_1/2, \dots, \delta_k/2\}$.

Given T, $|T| < \delta$. then $T \cap B(x_i, \frac{\delta_i}{2})$ for some i

but then $T \subset B(x_i, \delta_i)$ $\ast$

~~Fact~~: The set $J_\varepsilon = \{x \in [a,b] \mid \omega_f(x) \geq \varepsilon\}$ is closed (hence cpt).

pf: If $x \notin J_\varepsilon$ ie. $\omega_f(x) < \varepsilon$, then $\Omega_f(B_x \cap [a,b]) < \varepsilon$ for some B_x but then $B_x \cap J_\varepsilon = \phi$ $\ast$

~~Theorem (Lebesgue)~~: Let $f \in B([a,b])$ then $f \in R \iff D := Disc.(f)$ has $m = 0$.

pf: $\Rightarrow$: if $m(D) \neq 0$, then some $D_r := \{x \mid \omega_f(x) \geq 1/r\}$ has $m \neq 0$.

Any $P \in \mathcal{P}[a,b] \Rightarrow U(P,f) - L(P,f) = S_1 + S_2 \geq S_1$

sum on $[x_{k-1}, x_k]$ st. $(x_{k-1}, x_k) \cap D \neq \phi$

$\exists \varepsilon > 0$, any count. open cover of D_r has length $\geq \varepsilon$.

difference is only finite pts.

$\Rightarrow S_1 \geq \frac{\varepsilon}{r}$ $\forall P$ $\ast$

$\Leftarrow$: For P_ε to be determined later: $P \geq P_\varepsilon$

$U(P,f) - L(P,f) = S_1 + S_2$

$m(D_r) = 0$, D_r cpt (by fact) covered by finite U_i, $\sum |U_i| < \varepsilon_1$ st. $S_1 < (M - m)\varepsilon_1 < \frac{\varepsilon}{2}$

for S_2: subdivide $[a,b] - \bigcup U_i$ (cpt. set) into subintervals T_j st. $\Omega_f(T_j) < \frac{1}{r}$. $\exists$ finite partition $\Rightarrow S_2 < \frac{b-a}{r} < \frac{\varepsilon}{2}$

choose r.

This gives P_ε and then $S_1 + S_2 < \varepsilon$ $\ast$

Ch 8 ∞-Series / products. 10/20.

NO.

DATE

convergence in ℝ or ℂ / Cauchy sequence criterion

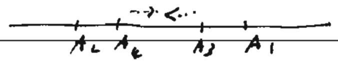
monotone < M ; lim sup, lim inf in ℝ.

sequence $a_i \mapsto$ series $A_k = a_1 + \cdots + a_k$; $a_k = A_k - A_{k-1} \to 0$.

+ - ( ) : Fact + ( ) ok. Thm : -( ) ok if length bounded & $a_k \to 0$.

pf: $|A_n - B| \le |B_{m+1} - B| + |B_{m+1} - A_n|$ $b_{n+1} := a_{p(n)+1} + \cdots + a_{p(n+1)}$

$\le (p(m+1) - p(m)) \cdot \varepsilon_1$ for $n \gg 0$.

Fact : alternating series :  in ℝ.

- Abs / conditional convergence. Abs conv ⇒ conv. (Cauchy) in ℂ

Tests : (for $a_i > 0$) comparison / integral

Abs. conv. test in ℂ : Ratio $r = \underline{\lim}\left|\frac{a_{n+1}}{a_n}\right|$; $R = \overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| < 1$

Root: $\rho = \overline{\lim}\sqrt[n]{|a_n|} \lessgtr 1$. $\vee$ 1 div. ⇒ abs. conv.

- Abel's partial sum formula: $\sum_1^n \underset{\Delta A_k}{a_k} b_k = A_n b_{n+1} - \sum_1^n A_k \cdot \underset{\Delta b_{k+1}}{(b_{k+1} - b_k)}$ . . . . .

Dirichlet test : For ℂ :

$\sum a_n$ has bounded A_n ; $b_n \searrow 0$ ⇒ $\sum a_n b_n$ conv.

eg. $\sum e^{ikx}$ bdd, not conv.

Abel : $\sum a_n$ conv. & b_n monotone conv.

- Re-arrangement : Thm: $\sum a_n$ abs. conv ⇒ $\sum \overset{a_{f(n)}}{\overset{\shortparallel}{b_n}}$ abs conv to $\sum a_n$.

pf : $|B_{\cancel{k}} - A| = |B_k - A_N| + |A_N - A| \le (|a_{N+1}| + \cdots) + \frac{\varepsilon}{2} < \varepsilon$ ※

key point : make this finite sum to compare.

Thm (Riemann) : condit. conv ⇒ ∃ rearrange $\underline{\lim}\, B_n = x$, $\overline{\lim}\, B_n = y$, $\underset{\text{any}}{x < y}$

- Subseries / countable decomp $\beta^{(k)} = \sum b_n^{(k)}$ $\sum a_i$ abs ⇒ $\sum b_n^{(k)}$ abs. ∀k

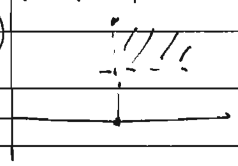
Moreover, $\sum \beta^{(k)}$ abs. conv. to A.

pf: $|\beta^{(1)} + \cdots + \beta^{(n)} - A| = |\beta^{(1)} + \cdots + \beta^{(n)} - \sum_{k=1}^{n} a_k| + |A_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ※

Double sequence. Thm : $\lim_{p,q \to \infty} f(p,q) = a$; $F(p) = \lim_{q \to \infty} f(p,q)$ ⇒ $\lim_{p \to \infty} F(p) = a$.

$|F(p) - a| = |F(p) - f(p,q)| + |f(p,q) - a|$

Double series / abs. conv ⇒ any rearrangement abs. conv. (Ex.)

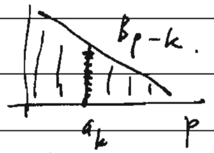


- Multiplication of series : (Cauchy product)

Thm (Mertens) : $\sum a_n$ abs. conv. $\sum b_n$ conv. ⇒ $\sum c_n \to AB$.

idea pf: let $d_n = B - B_n$, $e_n = \sum a_k d_{n-k}$, then

$c_p = \sum a_k B_{p-k} = A_p B - \sum a_k d_{p-k} = A_p B - \underline{e_p}$.



Summability method : Eg. Cesàro : $\sigma_n := \frac{1}{n}(A_1 + \cdots + A_n)$

Thm : $A_n \to A$ ⇒ $\sigma_n \to A$.

pf: $|\sigma_n - A| = \frac{1}{n}|(A_1 - A) + (A_2 - A) + \cdots (A_n - A)|$

↑ cut at some N. let $n \gg N$. ※

10/2 : <u>∞ product</u>.

$u_i \in \mathbb{R}$ or $\mathbb{C}$. $\quad p_n = \prod_{k=1}^{n} u_i$

<u>Def'n</u>: If $u_i \neq 0 \ \forall i$, then $\prod u_i$ conv. $\overset{\Delta}{\iff}$ $p_n \longrightarrow \overset{\exists}{p} \neq 0$.

if only finite zeros $(u_i \neq 0 \ \forall i \geq N)$ then conv $\overset{\Delta}{\iff}$ $\prod u_{N+i}$ conv.

otherwise call $\prod u_i$ diverges. (eg. ∞ - 0's)

Eg. $p_n = \prod_{k=1}^{n} (1+\frac{1}{n}) = n+1 \nearrow \infty$; $p_n = \prod_{k=1}^{n} (1-\frac{1}{n}) = \frac{1}{n} \to 0$

both diverge.

<u>Cauchy criterion</u>: $\prod u_i$ conv. $\iff$ $\forall \varepsilon > 0, \exists N$ st $n \geq N$ $\Rightarrow$ $|u_{n+1} \cdots u_{n+k} - 1| < \varepsilon$. for any $k \in \mathbb{N}$

pf: $\Rightarrow$: $\exists N, |p_n| > M \neq 0 \ \forall n \geq N$.

p_n is Cauchy $\Rightarrow$ $|p_{n+k} - p_n| < \varepsilon M \Rightarrow |\prod_{n+1}^{n+k} u_i - 1| < \varepsilon$.

$\Leftarrow$: clearly $n > N \Rightarrow u_n \neq 0$; Pick $\varepsilon = \frac{1}{2}$ with N_0

then $\xi_n := u_{N_0+1} \cdots u_n$ satisfies $\frac{1}{2} < |\xi_n| < \frac{3}{2}$.

Now for any $\varepsilon > 0$, $\exists N, n \geq N \Rightarrow \left| \frac{\xi_{n+k}}{\xi_n} - 1 \right| < \varepsilon$

ie. $|\xi_{n+k} - \xi_n| < |\xi_n| \cdot \varepsilon < \frac{3}{2}\varepsilon$ ie. $\{\xi_n\}$ is Cauchy $\to \xi \neq 0$.

<u>Thm</u>: let $a_n > 0$. Then $\prod(1+a_n)$ conv. $\iff$ $\sum a_n$ conv.

Cor. $u_n \to 1$ write $u_n = 1 + a_n$.

Pf: $s_n = a_1 + \cdots + a_n < p_n = \prod_{i=1}^{n}(1+a_i) < \prod_{i=1}^{n} e^{a_i} = e^{s_n}$ *

<u>Def'n</u>: $\prod(1+a_i)$ conv. absolutely $\overset{\Delta}{\iff}$ $\prod(1+|a_i|)$ conv.

<u>Fact: abs conv $\Rightarrow$ conv.</u>

Apply Cauchy: $|\prod(1+a_{n+i}) - 1| \leq |\prod(1+|a_{n+i}|) - 1|$ *.

<u>Thm</u>: Let $a_n > 0$. Then $\prod(1-a_n)$ conv. $\iff$ $\sum a_n$ conv.

pf: $\Leftarrow$: since then $\prod(1+|a_n|)$ conv. (abs.)

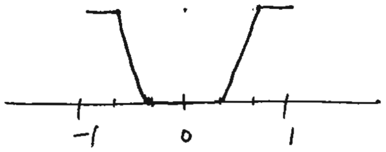
$\Rightarrow$: $\prod(1-a_n)$ conv. $\Rightarrow$ $\prod \frac{1}{1-a_n}$ conv. $\Rightarrow$ $\prod(1+a_n)$ conv *.

<u>Example</u>: <u>Riemann Zeta function</u>:

If $s > 1$, $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1-p_k^{-s}}$ conv. absolutely.

- Space-filling curve. first by Peano (1890).

$\phi(t) = \phi(t+2)$



$$x(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-2}t)}{2^n} \qquad y(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-1}t)}{2^n}$$

$f(t) = (x(t), y(t)) : [0,1] \to [0,1] \times [0,1]$ conti.

let $(a,b) \in [0,1]^2$, in binary system $a = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$; $b = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$.

Schoenberg (1938): $c = 2\sum_{n=1}^{\infty} \frac{c_n}{3^n}$: $\begin{cases} c_{2n-1} = a_n \\ c_{2n} = b_n \end{cases}$

Claim: $\phi(3^k c) = c_{k+1}$, $k = 0, 1, 2, \ldots$

If so, then $x(c) = a$, $y(c) = b$; i.e. f is surjective.

pf: $3^k c$ = even + d_k, $d_k = 2\sum_{n=1}^{\infty} \frac{c_{n+k}}{3^n}$

$\phi(3^k c) = \phi(d_k)$

if $c_{k+1} = 0$ then $d_k \leq 2 \cdot \frac{\frac{1}{3^2}}{1-\frac{1}{3}} = \frac{1}{3} \Rightarrow \phi(3^k c) = 0 = c_{k+1}$.

if $c_{k+1} = 1$ then $\frac{2}{3} \leq d_k \leq 1 \Rightarrow \phi(3^k c) = 1 = c_{k+1}$. □

- Thus the crucial thing is unif. conv.

Def^{n}: $f_n \to f$ unif on S $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists N$; $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in S$.

Thm: If all f_n conti at $a \in S$ and $f_n \to f$ unif. then f is also conti at a.

pf: Let a be accum. pt.

$d(f(x), f(a)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(a)) + d(f_n(a), f(a))$

$\varepsilon > 0$, choose unif n (fixed) then choose $B(a;r)$ #

Now also have unif. version of Cauchy criterion of conv.

- For series $F_n(x) = \sum_{k=1}^{n} f_k(x) \longrightarrow F(x)$ unif. on S has a def^{n}.

Unif. Cauchy criterion : $\forall \varepsilon > 0$, $\exists N$, $n \geq N \Rightarrow \left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon$ $\forall p$, $\forall x \in S$.

Cor (M-test) : $|f_n(x)| \leq M_n$ & $\sum_{n=1}^{\infty} M_n$ conv.

$\Rightarrow$ $\sum f_n(x)$ unif. conv. (absolutely)

Note : f_n conti, $F_n \to F$ unif. $\Rightarrow$ F is also conti.

$\Rightarrow$ also conti

- univ. conv. & R.S. integral

Thm. Let $\alpha \in BV([a,b])$, $f_n \in R(\alpha) \xrightarrow{\text{unif.}} f$ on $[a,b]$

Then (a) $f \in R(\alpha)$, (b) $g_n(x) := \int_a^x f_n \, d\alpha \xrightarrow{\text{unif}} g(x) := \int_a^x f \, d\alpha$.

pf: Assume $\alpha \nearrow$. Only need prove (a), which is trivial if $f \in C$.

in general, $U(P,f,\alpha) - L(P,f,\alpha)$

$$\leq \underset{\wedge\, \varepsilon/3}{U(P, f-f_n, \alpha)} - \underset{\wedge\, \varepsilon/3}{L(P, f-f_n, \alpha)} + \overset{② \vee\, \varepsilon/3}{(U(P, f_n, \alpha) - L(P, f_n, \alpha))}$$

for ① $|f - f_n| \cdot (\alpha(b) - \alpha(a)) < \varepsilon/3$.

Cor. Same for series $\Sigma f_n(x) = f(x)$.

The case for non-unif. conv. requires "unif. bounded": eg. x^n.
postponed to Lebesgue's bounded conv. thm (Thm 10.29)

- univ. conv. & Differentiation

eg. $f_n(x) = \sin(2^{2n} x)/2^n$; $f_n'(x) = 2^n \cos(2^{2n} x)$ at $x = 0$.

Thm Let f_n diff'ble on (a,b) and $\exists\, x_0$ $f_n(x_0)$ conv.
assume $f_n' \longrightarrow g$ unif on (a,b). Then $f_n \xrightarrow{\exists} f$ unif & $f' = g$.

pf: The pf is easy if f_n' is conti. (simply $\int$)

In general, recall $f_n(x) = f_n(x_0) + (x - x_0) f_n^*(x)$ wrt x_0.

$$\Rightarrow f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + (x - x_0) \cdot (f_n^*(x) - f_m^*(x))$$

$$\| \circledast$$

Now use Cauchy criterion ∗ the same x_1! $f_n'(x_1) - f_m'(x_1)$ why?

So $f_n \longrightarrow f$ unif. wrt any $c \in (a,b)$, $f_n(x) = f_n(c) + (x - c) f_n^*(x)$.

Consider $G(x) = \lim_{n\to\infty} f_n^*(x)$, it exists if $x \neq c$, $G(x) = \frac{f(x) - f(c)}{x - c}$

for $x = c$: $G(c) = \lim_{n\to\infty} f_n'(c) = g(c)$ also exists by assumption

Moreover, we just saw ⊛ f_n^* conv. unif. (to G) $\Rightarrow$ G conti at c.

$$\Rightarrow f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} G(x) = g(c)$$, i.e. f' exists and $= g$ ∗

- Some Tests for unif. conv for series.

Weierstrass M test

Dirichlet/Abel test : $F_n = \sum_{k=1}^{n} f_k$ unif bdd $\mathbb{C}$-valued, $\overset{\mathbb{R}\text{-valued}}{g_n \searrow 0}$ unif.

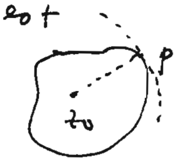
power series $\Rightarrow \Sigma f_n(x) g_n(x)$ conv. unif.

eg. $\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$.

power series 11/3

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in $\mathbb{C}$ conv. radius $r = \overline{\lim_{n\to\infty}} |a_n|^{-1/n}$

conv. abs. for $|z-z_0| < r$, div for $> r$. unif. on K cpt in $B(z_0, r)$ by M test



Eg. $\sum_{n=1}^{\infty} z^n$ div. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ conv.

on $|z| = 1 = r$. div at $z=1$ but conv. if $z \neq 1$ by Dirichlet test.

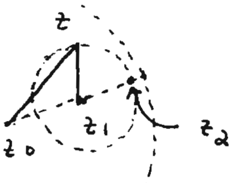
Thm : Power series is conti, $f'(z) = \sum_{n=1}^{\infty} n \cdot a_n (z-z_0)^{n-1}$, hence $f \in C^\infty$.

Pf : On $\mathbb{R}$ easy : $\underbrace{\sum n a_n (z-z_0)^{n-1}}_{\text{conti}} \xrightarrow{\text{unif.}} g(z) \Rightarrow f'(z)$ exists $= g(z)$ #

In fact, this $\int^z$ method also works for $\mathbb{C}$.

Method 2 : $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$= \sum_{n=0}^{\infty} a_n ((z-z_1)+(z_1-z_0))^n = \sum_{n=0}^{\infty} a_n \cdot \sum_{k=0}^{n} C^n_k (z_1-z_0)^{n-k} \cdot (z-z_1)^k$



the double series is abs. conv. to $\sum |a_n| \cdot |z_2 - z_0|^n$

hence can change order $= \sum_{k=0}^{\infty} \underbrace{\left(\sum_{n=k}^{\infty} C^n_k a_n (z_1-z_0)^{n-k} \right)}_{b_k} (z-z_1)^k$

$\Rightarrow f'(z_1) = \lim_{z\to z_1} \frac{f(z)-f(z_1)}{z-z_1} = b_1$.

Thm : $\sum a_n z^n \cdot \sum b_n z^n = \sum c_n z^n$; $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ (Mertens')
in the smaller conv. radius

Thm : Composition / substitution : $f = \sum a_n z^n$ $|z| < r$
if for $g = \sum b_n z^n$, $|z| < R$ has $\sum |b_n z^n| < r$ then
$f(g(z)) = \sum_{k=0}^{\infty} c_k z^k$; $c_k := \sum_{n=0}^{\infty} a_n b_k(n)$ with $g^n = \sum_{k=0}^{\infty} b_k(n) z^k$.

pf: Write out, check $|\ |$ conv. exchange order of sum. #

Thm / Cor : Division Let $p_0 = 1$, $\frac{1}{p(z)} = \frac{1}{1-(1-p(z))} = \sum_{n=0}^{\infty} (1-p(z))^n$.

Eg. If $f \in C^\infty([a,b])$ with $|f^{(n)}(x)| \leq M^n$ on $B(c) \Rightarrow f = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$
since Lagrangian form $R_n \longrightarrow 0$

Thm (Bernstein) If $f^{(n)} \geqslant 0$ on $[b, b+r] \Rightarrow$ conv. on $[b, b+r)$ at $x = b$.

pf: May assume $b = 0$. Using integral form

of $R_n(x) = \frac{1}{n!} \int_{\underset{0}{\overset{\shortparallel}{b}}}^{x} (x-t)^n f^{(n+1)}(t)\, dt \overset{\uparrow}{=} \frac{x^{n+1}}{n!} \int_0^1 u^n f^{(n+1)}((1-u)x)\, du$

let $t = (1-u)x$
$dt = -x\, du$

this $\nearrow$ in x

$\Rightarrow R_n(x) \leq \left(\frac{x}{r}\right)^{n+1} \underset{\underset{f(r)}{\wedge}}{R_n(r)}$ by def'n

$\Rightarrow R_n(x) \longrightarrow 0$ as $n \to \infty$ $\square$

11/17 Lebesgue integral 1902

general interval, could be ∞.

Def'n: "$f \in S(I)$" if f is a step fcn on $[a,b] \subset I$, $= 0$ outside.

$\int_I f = \sum_{k=1}^{n} c_k \Delta x_k$ ind. of $f(x_k)$.

- Thm: $s_n \in S(I)_{\geq 0}$, $s_n \searrow 0$ a.e. on I $\Rightarrow$ $\lim_{n\to\infty} \int_I s_n = 0$.

Cor/Def'n: If $s_n \nearrow f$ a.e. on I and $\underbrace{\lim_{n\to\infty} \int_I s_n}_{= A} < \infty$, ie. "$f \in U(I)$", then $\int_I f := A$ is well-defined.

pf: Claim: If $t \in S(I)$, $t \leq f$ a.e. on I $\Rightarrow$ $\int_I t \leq A$.

let $\tilde{s}_n(x) = \max(t(x) - s_n(x), 0) \xrightarrow[\text{a.e.}]{} \searrow \underbrace{\max(t(x) - f(x), 0)}_{= 0 \text{ a.e. on } I}$

$\Rightarrow \int_I t - \int_I s_n \leq \int_I \tilde{s}_n \to 0$ ✲

Now if $t_n \nearrow f$ a.e. $\Rightarrow$ $t_n \leq A$ $\forall n$ $\Rightarrow$ $\lim_{n\to\infty} \int_I t_n \leq A$, use Sym ✲

Facts: $f, g \in U(I)$ $\Rightarrow$ ① $\int f+g = \int f + \int g$; ② $\int cf = c \int f$ $(c \geq 0)$

$c < 0$ OK for $S(I)$ but not for $U(I)$

③ $f \leq g$ a.e. $\Rightarrow$ $\int f \leq \int g$ ("=" $\Rightarrow$ "=").

④ $\max(f,g)$, $\min(f,g) \in U(I)$.

pf: $s_n \underset{\text{a.e.}}{\nearrow} f$, $t_n \underset{\text{a.e.}}{\nearrow} g$ $\Rightarrow$ $u_n := \max(s_n, t_n) \nearrow \max(f,g)$ a.e.

$v_n := \min(s_n, t_n) \nearrow \min(f,g) \leq f$ a.e. $\Rightarrow$ $\min(f,g) \in U(I)$. by Claim.

Now $u_n + v_n = s_n + t_n$, take $\lim_{n\to\infty} \int$. ✲

⑤ $f \in U(\underset{I_1 \cup I_2}{\overset{\shortparallel}{I}})$, $\underline{\geq 0 \text{ a.e.}}$ $\Rightarrow$ $f \in U(I_1), U(I_2)$ & $\int_I f = \int_{I_1} f + \int_{I_2} f$.

pf: $s_n \nearrow f$ $\Rightarrow$ $s_n^+ := \max(s, 0) \underset{\text{a.e.}}{\nearrow} f \geq 0$ $\Rightarrow$ $\int_J s_n^+ \leq \int_I s_n^+ \leq \int_I f$

$\forall J \subset I$ $\Rightarrow$ $f \in U(J)$. Also $\int_I s_n^+ = \int_{I_1} s_n^+ + \int_{I_2} s_n^+$. Let $n \to \infty$ ✲

- Thm: $f \in R[a,b]$ $\Rightarrow$ $f \in U([a,b])$ and $\int_{[a,b]} f = \int_a^b f$.

Def'n: $L(I) = \{ f = u - v \mid u, v \in U(I) \}$; $\int f = \int u - \int v$.

Facts*: $f, g \in L(I)$ $\Rightarrow$ ① $af + bg \in L(I)$ ② $f \geq 0$ a.e. $\Rightarrow$ $\int f \geq 0$

③ $f \leq g$ a.e. $\Rightarrow$ $\int f \leq \int g$ & ("=") ④ $f^+, f^-, |f|, \max(f,g)$, $\min(f,g)$; $\int |f| \geq |\int f|$

⑤ $\int_I f = \int_{I_1} f + \int_{I_2} f$.

pf of ④: $f = u - v$ $\Rightarrow$ $f^+ = \max(u - v, 0) = \max(u, v) - v \in L(I)$,

$f^- = f^+ - f \in L(I)$, $|f| = f^+ + f^- \in L(I)$; $-|f| \leq f \leq |f|$ $\Rightarrow$ int.

$\max(f,g) = \frac{1}{2}(f + g + |f - g|)$; $\min = \frac{1}{2}(f + g - |f - g|)$.

Fact (Thm): Translation / Scaling / Reflection invariance of $\int$.

Pf of thm: May set $I=[a,b]$. Let $D=\cup D_n$ - end pts of s_n

$F=D\cup E$ - pts s_n does not conv, has measure 0.

$x\in[a,b]-F \Rightarrow \exists N=N_x$ st $s_N(x)<\varepsilon$, hence $\exists B(x)$ st $s_N<\varepsilon$ on $B(x)$, true $\forall n\geq N$

let F_i open cover of F, $\Sigma|F_i|<\varepsilon$, "$F_i, B(x)$" cover $[a,b]$

$\Rightarrow [a,b]\subset B(x_1)\cup\cdots\cup B(x_p)\cup F_1\cup\cdots\cup F_q$

let $N_0=\max(N(x_1),\cdots,N(x_p))$

$B:=F_1\cup\cdots\cup F_q$; $A:=[a,b]-B$ finite disj. intervals

$\int_I s_n = \underset{\underset{\varepsilon\cdot(b-a)}{\wedge}}{\int_A s_n}+\underset{\underset{\varepsilon\cdot M}{\wedge}}{\int_B s_n}$ $\forall n\geq N_0$. done *. where $s_1\leq M$

Pf of $R[a,b]\subset U([a,b])$:

Let $P_n=\{x_0,x_1,\cdots,x_{2^n}\}$, P_{n+1} bi-sects P_n, of $[a,b]$

$s_n(x):=m_k$ on $(x_{k-1},x_k]$, $m_k=\inf f$ on $[x_{k-1},x_k]$

claim: $s_n(x)\nearrow f(x)$ at a conti pt x.

$\forall\varepsilon>0$, $\exists\delta=\delta_x$ st. $|y-x|<\delta\Rightarrow|f(y)-f(x)|<\varepsilon$

$\exists P_N$, $x\in[x_{k-1},x_k]\subset(x-\delta,x+\delta)\Rightarrow s_N(x)\leq f(x)\leq s_N(x)+\varepsilon$

$\Rightarrow s_n(x)\leq f(x)\leq s_n(x)+\varepsilon$ $\forall n\geq N$.

Now $\int_{[a,b]}s_n=\sum_{k=1}^{2^n}m_k\Delta x_k=L(P_n,f)$ done.

$\Rightarrow \lim_{n\to\infty}\int_{[a,b]}s_n=\int_a^b f$ *

Example: ① $f(x)=\begin{cases}1/p & \text{if } x=\frac{q}{p}\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$ $\Rightarrow f$ conti at $x\notin\mathbb{Q}$ hence $f\in R[0,1]$

② $f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$ $\Rightarrow f\notin R$ but $f\in L([0,1])$ in fact U.

Approximation lemma: Let $f\in L(I)$. Then $\forall\varepsilon>0$

a) $\exists u,\underset{\underset{0\ \text{a.e.}}{\vee}}{v}\in U(I)$, $f=u-v$ with $\int_I v<\varepsilon$. If $f\underset{\text{a.e.}}{\geq}0$, $u\underset{\text{a.e.}}{\geq}0$ too.

b) $\exists s\in S(I)$, $g\in L(I)$ st. $f=s+g$ with $\int_I|g|<\varepsilon$.

pf: a) $f=u_1-v_1$; $t_n\nearrow v_1$ st. $0\leq\int v_1-\int t_N<\varepsilon\Rightarrow f=(u_1-t_N)-(v_1-t_N)$.

b) Choose $f=u-v=(u-s')-(v-t')+(s'-t')=:g+s$

with $\int|g|\leq\int u-s'+\int v-t'<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ *.

Levi Monotone conv. thm

$f_n \in L(I)$, $f_n \nearrow$ a.e. $\lim_{n\to\infty} \int_I f_n = A < \infty \Rightarrow f_n \underset{a.e.}{\nearrow} f \in L(I)$, $\int_I f = A$.

Step 1. $f_n \in S(I) \nearrow \Rightarrow f_n \underset{a.e.}{\nearrow} f \in U(I)$. May assume $f_n \geq 0$.

let $D = \{x \in I \mid f_n(x)$ div. $\}$. Given $\varepsilon > 0$:

$t_n(x) := \left[\frac{\varepsilon}{2A} f_n(x)\right] \in \mathbb{Z}_{\geq 0}$. $x \notin D \Rightarrow f_n(x)$ bdd, $t_{n+1}(x) = t_n(x)$ for $n \gg 0$

$x \in D \Rightarrow t_{n+1}(x) - t_n(x) \geq 1$ for (∞ many) some n

\ $D_n \equiv$ such $x \in I$ = finite intervals $\subset I$

$$\sum_{n=1}^{m} |D_n| \leq \sum_{n=1}^{m} \int_I (t_{n+1} - t_n) \leq \int_I t_{m+1} \leq \frac{\varepsilon}{2A} \int_I f_{m+1} \leq \frac{\varepsilon}{2}$$

$m \to \infty \Rightarrow D \subset \bigcup_{n=1}^{\infty} D_n$ have measure 0.

Now simply define $f(x) = \lim_{n\to\infty} f_n(x)$ $x \notin D$; $f(x) = 0$, $x \in D$.

Step 2. $f_n \in U(I) \Rightarrow f_n \underset{a.e.}{\nearrow} f \in U(I)$ & $\int_I f = A$.

for each n, let $s_{n,m} \in S(I) \nearrow f_n$ $(m \in \mathbb{N})$

then define $t_n(x) = \max(s_{1,n}(x), \cdots, s_{n,n}(x)) \nearrow$ & $\leq f_n(x)$

$\overset{\cap}{S(I)}$ $\Rightarrow \int_I t_n \leq \int_I f_n \leq A$ ① $\overset{\cap}{U(I)}$

step 1 $\Rightarrow t_n \underset{a.e.}{\nearrow} f \in U(I)$. But $f_n(x) \underset{a.e.}{\leq} f(x)$ $(\Leftarrow s_{n,k} \leq t_n, k\to\infty)$

Now: $\int_I f \leq A$: ① $\Rightarrow f_n \underset{a.e.}{\nearrow} g \underset{a.e.}{\leq} f \underset{a.e.}{\leq} g \Rightarrow f \underset{a.e.}{=} g$.

$A = \lim_{n\to\infty} \int_I f_n \leq \int_I f$: ② $\Rightarrow$ "=". ②

\ basic fact for $U(I)$.

Step 3 The series analogue of Levi thm: $f_n = \sum_{k=1}^{n} g_k$; $(g_n = f_n - f_{n-1})$ $\overset{0}{\wedge}$

Approximation lemma $\Rightarrow g_k = \underset{0}{\underset{\vee}{u_k}} - \underset{0}{\underset{\vee}{v_k}}$ in $U(I)$, $\int_I v_k < \frac{1}{2^k}$.

$\int_I \sum_{k=1}^{n} v_k < 1 \underset{\text{step 2}}{\Rightarrow} V(x) = \sum_{k=1}^{\infty} v_k(x) \in U(I)$ exists & $\int_I V = \sum_{k=1}^{\infty} \int_I v_k$.

similarly, $\int_I \sum_{k=1}^{n} u_k \leq A + 1 \Rightarrow U(x) = \sum_{k=1}^{\infty} u_k(x)$, $\int_I U = \sum_{k=1}^{\infty} \int_I u_k$.

Finally, $\sum_{k=1}^{n} g_k = \sum u_k - \sum v_k \longrightarrow f = U - V \in L(I)$

$$\int_I f = \int_I U - \int_I V = \sum_{k=1}^{\infty} \int_I u_k - \sum_{k=1}^{\infty} \int_I v_k = \sum_{k=1}^{\infty} \int_I g_k$$ #

Cor. $\int_I f = 0 \Leftrightarrow f = 0$ a.e.

Cor. $\searrow$ version holds for $L(I)$, but not for Step 1, 2.

11/24.

<u>Fatou's lemma</u>: $f_n \in L(I)$, $\geqslant 0$, then $\inf f_n \in L(I)$.

If $\underline{\lim} \int f_n < \infty$ then $\underline{\lim} f_n \in L(I)$ too, & $\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$.

pf: (cf. Ex 10.8) let $g_n = \min(f_1, \cdots, f_n) \in L(I) \geqslant 0 \searrow \inf f_n =: g$

Apply Levi's monotone thm to $-g_n \nearrow -g \Rightarrow g \in L(I)$.

- Now let $h_n := \inf\limits_{k \geqslant n} f_k \in L_1(I)$ for each $n \in \mathbb{N}$

then $h_n \nearrow \underline{\lim} f_n$. But $h_n \leq f_k \ \forall k \geqslant n \Rightarrow \lim\limits_{n\to\infty} \int h_n \leq \underline{\lim} \int f_n$

Levi's $\Rightarrow \underline{\lim} f_n \in L(I)$ & $\int \underline{\lim} f_n = \lim\limits_{n\to\infty} \int h_n \leq \underline{\lim} \int f_n \overset{\wedge}{\infty}$ *.

<u>Lebesgue's Dominated Convergence Theorem</u>.

Let $f_n \in L(I)$ & $f_n \xrightarrow[a.e.]{} f$ with $|f_n(x)| \underset{a.e.}{\leq} g(x) \in L(I)$.

Then $f \in L(I)$ and $\int f = \lim\limits_{n\to\infty} \int f_n$.

pf: $f_n + g \geqslant 0$ & $\int f_n + g \leq 2 \int g \underset{\text{Fatou}}{\Rightarrow} f + g \in L(I) \Rightarrow f \in L(I)$.

and $\int f + g \leq \underline{\lim} \int f_n + g = \underline{\lim} \int f_n + \int g \Rightarrow \int f \leq \underline{\lim} \int f_n$.

But can also do $g - f_n \geqslant 0 \underset{\text{Fatou}}{\Rightarrow}$

$$\int g - f \leq \underline{\lim} \int g - f_n = \int g - \overline{\lim} \int f_n \Rightarrow \int f \geqslant \overline{\lim} \int f_n .$$

Hence $\lim\limits_{n\to\infty} \int f_n = \int f \quad (\equiv \int \lim\limits_{n\to\infty} f_n)$ *

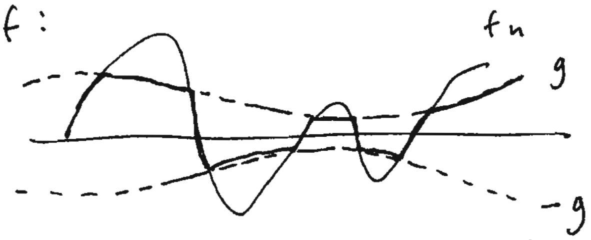
<u>Cor. Lebesgue's bounded conv. thm.</u>

If $|I| < \infty$, $f_n \in L(I)$, $f_n \xrightarrow[a.e.]{} f$ (uniformly) boundedly convergent on I. i.e. $|f_n(x)| \underset{a.e.}{\leq} M$

then $f \in L(I)$, $\int f = \lim\limits_{n\to\infty} \int f_n$.

<u>Cor.</u> $f_n \in L(I)$, $f_n \xrightarrow[a.e.]{} f$ st. $\underline{|f(x)| \leq g(x) \in L(I)} \Rightarrow f \in L(I)$.

pf:



Let $g_n = \max(\min(f_n, g), -g)$

then $|g_n| \leq g$ and $g_n \xrightarrow[a.e.]{} f$

hence $f \in L(I)$. *

<u>Q:</u> We do not know if $\lim\limits_{n\to\infty} f_n$ exists at all !

IV29.

Unbounded intervals.

Thm let f on $I = [a,\infty)$, $f \in L[a,b]$ for all $b \geq a$ and $\int_a^b |f| \leq M \quad \forall b \geq a$, then $f \in L(I)$, $\lim_{b\to\infty} \int_a^b f = \int_I f$.

Pf. Pick $a \leq b_n \nearrow \infty$, $f_n(x) = \begin{cases} f(x) & \text{on } [a,b_n] \\ 0 & \text{otherwise} \end{cases} \in L(I)$

$f_n \to f$ on $I \Rightarrow |f_n| \nearrow |f|$ on I

$\int_I |f_n| \leq M \underset{\text{Levi}}{\Rightarrow} |f| \in L(I)$. But $|f_n| \leq$ "$|f|$" $\underset{\text{Lebesgue}}{\Rightarrow} f \in L(I)$

and $\lim_{n\to\infty} \int_a^{b_n} f = \lim_{n\to\infty} \int_I f_n = \int_I f$ ✱

Cor True for $I = (-\infty, a]$. Thus if $I = (-\infty,\infty) = \mathbb{R}$, $\int_{[c,b]} |f| \leq M$ $\forall c \leq b$, then $f \in L(\mathbb{R})$ and $\int_{-\infty}^{\infty} f = \lim_{c\to-\infty} \int_c^a f + \lim_{b\to\infty} \int_a^b f$.

Cor. If $f \in R[a,b] \;\forall b \geq a$ and $\int_a^b |f|\,dx \leq M$ (Riem sense) Then the improper Riem int for f, $|f|$ both exist on (a,∞). Moreover, $f \in L([a,\infty))$ with equal integral.

Pf: $b \to \infty \Rightarrow \int_a^\infty |f|\,dx$ exists. $\underset{\text{Cauchy criterion}}{\Longrightarrow} \int_a^\infty f\,dx$ exists.

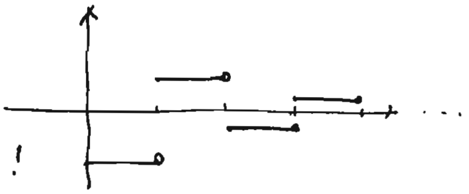
(Alternatively, $\int_a^b \underset{\vee\,0}{(|f| - f)} \leq \int_a^b 2|f|$ let $b \to \infty$)

Now use $R[a,b] \subset L[a,b]$ & Thm. ✱

Example: Not true without $|\cdot|$.

$f(x) = \frac{(-1)^n}{n}$ on $x \in [n-1, n)$.

We never talk about improper Lebesgue int!



Def^n: Measurable functions.

$f \in M(I) \triangleq \exists s_n \in S(I)$ st. $s_n \xrightarrow{a.e.} f$

Notice that $1 \in M(\mathbb{R})$ but $1 \notin L(\mathbb{R})$. Thus $M(I) \supsetneq L(I)$.

Facts:
- $f \in M(I)$, $|f| \underset{a.e.}{\leq} g \in L(I) \Rightarrow f \in L(I)$
  in particular this holds if $|f| \in L(I)$ or if $|f| \leq M$ and $|I| < \infty$.
- $f, g \in M(I)$, $\varphi: \mathbb{R}^2 \to \mathbb{R} \Rightarrow \varphi(f,g) \in M(I)$. eg $f+g$, fg etc... $\max(f,g)$, $\min(f,g)$
- If $\int |s_n| < A$ then $|f| \overset{a.e.}{=} \lim_{n\to\infty} |s_n| \in L(I)$ by Fatou's lemma, hence $f \in L(I)$. Conversely, $f \in L(I)$ $\Rightarrow |f| \in L(I)$, $|s_n| \underset{a.e.}{\to} |f|$.

Examples continued.

Thm. $F(y) := \int_X f(x,y)\,dx$ ie. $f_y(x) = f(x,y) \in L(x)$

(1) if $|f_y| \underset{a.e.}{\leq} g \in L(x)$ and f conti in y for almost all x

then F conti in y.

pf: Apply Lebesgues' domi thm to $f_n(x) := f(x, y_n)$ $\not\!\!*$
(fixed $y \in Y$. pick $y_n \to y$)

(2) Moreover if $\frac{\partial f}{\partial y}$ exists and $\left|\frac{\partial f}{\partial y}\right| \leq g \in L(x)$,

then $F'(y) = \int_X \frac{\partial f}{\partial y}(x,y)\,dx$.

pf: $$\frac{F(y_n) - F(y)}{y_n - y} = \int_X \frac{f(x,y_n) - f(x,y)}{y_n - y}\,dx$$

Apply Lebesgue to $\overset{!!}{f_n(x)}$ $\overset{\shortparallel}{D_2 f(x, u'(x))}$ between y_n & y since $|f_n| \leq g$

and notice $f_n(x) \longrightarrow D_2 f(x,y)$ by def^{n} $\not\!\!*$

Rmk: Compare Courant & John Vol II. § 4.12 on improper R int.
the above thm holds for unif conv case. ie. $\left|\int_A^\infty\right| < \epsilon$.

~~Example 1~~ $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx$ both improper R & L ; $s > 0$
(+ case) $\Gamma'(s) = \int_0^\infty e^{-x} x^{s-1} \log x\,dx$
since for $s \geq a$, $D_2 f$ has bound $g(x) = \begin{matrix} M e^{-x/2} & x \geq 1 \\ x^{a-1}|\log x| & 0 < x < 1 \end{matrix}$.

Example 2 $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ improper R, but not $L(\mathbb{R})$
(± case) $F(y) := \int_0^\infty e^{-xy} \frac{\sin x}{x}\,dx$ $\underset{*}{\text{IR for } y \geq 0}$ but L for $y > 0$.

$y > 0 \Rightarrow F'(y) = -\int_0^\infty e^{-xy} \sin x\,dx = \frac{-1}{1+y^2}$

$\Rightarrow F(y) = c - \tan^{-1} y$, $y \to \infty$ $F(y) \to 0 \Rightarrow c = \frac{\pi}{2}$.

pf of $*$: Alt. test $\Rightarrow \left| \int_A^\infty e^{-xy} \frac{\sin x}{x}\,dx \right| < \int_A^{(k+1)\pi} \frac{e^{-xy}}{x}\,dx < \underset{\varepsilon}{\overset{\wedge}{\frac{2\pi}{A}}}$.
let $A \in ((k-1)\pi, k\pi]$.

Hence $F(y)$ is conti for $y \geq 0 \Rightarrow F(0) = \frac{\pi}{2}$.

$$\left| F(y_1) - F(y) \right| < \left| \int_0^A \left(f(x,y_1) - f(x,y)\right) dx \right| + 2\varepsilon$$

now pick $y_1 \sim y$.

(3) Fubini's thm, proved later.

12/1

Thm: $f_n \in M(I)$, $f_n \xrightarrow[a.e.]{} f \Rightarrow f \in M(I)$.

Pf: The idea "$s_{n,m} \longrightarrow f_n \longrightarrow f$" does not work!

let $g \in L(I)$, e.g. $1/(1+x^2) > 0$

$F_n = g \frac{f_n}{1+|f_n|} \xrightarrow[a.e.]{} F = g \frac{f}{1+|f|}$ $\quad |F| < g$

$F_n \in M(I)$ & $|F_n| < g \Rightarrow F_n \in L(I) \Rightarrow F \in L(I)$

But then $|F| = g \frac{|f|}{1+|f|} \Rightarrow 1+|f| = \frac{g}{g-|F|} \Rightarrow f = \frac{F}{g}(1+|f|) = \frac{F}{g-|F|}$

hence $f \in M(I)$ since $g-|F| > 0$ ✱

Lebesgue measure on $\mathbb{R}$:

$S \subset \mathbb{R}$, characteristic function $\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$

Def'n: S is measurable $\overset{\Delta}{=} \chi_S \in M(\mathbb{R})$

if $\chi_S \in L(\mathbb{R})$ then $\mu(S) := \int_{\mathbb{R}} \chi_S$, otherwise $\mu(S) = \infty$.

- since $A \subset B \Rightarrow \mu(A) \leq \mu(B)$, $\mu(S \cap [a,b]) \leq \mu([a,b]) = b-a$.
- S has measure 0 $\Longleftrightarrow \mu(S) = 0$. ($\Leftarrow$ by Levi for $f_n = n\chi_S$)

Thm: Let $\mathcal{M}$ be the collection of measurable sets (in $\mathbb{R}$)

(1) $\mathcal{M}$ is a σ-ring: $S, T \in \mathcal{M} \Rightarrow S \setminus T \in \mathcal{M}$

$S_i \in \mathcal{M}$, $i \in \mathbb{N} \Rightarrow \cup_{i=1}^{\infty} S_i$, $\cap_{i=1}^{\infty} S_i \in \mathcal{M}$.

(2) μ is countably additive: $A_i \in \mathcal{M}$ disjoint $\Rightarrow \mu(\cup A_i) = \sum \mu(A_i)$

Pf: (1) $\chi_{S-T} = \chi_S(1-\chi_T) \in M(\mathbb{R})$.

$\chi_{\cup S_i} = \lim_{n\to\infty} \max(\chi_{S_1}, \cdots, \chi_{S_n})$ as limit of measurable fcn's.

$\cap S_i = \mathbb{R} \setminus \cup S_i \in \mathcal{M}$.

(2) $\chi_{A_1 \cup \cdots \cup A_n} = \chi_{A_1} + \cdots + \chi_{A_n} \geq \chi_{A_i}$. $\int \Rightarrow \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

Now let $T_n = A_1 \cup \cdots \cup T_n$, $\chi_{T_n} \nearrow \chi_T$, $T := \cup A_i$.

if $\mu(T) < \infty$ then $\chi_T, \chi_{T_n} \in L(\mathbb{R})$

the result follows from Lebesgue dominated conv. thm.
or (Levi's monotone)

if $\mu(T) = \infty$, then either $\mu(T_n) = \infty$ (some $\mu(A_i) = \infty$) or $\mu(A_i) < \infty$ but $\lim_{n\to\infty} \mu(T_n) = \infty$ (by Levi's thm series version)

Fact: If $0 \neq \mu(S) < \infty$ then $\exists\, T \subset S$ with $T \notin \mathcal{M}$. (Ex. 10.36)

12/6

Def^{n}: $L^2(I) := \{ f \in M(I) \mid |f|^2 = u^2 + v^2 \in L(I) \}$ ie $u^2, v^2 \in L$.

$f, g \in L^2 \Rightarrow \overset{u+iv}{\overset{\shortparallel}{f}} \cdot g \in L^2$ since $|fg| \leq \frac{1}{2}(|f|^2 + |g|^2)$

$af + bg \in L^2$ since $|af + bg|^2 = |a|^2|f|^2 + 2\mathrm{Re}(af\overline{bg}) + |b|^2|g|^2$

inner product (hermitian) : $(f, g) := \int_I f\bar{g} \in \mathbb{C}$

Facts: ~~Cauchy inequality~~ $|(f,g)| \leq \|f\| \cdot \|g\|$, $\|f + g\| \leq \|f\| + \|g\|$.

pf: Use $\|f + tg\|^2 \geqslant 0 \quad \forall t \in \mathbb{R}$.

Rmk: More basic Cauchy : $f = u + iv \in L \Rightarrow |\int f| \leq \int |f|$. (pf = ?)

$\|f\| = 0 \Leftrightarrow f = 0$ a.e. ie. $L^2(I)/\sim$ is a metric space.

Theorem (Riesz - Fischer) : L^2 is complete. (Hilbert space)

Prop : $g_n \in L^2$ st. $A = \sum_{n=1}^{\infty} \|g_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} g_n \xrightarrow{a.e.} g \in L^2$

with $\|g\| = \lim_{n \to \infty} \|\sum_{k=1}^{n} g_k\| < A$.

pf: let $f_n = \left(\sum_{k=1}^{n} |g_k|\right)^2 \in L^2$, $\nearrow$; $\int f_n = \| |g_1| + \cdots + |g_n| \|^2$

$\leq (\|g_1\| + \cdots + \|g_n\|)^2 = A^2$

Levi $\Rightarrow f_n \xrightarrow{a.e.} f \in L(I)$.

But then $\sum_{k=1}^{\infty} g_k(x)$ converges absolutely on I. a.e. $\longrightarrow g(x)$

$G_n := |\sum_{k=1}^{n} g_k|^2 \leq f$ and $\longrightarrow |g|^2$, Lebesgue $\Rightarrow |g|^2 \in L(I)$.

and $\int |g|^2 = \lim_{n \to \infty} \int G_n \leq \lim_{n \to \infty} \int f_n \leq A^2$ ✱.

pf of R-F thm : Let $\{f_n\}$ be Cauchy in $L^2(I)$.

$\exists\, n_1 < n_2 < n_3 < \cdots$ st. $\|f_m - f_{n_k}\| < \frac{1}{2^k} \quad \forall m \geqslant n_k$

Let $g_1 = f_{n_1}$; $g_k = f_{n_k} - f_{n_{k-1}} \quad \forall k \geqslant 2 \Rightarrow \sum_{k=1}^{\infty} \|g_k\| = A < 1 + \|f_{n_1}\|$

Prop $\Rightarrow \underset{\circ\, f_{n_\ell} - \cancel{f_1} + \cancel{f_1}}{\sum_{k=1}^{\ell} g_k} \xrightarrow{a.e.} f \in L^2(I)$.

Now $\|f_m - f\| \leq \underset{1/2^k}{\underset{\wedge}{\|f_m - f_{n_k}\|}} + \|f_{n_k} - f\|$

$\overset{\shortparallel}{\|\sum_{\ell=k}^{\infty} (f_{n_\ell} - f_{n_{\ell+1}})\|} \leq \sum_{\ell=k}^{\infty} \frac{1}{2^\ell} = \frac{1}{2^{k-1}}$

pick k large st. $1/2^{k-1} < \varepsilon/2$. and $m \geqslant n_k$. □

Rmk: We actually proved $f_{n_\ell} \xrightarrow[a.e.]{} f$ (a sub-sequence)

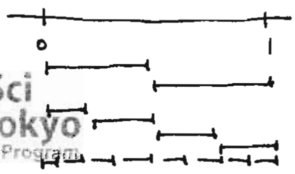
but the full sequence $f_m(x)$ may not conv. for any $x \in I$!

eg. consider $f_n = \chi_{I_n}$ with

$\|f_n\| \to 0$ but

$\lim_{n \to \infty} f_n(x)$ does not conv.

for any $x \in [0,1]$.



Linear alg on H: inner prod space

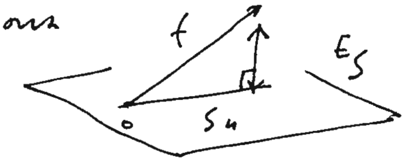
$\mathcal{S}$: $\varphi_0, \cdots, \varphi_n \in H$ orthonormal system : $(\varphi_i, \varphi_j) = \delta_{ij}$.

$f \in H$, proj to $E_S = \langle \varphi_0, \cdots, \varphi_n \rangle$: $s_n = \sum_{i=0}^{n} \underset{c_i}{\underset{\parallel}{(f, \varphi_i)}} \varphi_i$

$f - s_n \perp E_S$, hence $\forall t \in E_S$

$\|f - t\|^2 = \|f - s_n\|^2 + \|s_n - t\|^2$ Phytagoras

i.e. s_n is the best approximation.



Fact: If φ_i, $i \in \mathbb{N}$ ONS then

Bessel's inequality : $\sum_{i=0}^{\infty} |c_i|^2 \leq \|f\|^2$

equality holds $\iff \lim_{n \to \infty} \|f - s_n\| = 0$ (Parseval)

pf: $\|f\|^2 = \|f - s_n\|^2 + \|s_n\|^2$, but $\|s_n\|^2 = \sum_{i=0}^{n} |c_i|^2$.

Example: $L^2(I)$, eg. $I = [0, 2\pi]$, $\varphi_n(x) = e^{inx}/\sqrt{2\pi}$

$c_n = \int_0^{2\pi} f(x) e^{-inx} dx \longrightarrow 0$ (Riemann-Lebesgue Lemma).

Q: Is $\mathcal{S} = \{\varphi_n\}_{n=0}^{\infty}$ a "basis" in any sense?

Fact (Riesz-Fischer). Let H be complete. (eg. $L^2(I)$)

Given c_i with $\sum |c_i|^2$ converges. Then $\exists f \in H$ st.

$\|f - s_n\| \longrightarrow 0$ and $(f, \varphi_i) = c_i$ $\forall i \geq 0$.

Pf: Define $s_n = \sum_{i=0}^{n} c_i \varphi_i$, $\{s_n\}$ is Cauchy since

$m > n \Rightarrow \|s_m - s_n\|^2 = \sum_{i=n+1}^{m} |c_i|^2 < \varepsilon$ when n large.

hence $\exists f \in H$ st. "$s_n \longrightarrow f$" in the metric sense.

Now $|(f, \varphi_i) - c_i| = |(f - s_n, \varphi_i)| \leq \|f - s_n\| \cdot \underset{1}{\underset{\parallel}{\|\varphi_i\|}} \longrightarrow 0$.

for $n \geq i$

Rmk: for $H = L^2(I)$, Riesz-Fischer actually shows that

s_n has a subsequence $s_{n_\ell} \xrightarrow[\text{a.e.}]{} f$ (choose this as f)

but the full $s_n(x)$ may not conv.

Q: Namely, if $I = [0, 2\pi]$, $f \in L^2$, $x \in [0, 2\pi]$

is the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ conv. at x?

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt$, $b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt$.

12/8 Fourier Series

~~Def'n~~: $D_n(t) = \frac{1}{2} + \cos t + \cdots + \cos nt = \begin{cases} \frac{\sin(n+\frac{1}{2})t}{2\sin t/2} & t \neq 2m\pi \\ n + 1/2 & t = 2m\pi \end{cases}$

~~Dirichlet kernel~~

~~Prop~~: $f \in L[0, 2\pi]$, then, extend periodically.

$$S_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kt + b_k \sin kt \right) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} D_n(t) dt.$$

~~Thm~~: (~~Riemann's localization thm~~)

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{2}{\pi} \int_0^{\delta} \frac{f(x+t) + f(x-t)}{2} \frac{\sin(n+\frac{1}{2})t}{t} dt$$

∃ iff ∃ δ st RHS exists

pf: Since $\frac{1}{t} - \frac{1}{2\sin t/2} \in C[0,\pi]$, may replace D_n by $\frac{\sin(n+\frac{1}{2})t}{t}$

by Riemann-Lebesgue lemma. Moreover $\int_{\delta}^{\pi} \to 0$ by R-L again. ※

~~Riemann-Lebesgue lemma~~: $f \in L(I) \Rightarrow \lim_{\alpha\to\infty} \int_I f(t) \sin(\alpha t + \beta) = 0$

pf: $\int_a^b \sin(\alpha t + \beta) = \frac{-\cos(\alpha t+\beta)}{\alpha}\Big|_a^b \to 0$ as $\alpha \to \infty$ $\Rightarrow$ true for $f \in S(I)$.

for $f \in L(I)$, by approx. lemma $f = g + s$, $\int |g| < \frac{\varepsilon}{2}$

$\exists M$ st $\alpha \geq M \Rightarrow \left| \int s(t) \sin(\alpha t + \beta) \right| < \frac{\varepsilon}{2}$

$\Rightarrow \left| \int f \sin(\alpha t + \beta) \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ※.

~~Rmk~~: Where to guess the limit of the (Dirichlet) integral $\int_0^{\delta} g(t) \frac{\sin \alpha t}{t} dt$?

idea: $\lim_{\alpha\to\infty} \int_0^{\delta} \frac{\sin \alpha t}{t} dt = \lim_{\alpha\to\infty} \int_0^{\alpha\delta} \frac{\sin t}{t} dt = \frac{\pi}{2}$!

So we expect $\frac{\pi}{2} g(0+)$.

Thm (Jordan's test). $g \in BV[0,\delta]$ is OK.

pf: May assume $g \nearrow$. Given $\varepsilon > 0$: $\left(\int_0^{\delta} \frac{\sin \alpha t}{t} - \frac{\pi}{2} \right) g(0^+) < \frac{\varepsilon}{3}$ for $\alpha >> 0$.

$$\int_0^{\delta} g(t) \frac{\sin \alpha t}{t} dt - \frac{\pi}{2} g(0^+) \leq \frac{\varepsilon}{3} + \int_0^{h} \left(g(t) - g(0^+) \right) \frac{\sin \alpha t}{t} dt + \int_h^{\delta}$$

$$= \left(g(h) - g(0^+) \right) \int_c^{h} \frac{\sin \alpha t}{t} dt + \int_h^{\delta} \frac{g(t) - g(0^+)}{t} \sin \alpha t \, dt + \frac{\varepsilon}{3}$$

Pick M st $\left| \int_a^b \frac{\sin t}{t} dt \right| \leq M$ $\forall b \geq a \geq 0$

find h st. $\left| g(h) - g(0^+) \right| < \frac{\varepsilon}{3M}$ $\Rightarrow$ $< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ for $\alpha >> 0$

by R-L. ※

Cor. If $f \in BV[x-\delta, x+\delta]$ for $\delta > 0$ then

$g(t) := \frac{1}{2}\left(f(x+t)+f(x-t)\right) \in BV[0,\delta]$ and $\lim_{n\to\infty} S_n(x) = g(0^+) = \frac{1}{2}\left(f(x^+)+f(x^-)\right)$.

Thm (Dini's test) If $g(0^+)$ exists & $\int_0^\delta \frac{g(t)-g(0^+)}{t}\,dt$ exists for some $\delta>0$ then also OK. In particular, if $|g(t)-g(0+)| \leq M t^p$, $p>0$. which holds if $g'(0+)$ exists. (since can let δ small).

pf: R-L $\Rightarrow$ $\int_0^\delta \frac{g(t)-g(0+)}{t} \sin \alpha t \, dt \xrightarrow[\alpha\to\infty]{} 0$ $\divideontimes$

Thm (Cesàro sum): $f \in L[0,2\pi]$, periodic, then

$$\sigma_n(x) := \frac{1}{n}\left(S_0(x)+\cdots+S_{n-1}(x)\right) = \frac{1}{n\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt.$$

in particular, $f \equiv 1 \Rightarrow \frac{1}{n\pi}\int_0^\pi \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt = 1$.

Theorem (Fejér): $f \in L[0,2\pi]$, periodic, $S(x) := \frac{1}{2}\left(f(x^+)+f(x^-)\right)$ if it exists. Then $\sigma_n(x) \longrightarrow S(x)$.
If $f \in C[0,2\pi]$, then the convergence is uniform*.

pf: Let $g_x(t) = \frac{1}{2}\left(f(x+t)+f(x-t)\right) - S(x) \longrightarrow 0$ as $t \to 0^+$

Given $\varepsilon>0$, $\exists\,\delta$ st $|g_x(t)| < \frac{\varepsilon}{2}$ for $0<t<\delta$

If $f \in C$ outside finite pts p_i, then δ is uniform outside nbd of p_i.
in fact unif on each segment.

$$|\sigma_n(x) - S(x)| \leq \frac{1}{n\pi}\int_0^\delta |g_x(t)| \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt + \frac{1}{n\pi}\int_\delta^\pi |g_x(t)| \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$< \frac{\varepsilon}{2} + \frac{1}{n\pi}\cdot\frac{1}{\sin^2\delta/2}\int_\delta^\pi |g_x(t)|\,dt < \varepsilon \quad \text{for } n \text{ large.}$$

$\divideontimes$

Applications to continuous functions:

① Thm: $f \in C[0,2\pi]^*$, periodic, then (may allow finite p_i's st $S(x)$ exists)

① $\|S_n - f\| \longrightarrow 0$

② $\|f\|^2 = \frac{a_0^2}{2} + \sum_{n=1}^\infty (a_n^2+b_n^2)$

③ If $S_n(x)$ conv at x then $S_n(x) \longrightarrow S(x)$.

④ $\int_0^x f = \lim_{n\to\infty}\int^x S_n$ even if S_n diverges. unif in x.

pf: ①: $\|f - S_n\| \leq \|f-\sigma_n\| \longrightarrow 0$ since $\sigma_n \to S$ unif.

① $\Rightarrow$ ②. $S_n(x) \longrightarrow A$ then $\sigma_n(x) \longrightarrow A$, hence $A = S(x)$.

① $\Rightarrow$ ④ by $\left|\int_0^x (f-S_n)\right| \leq \|f-S_n\|\cdot\|1\| \longrightarrow 0$ when $x \leq 2\pi$ $\divideontimes$

② Weierstrass approx thm of conti. functions by polynomials.

pf: Fejér + Taylor expansion $\divideontimes$

for f on $[-B, B] \ni x$ let $y = \frac{\pi}{B} x \in [-\pi, \pi]$

$g(y) := f(x)$ has $g(y) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} g(t) e^{-int} dt \right) e^{iny}$

$e^{-in(t-y)}$

i.e. $f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\pi}{B} \int_{-B}^{B} f(s) e^{-in\frac{\pi}{B}(s-x)} ds$

Set $t = \frac{\pi}{B} s$ let $\frac{\pi}{B} = \Delta u$; $n\frac{\pi}{B} =: v_n$

Now let $B \to \infty$, so $\Delta u \to 0$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} f(s) e^{-iv(s-x)} ds$

i.e. $f(x) = \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ivx} du} \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-ius} ds} = \underline{\mathcal{F}^{-1}(\mathcal{F} f)}$.

equivalent real version:

(*) $f(x) = \frac{1}{\pi} \int_0^{\infty} dv \int_{-\infty}^{\infty} f(s) \cos[v(s-x)] ds$. since $\sin[u(s-x)]$ is odd in s.

Theorem: Assume $\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} f(x+t) \frac{\sin \alpha t}{t} dt = \frac{1}{2}(f(x+) + f(x-))$

let $f \in L(\mathbb{R})$. e.g. with Jordan or Dini's test.

then (*) holds with $f(x)$ being replaced by $(+,-)$ mean.

and $\int_0^{\infty} = \lim_{\alpha \to \infty} \int_0^{\alpha}$ being improper R.

Pf:

$$\int_{-\infty}^{\infty} f(x+t) \frac{\sin \alpha t}{t} dt = \int_{-\infty}^{\infty} f(u) \frac{\sin \alpha(u-x)}{u-x} du \quad \text{——} ①$$

$u = x+t$

$$= \int_{-\infty}^{\infty} f(u) \left(\int_0^{\alpha} \cos v(u-x) \, dv \right) du$$

if can change order of integration $= \int_0^{\alpha} dv \int_{-\infty}^{\infty} f(u) \cos v(u-x) du$ —— ①'

(*). By the approx. lemma, $\exists\, s \in S(I), g \in L$

st. $f = s + g$ with $\int_{\mathbb{R}} |g| < \varepsilon$

then in ① and ①' but contr. from g are small *

Def'n: Convolution: $f, g \in L(\mathbb{R})$, $f * g(x) := \int_{\mathbb{R}} f(t) g(x-t) dt$

Facts: $f*g = g*f$ ① $|g| \le M \Rightarrow f*g(x)$ exists $\forall x \in \mathbb{R}$ and is bdd.

② or $f, g \in L^2$

③ $|g| \le M$ and $g \in C$, then $f*g \in C(\mathbb{R}) \cap L(\mathbb{R})$, and bdd.

pf of ③: $\in C$ is clear. for any $[a,b]$: using (*) again

$$\int_a^b |f*g| dx \le \int_a^b \int_{-\infty}^{\infty} |f(t)| |g(x-t)| dt\, dx = \int_{-\infty}^{\infty} |f| \int_a^b |g(x-t)| \le \int_{-\infty}^{\infty} |f| \int_{-\infty}^{\infty} |g|$$

*

Thm: If $|g| \leq M$, $g \in C$ then $\mathcal{F}(f*g) = \mathcal{F}(f)\cdot\mathcal{F}(g)$.

i.e. $\int_{-\infty}^{\infty} (f*g)(x)\, e^{-ixu}\, dx = \int_{-\infty}^{\infty} f(t)\, e^{-itu}\, dt \int_{-\infty}^{\infty} g(y)\, e^{-iyu}\, dy$

LHS is both L and improper Riemann.

pf: the pf simply refines part ③ by Lebesgue DCT.

$$\int_{a_n}^{b_n} (f*g)(x)\, e^{-ixu}\, dx = \int_{a_n}^{b_n}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\; e^{-ixu}\, dx$$

$$(\text{same reason}) = \int_{-\infty}^{\infty} f(t)\, e^{-itu}\left(\int_{a_n}^{b_n} \underset{\shortparallel}{g(x-t)\; e^{-i(x-t)u}}\, dx\right) dt$$

$$g_n(t) \equiv \int_{a_n+t}^{b_n+t} g(y)\, e^{-iyu}\, dy$$

~~Key point~~: $\left| f(t) \overset{e^{-itu}}{\cdot} g_n(t) \right| \leq |f(t)| \cdot \int_{-\infty}^{\infty} |g| \in L(\mathbb{R})$

Lebesgue's DCT $\Rightarrow$ May pass $\lim\limits_{n\to\infty}$ across the integral $*$.

Example: $\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1} =: B(p,q)$ for $p, q > 0$.

Beta function

Let $f_p(t) = e^{-t}\, t^{p-1}$ $(t>0)$; 0 $(t \leq 0)$

$f_p \in L(\mathbb{R})$ and $\Gamma(p) = \int_{\mathbb{R}} f_p(t)\, dt$

the case $u=0$ in Thm $\Rightarrow$ $\int_{-\infty}^{\infty} f_p * f_q = \Gamma(p)\Gamma(q)$
if $q > 1$.

But $(f_p * f_q)(x) = \int_0^x f_p(t)\, f_q(x-t)\, dt = \int_0^x t^{p-1}(x-t)^{q-1}\, dt \cdot e^{-x}$ if $x > 0$.

$= 0$ if $x \leq 0$

change variable: $t = ux$ get $e^{-x}\, x^{p+q-1}\, B(p,q)$

$\Rightarrow$ LHS $= \int_{-\infty}^{\infty} (\cdots) = \Gamma(p+q)\cdot B(p,q)$.

For the case $0 < q \leq 1$, use $B(p, q+1) = \dfrac{q}{p+q} B(p,q)$ $*$

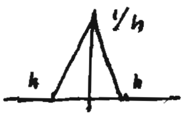
~~Final Remark~~: In

Had seen, by Fejér:

$$\begin{tikzcd} L^2[0,2\pi] \arrow[r, "F"] & \ell^2 & \\ \cup & \cup & \\ C \arrow[r] & F(C) & \text{isometry in } L^2 \text{ norm} \\ \cup & \cup & \\ C^1 \arrow[r] & F(C^1) & \text{pointwise conv.} \end{tikzcd}$$

Even to establish $\|f - S_n\| \to 0$ for all $f \in L^2$ requires

~~approx. thus in L^2~~: C^0 (in fact C^∞) is dense in L^2 (in fact L^p $\forall\, 1 \leq p < \infty$)

This can be done by ~~$f_h := f * K_h$~~ with $K_h =$  as $h \to 0$.

Thm: $S \subset L^2$ is dense (already know true in L^1)

pf: $f \in L^2$, may assume $f \geq 0$ (both $f^+, f^- \in L^2$)

$E_n = \{x \mid \frac{1}{n} < f \leq n\}$ has finite measure $\qquad f_n(x) = \begin{cases} f(x) & x \in E_n \\ 0 & x \notin E_n \end{cases}$

(since $\chi_{E_n} = \chi_{E_n}^2 \leq n^2 f^2$) $\quad \|f - f_n\| \longrightarrow 0$

$$\int f_n = \int \chi_{E_n} f \leq \left(\int \chi_{E_n}^2\right)^{1/2} \|f\| \Rightarrow f_n \in L$$

Given $\varepsilon > 0$, pick n st. $\|f - f_n\| < \frac{\varepsilon}{2}$

pick $\underset{\substack{\vee \\ 0}}{s_k} \longrightarrow f_n$ in L^1-norm, may assume $s_k \leq n$.

But then $s_k \longrightarrow f_n$ in L^2-norm as well since $|f_n - s_k| \leq n$

ie. may choose k st. $\|f_n - s_k\| < \frac{\varepsilon}{2}$, $\Rightarrow \|f - s_k\| < \varepsilon$ ✱

Cor. S is approx by C in L^2-norm, hence $C \subset L^2$ dense.

e.g.

Let $\underset{\substack{\cap \\ C}}{g_m} \longrightarrow \underset{\substack{\cap \\ L^2}}{g}$ in $L^2([0,2\pi])$

$$\|g - S_n(g)\| \leq \|g - g_m\| + \|S_n(g_m) - S_n(g)\| + \|g_m - S_n(g_m)\|$$

$$\|S_n(g_m) - S_n(g)\| = \|S_n(g_m - g)\| \leq \|g_m - g\|$$

pick m st. $\|g - g_m\| < \varepsilon/3$

then pick N st. $n > N \Rightarrow \|g_m - S_n(g_m)\| < \varepsilon/3$

$\Rightarrow \|g - S_n(g)\| < \varepsilon$ ✱

Thm (Cor). The Fourier basis in $L^2([0,2\pi])$ is complete.

Poisson Summation Formula. (Intro)

$f \geqslant 0$ $^{(*)}$good function on $\mathbb{R}$ $\Rightarrow \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$

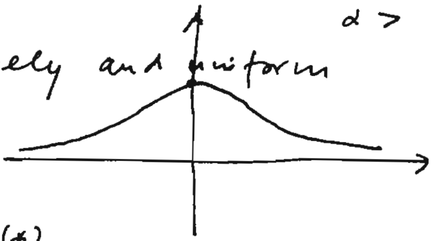
where $\hat{f}(y) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x y}\, dx = \sqrt{2\pi}\, \mathcal{F}f(2\pi y)$.

pf: Make f periodic : of period 1.

$$F(x) := \sum_{m\in\mathbb{Z}} f(x+m)$$

~ need conv. absolutely and uniform using $(*)_1 + (*)_2$

eg. $f(x) = e^{-\alpha x^2}$, $\alpha > 0$



F is of BV near any point x.

$$F(x) = \underset{\nearrow}{\sum_{m=0}^{\infty} f(x+m)} - \underset{\nearrow}{\sum_{m=-\infty}^{-1} \big(-f(x+m)\big)}$$

on $[0, \frac{1}{2}]$ using $(*)_2$

Similarly on $[-\frac{1}{2}, 0]$, then use periodicity.

$(*)_1$ $\int_{-\infty}^{\infty} f(x)\,dx$ exists as improper R

$(*)_2$ $f \nearrow$ on $\mathbb{R}_{\leq 0}$
$f \searrow$ on $\mathbb{R}_{\geqslant 0}$

$\Rightarrow$ Jordan's test

$$F(x) = \sum_{n=-\infty}^{\infty} \left(\int_0^1 \underset{\overset{\shortparallel}{\sum_{m=-\infty}^{\infty} f(x+m)}}{F(t)}\, e^{-2\pi i n t}\, dt \right) e^{2\pi i n x}$$

int. term by term

$$= \sum_{n=-\infty}^{\infty} \Big(\sum_{m=-\infty}^{\infty} \underset{\overset{\shortparallel}{\int_m^{m+1} f(t)\, e^{-2\pi i n t}\, \underset{\overset{\equiv}{1}}{e^{2\pi i n m}}\, dt}}{\int_0^1 f(t+m)\, e^{-2\pi i n t}\, dt} \Big) e^{2\pi i n x}$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, e^{-2\pi i n t}\, dt \right) e^{2\pi i n x}$$

Now set $x = 0$ #

Review of x_1 Differential Calculus 12/22

<u>Formula for $(f^{-1})'$:</u>

$$\underset{y_1}{\underset{\|}{\underline{f(x_0+h)}}} - \underset{y_0}{\underset{\|}{f(x_0)}} = f'(x_0)\underset{x_1-x_0}{\underset{\|}{h}} + o(h)$$

$$f'(x_0)^{-1}(y_1-y_0) = x_1-x_0 + f'(x_0)^{-1}\cdot o(\underset{\|x_1-x_0\|}{\underset{\|}{h}})$$

Need: $\|x_1-x_0\| \le M\cdot\|y_1-y_0\|$

in fact, $\|x_1-x_0\| \le L\|y_1-y_0\| + \underline{L\cdot\varepsilon}\,\|x_1-x_0\|$ make this < 1/2.

$\Rightarrow \underline{\|x_1-x_0\| \le 2L\|y_1-y_0\|}$

This key step already appeared in <u>the pf of chain rule</u>:

$$f(x+h) - f(x) = Ah + |h|\cdot\rho(h)$$
$$g(y+k) - g(y) = Bk + |k|\sigma(k)$$

$$g(\underset{f(x)+k}{\underset{\|}{f(x+h)}}) - g(\underline{f(x)}) = Bk + |k|\sigma(k)$$
$$= BAh + \underline{B}|h|\underline{\rho}(h) + |k|\sigma(k)$$

~ operator norm for B is needed!

<u>changing order of diff</u>: $D_{ij}f \underset{?}{=} D_{ji}f$

(*) $g(x,y) := f(x+h,y) - f(x,y)$

$$g(a,b+k) - g(a,b) = D_2 g(a, b+\theta k)\,k$$
$$= D_2 f(a+h, b+\theta k)k - D_2 f(a, b+\theta k)k$$

by (*) $\|$ $= D_{12} f(a+\theta_1 h, b+\theta k)\,hk$

Let

$A := f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$

$$= D_2 f(a+h, b+\theta k)k - D_2 f(a, b+\theta' k)k$$

$$\downarrow = \cancel{D_2 f(a,b)k} + \underline{D_{12} f(a,b)hk} + D_{22} f(a,b)\underline{\theta}kk + o(h^2+\theta^2k^2)k$$
$$- \cancel{D_2 f(a,b)k} - D_{22} f(a,b)\underline{\theta'}kk + o(\theta'^2k^2)k$$

The direct method does not work!

(**) : $h(x,y) := f(x,y+k) - f(x,y)$

$$h(a+h,b) - h(a,b) = D_1 h(a+\theta h, b)\,h$$
$$= (D_1 f(a+\theta h, b+k) - D_1 f(a+\theta h, b))h$$

In fact: <u>the following 3 all work</u>:

(i) $D_i f$, $D_j f$ both diff at (a,b).

(ii) $D_{ij} f$, $D_{ji} f$ conti at (a,b).

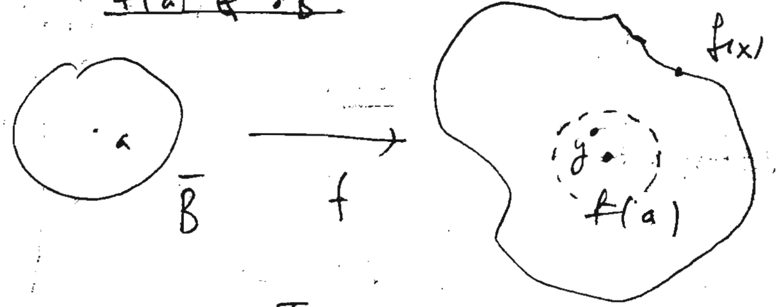
(iii) $D_i f$, $D_j f$, $D_{ij} f$ cont at (a,b) ($\Rightarrow D_{ji}$ exists).

Mappings with $J \neq 0$:

12/27

Thm [Onto] $f \in C(\overline{B})$, $J_f \neq 0$ on B ⇒ $f(B) \supset B(f(a), r')$ some r'.

$f(a) \notin \partial \overline{B}$



in particular, f is an open map. if f is 1-1.

$J_f \neq 0$

(可解方程式)

pf: · $\ell^2(x) = \| f(x) - y \|^2 = (f-y)^t \cdot (f-y)$

$$\frac{\partial \ell^2}{\partial x_i} = 2 \underset{\overset{\uparrow}{\det \neq 0}}{\frac{\partial f}{\partial x_i}^t} \cdot (f-y) \overset{\text{if } p \in \text{int}}{=} 0 \quad \text{at minimum } p$$

det ≠ 0 ⇒ $f(p) = y$.

· How to make sure $p \notin \partial B$?

let $m = \min \| f(x) - f(a) \|$

then $T := B(f(a); \frac{m}{2}) \ni y$

$$\| f(x) - \underline{y} \| + \underset{\overset{\wedge}{m/2}}{\| y - f(a) \|} \geq \| f(x) - f(a) \| \geq m$$

$\Rightarrow \| f(x) - y \| > \frac{m}{2}$ larger than $x = a$.

Thm [One-one].

if $f \in C'$ & $J_f(a) \neq 0$ ⇒ f is 1-1 on some $B(a, r)$.

pf: if $\exists\, x, y$ $0 = f_i(x) - f_i(y) = \nabla f_i(\xi_i) \cdot (x-y)$

Need $\det \begin{pmatrix} \nabla f_1(\xi_1) \\ \vdots \\ \nabla f_n(\xi_n) \end{pmatrix} \neq 0$ there is such a $B(a)$ nbhd a st this works.

12/29 Thm (IFT). $f \in C^1(S)$, S open in $\mathbb{R}^n$ $f : S \to \mathbb{R}^n$

$J_f(a) \neq 0 \Rightarrow \exists$ open $X \subset S$, $Y \subset \mathbb{R}^n$ st. f has inverse f^{-1} on Y

$f^{-1} \in C^1(Y)$. $(f^{-1})'(y) \cdot \overset{\& }{f'}(x) = id_{\mathbb{R}^n}$.

pf: find f 1-1 on $B(a)$, onto $B(f(a), r')$, set $X = f^{-1}(\underset{\overset{\parallel}{Y}}{B(f(a), \tfrac{r'}{2})})$

f^{-1} on $\bar{X}$ cpt $\Rightarrow$ g is conti.

To prove $f^{-1} \in C^1$: At a fixed $x_0 \in X$,

$$\underset{\overset{\parallel}{y_1}}{f(x_0+h)} - \underset{\overset{\shortparallel}{y_0}}{f(x_0)} = f'(x_0)\underset{\overset{\parallel}{x_1 - x_0}}{h} + o(h)$$

$$\Rightarrow f'(x_0)^{-1}(y_1 - y_0) = x_1 - x_0 + f'(x_0)^{-1} \cdot o(h) \quad (*)$$

Need $$\frac{\| f^{-1}(x_0)^{-1} \cdot o(h) \|}{\| y_1 - y_0 \|} \leq L \frac{o(h)}{\| x_1 - x_0 \|} \cdot \frac{\| x_1 - x_0 \|}{\| y_1 - y_0 \|} \longrightarrow 0 \ ?$$

say $\| x_1 - x_0 \| \underset{?}{\leq} M \| y_1 - y_0 \|$ ($\Rightarrow$ conti of f^{-1})

In fact, $(*) \Rightarrow \| x_1 - x_0 \| \leq L \| y_1 - y_0 \| + L \cdot \varepsilon \| x_1 - x_0 \|$, make $L \cdot \varepsilon < 1/2$

$\Rightarrow \| x_1 - x_0 \| \leq 2L \| y - y_0 \|$. done *

Notice that $A \mapsto A^{-1}$ is conti by Cramer's rule in the finite dim case.

IFT for Banach spaces: (replace $\mathbb{R}^n$ by $(V, \| \cdot \|)$)

pf: Assume $a = 0$, $f(a) = 0$, also $f'(0) = id_V$ (by $f'(0)^{-1} f(x)$)

given $b \in V$, to solve $f(x) = b$

is equiv. to fixed pt of $g(x) := x - (f(x) - b)$

(Newton's method)

$g'(0) = id_V - f'(0) = 0$ by $g \in C^1(V)$

$\Rightarrow$ Given $\varepsilon > 0$, $\exists \delta$ st. $\| g'(x) \| < \varepsilon$ for $\| x \| < \delta$

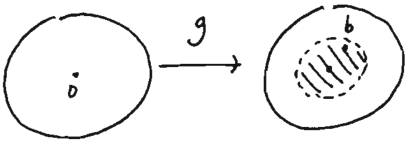
Set $\varepsilon = 1/2$. Get $\delta > 0$ fixed. Consider $\bar{B}_\delta$ and $\| b \| < \frac{\delta}{2}$

claim: $g : \bar{B}_\delta \longrightarrow \bar{B}_\delta$ and is a contr mapping.

(i) $\| g(x_1) - g(x_2) \| \leq \sup \| g'(\xi) \| \cdot \| x_1 - x_2 \| \leq \frac{1}{2} \| x_1 - x_2 \|$

(ii) $g(0) = b \Rightarrow \| g(x) - b \| \leq \frac{1}{2} \| x \|$

$\| g(x) \| \leq \| g(x) - b \| + \| b \| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$



$\bar{B}_\delta \subset V$ is closed, hence also complete.

$\Rightarrow \exists!$ fixed pt $a \in \bar{B}_\delta$ st $g(a) = a$, ie. $f(a) = b$

ie. f^{-1} exists on $B_{\delta/2}$. $f^{-1} \in C, C'$ as above. C' need $A \mapsto A^{-1}$ being C^0.

The implicit fcn thm.

$$f : \underset{\underset{(x,y)}{\downarrow}}{\mathbb{R}^{n+m}} \longrightarrow \mathbb{R}^n \qquad C^1$$

$$D_x f(\underset{\underset{(a,b)}{\shortparallel}}{p}) \text{ invertible.}$$

" non-linear Steniz
replacement theorem."

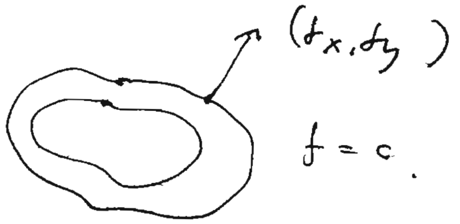
pf: $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} f(x,y) \\ y \end{pmatrix}$$

$$DF(p) = \left(\begin{array}{c|c} D_x f & D_y f \\ \hline 0 & I_m \end{array} \right) \text{ inv. at } p.$$

$$\text{IFT} \Rightarrow \quad G = F^{-1} \in C^1 : \begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow[G]{} \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix} \xrightarrow[F]{} \begin{pmatrix} f(g(u,v), h(u,v)) \\ h(u,v) \end{pmatrix}$$

$f(x,y) = 0$ get $h(u,v) = v$



$f = c.$

$$f(g(u,v), v) = u$$

$$\text{i.e.} \quad f(\underline{g(c,y)}, y) = c$$

✲

Extremal problem:

$f : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ candidate for min/max : ∂S, Singular, $f' = 0$

If $f'(a) = 0$, $f \in C^2$ near a:

$$f(a+t) = f(a) + \underset{\underset{0}{\shortparallel}}{f'(a)t} + \tfrac{1}{2} f''(z;t)$$

$$\overset{\shortparallel}{\tfrac{1}{2} f''(a;t) + \|t\|^2 E(t)}$$

Fact: $\lim_{t \to 0} E(t) = 0$.

$$Q(t) = \tfrac{1}{2} \sum D_{ij} f(a)\, t_i t_j$$

(since $|D_{ij} f(z) - D_{ij} f(a)| \frac{t_i t_j}{\|t\|^2} \longrightarrow 0$)

Prop: $Q(h) > 0 \Rightarrow f(a + \varepsilon h) > f(a)$ for ε small

$Q(h) < 0 \Rightarrow$ " $<$ "

Hence local max/min saddle pt dep only on Q if non-deg.

Application of implicit FT :

Extremal problem with side condi / Lagrange multi.

Thm $f : S \subset \mathbb{R}^{n+m} \to \mathbb{R}$, $g : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ All C^1 maps.

Then on the "surface" $g = c$, a "smooth" point p is an interior extr. pt for f $\Rightarrow$ $\exists\, \lambda_1, \dots, \lambda_n$ st. $\nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_n \nabla g_n$.

Def^{n}: p is a smooth pt on $\{ g = c \}$ $\Longleftrightarrow$ $g'(p)$ has rank $= n$.

pf: WLOG, may assume coor (x, y) : $p = (a, b)$

$g'(p)$ has its first $n \times n$ block invertible :

IFT $\Rightarrow$ $g(h(y), y) = c$

$$g'(p) = n\left(\underset{n}{\boxed{D_x g}} \,\Big|\, \underset{m}{D_y g} \right)$$

$x_1, \dots, x_n ; y_1, \dots, y_m$

for $1 \le j \le m$, $1 \le k \le n$: $\displaystyle\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \cdot \frac{\partial h_i}{\partial y_j} + \frac{\partial g_k}{\partial y_j} = 0$ (1)

Now for $w(y) := f(h(y), y) : \mathbb{R}^m \longrightarrow \mathbb{R}$, we must have

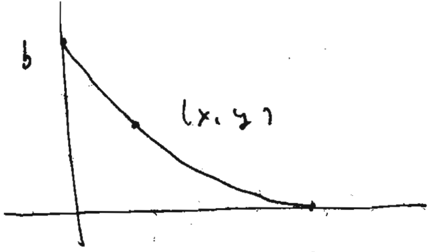
$$0 = \frac{\partial w}{\partial y_j}(b) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial h_i}{\partial y_j} + \frac{\partial f}{\partial y_j} \quad (2)$$

Since $\nabla_x g_1, \dots, \nabla_x g_k$ are linearly indep. $\Rightarrow$ $\nabla_x f = \displaystyle\sum_{k=1}^{n} \lambda_k \nabla_x g_k$

Hence $$\frac{\partial f}{\partial y_j} \underset{(1)}{=} - \sum_{k=1}^{n} \lambda_k \nabla_x g_k \cdot \frac{\partial h}{\partial y_j} \underset{(2)}{=} \sum_{k=1}^{n} \lambda_k \frac{\partial g_k}{\partial y_j}$$

ie. $\nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_n \nabla g_n$ at p for total ∇ *

Example of derivative in ∞-dim spaces : 1/5 2012. 36



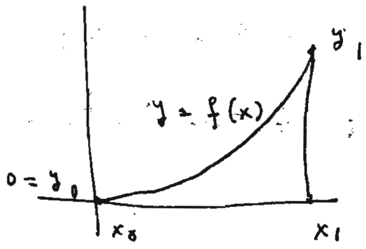
$$\frac{1}{2}mv^2 = mg(b-y)$$

$$\frac{ds}{dt} = v = \sqrt{2g(b-y)}$$

$$T = \int_0^a dt = \int_a^a \frac{\sqrt{1+y'^2}}{\sqrt{2g(b-y)}}\,dx$$

change var. Let's do

$$I(f) = \int_{x_0}^{x_1} \sqrt{\frac{1+f'^2}{f}}\,dx$$



$$I(f+tg) = \int_{x_0}^{x_1} \sqrt{\frac{1+(f'+tg')^2}{f+tg}}\,dx$$

$$\frac{d}{dt} I\Big|_{t=0} = \int \frac{f'g'}{\sqrt{1+f'^2}\sqrt{f}} + \sqrt{1+f'^2}\left(-\frac{1}{2}\right)\frac{g}{\sqrt{f}^3}\,dx$$

$$= -\int\left[\left(\frac{f'}{\sqrt{1+f'^2}\sqrt{f}}\right)' + \frac{1}{2}\frac{\sqrt{1+f'^2}}{\sqrt{f}^3}\right] g\,dx$$

How to solve the critical pt (differential) equation ?

~~General Euler-Lagrange eq'n~~ :

$$I(u) = \int_a^b F(x, u, u')\,dx \qquad u(a),\ u(b) \text{ fixed}$$

$$\frac{d}{dt} I(u+th)\Big|_{t=0} = \int_a^b \frac{d}{dt} F(x, u+th, u'+th')\,dx$$

$$h(a) = 0 = h(b) \qquad = \int_a^b (F_u h + F_{u'} h')\,dx$$

$$\searrow \quad \int F_{u'}\,dh = \underset{\overset{\shortparallel}{0}}{F_{u'} h\Big|_a^b} - \frac{d}{dx}(F_{u'})\,h$$

$$= \int_a^b L(u)\,h\,dx \equiv 0 \quad \forall h$$

$$\iff \quad L(u) := F_u - \frac{d}{dx}F_{u'} = 0 \quad (\text{why ?})$$

Noether Symmetry

Special cases: ① $F = F(x, u) \Rightarrow F_u(x, u) = 0$ alg. eq'n

② $F = F(x, u') \Rightarrow F_{u'} = c \Rightarrow$ solve u' $\Rightarrow$ solve u

③ $F = F(u, u') \Rightarrow$ 1st integral $E = F - u'F_{u'} = \text{const}.$

e.g. $F(u, u') = \sqrt{1+u'^2}/\sqrt{u} \Rightarrow \dfrac{\sqrt{1+u'^2}}{\sqrt{u}} - u'\dfrac{u'}{\sqrt{1+u'^2}\sqrt{u}} = c$

$$\Rightarrow \frac{1}{\sqrt{u}\sqrt{1+u'^2}} = c \Rightarrow \frac{dx}{du} = \sqrt{\frac{u}{c-u}} \Rightarrow \begin{cases} y(t) = c\sin^2 t \\ x(t) = c\left(t - \frac{1}{2}\sin 2t\right) \end{cases}$$

(cycloid)

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