Complex Geometry II. Algebraic Surfaces

lectures given at Taiwan University. 1999 Spring.

by Chin-Lung Wang (王金龍)

I. Fundamentals in Alg. Surfaces

Int. theory / R-R / Hodge index / Nakai-Moishezon / Castelnuovo's thm on (-1)

II. Basic Structure of Ruled Surfaces

str. of rat'l map / Enrique-Noether thm / Grothendieck lemma / formulae.

III. Structure Theorems for Rational Surfaces

Castelnuovo's thm A : rat'l $\Leftrightarrow q=0=P_2$ / thm B : min. rat'l $\Leftrightarrow \mathbb{P}^2, \mathbb{F}_n \ (n\neq 1)$.

IV. Classical Rational Surfaces : Projective Geometry

linear sys. of conics in $\mathbb{P}^2$ / L.S. of cubics / del Pezzo's / 27 lines.

V. Albanese Variety

Abel-Jacobi / Albanese map / uniqueness of min. model for non-ruled.

VI. Minimal Surfaces with $K^2<0$

statement of Enriques' thm / Iitaka's point of view / pf for case $K^2<0$.

VII. Surfaces with $p_g=0$, $q\geq 1$: Enriques' thm on $P_{12}(X)$.

χ for fiber space / χ for sing. curve / isotriviality / classification / pf of P_{12}.

VIII. Surfaces with $\kappa=0$

characterization of ab. v. / classification / Ex. of K3 : Kummer / $\exists$ Enrique.

IX. Elliptic Surfaces I : Easy Version

Refined Zariski lemma / $\kappa=1 \Rightarrow$ elliptic / Abundance theorem

X. Miyaoka-Yau inequalities for non-ruled Surfaces

"nef" condition / $c_2(X)\geq 0$ / Covering Trick / Bogomolov-Miyaoka's pf

XI. Geography of Surfaces of General Type

Kodaira lemma / $c_2(X)>0$ / Noether ineq. / Geography of (c_1^2, c_2).

XII. Intro. to Hermitian-Yang-Mills / Kähler-Einstein Geometry

Chern forms $c_i(E,\nabla)$ / Lübke's computation / Donaldson-UY; Yau's thm / Uniformi.

XIII. K3 Surfaces I : Torelli Theorem

Lattices / period map / Weyl transf. / exc. Kummer / Density / Limits / Moduli.

NO.
DATE

Un completed topics :

I. Elliptic Surfaces II : Advanced Version
rel. duality / Kodaira's classification for sing. fiber / can. bundle formula

II. Iitaka's Conjecture for $C_{2,1}$

III. K3 Surfaces II : Surjectivity of Period Map

IV. Noether's Formula for Riemann - Roch

Rmk : Notes on str. of rat'l map / Enrique - Noether thm is not written.

Paper House

Algebraic Surfaces (Fundamentals)

1. Intersection theory
2. Riemann-Roch
3. Hodge index theorem
4. Nakai-Moischezon criterion for ampleness
5. ~~Enrique-Castelnuovo's contractibility of~~ (-1) curves.

intersection theory: ($/\mathbb{C}$)

case 1. dim $X = 2$. cpx mfd

$$\begin{array}{ccc} L & & L' \\ & \searrow \ \swarrow & \\ & X & \end{array}$$

line bundles, then

$$L.L' := \left(c_1(L) \cup c_1(L')\right)[X]$$
$$= \int_X c_1(L) \wedge c_1(L')$$

If $L = \mathcal{O}(D)$. Def. then $= \int_D c_1(L')$

$$= \deg(L'|_D)$$

If $L' = \mathcal{O}(D')$ too. then $=: \underline{D.D'}$

Rmk: $\cap$ of cycles $\xleftrightarrow{P-D}$ cup product.
($\wedge$ of diff forms)

case 2. dim $X = n$. cpx mfd.

$C \hookrightarrow X$ curve. $L \to X$ line bundle

$L.C := \deg(\varphi^* L|_{\tilde{C}})$: $\varphi: \tilde{C} \to C$ normalization.

EX16. alternative way:

$C, D \subset X$ any two divisors.

$$C.D = \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1}) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{O}_X)$$

$\boxed{\text{Rieman-Roch for surface :}}$ (even not algebraic or Kähler)

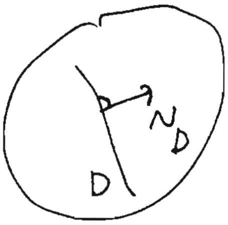
$D \subset X$ smooth curve

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \quad \text{get}$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0 \quad \text{hence}$$

$$\chi(\mathcal{O}_X(D)) = \underbrace{\chi(\mathcal{O}_X)}_{\text{const. of } X} + \underbrace{\chi(\mathcal{O}_D(D))}_{\text{R-R for curve } D}$$

$$\chi(\mathcal{O}_D(D)) = \deg(\mathcal{O}_D(D)) + 1 - g(D)$$



normal bundle

$\deg(\mathcal{O}_D(D)) = D^2$, and $1 - g(D)$ is given by the

adjunction formula $2g-2 = (K+D).D$

$$= \chi(\mathcal{O}_X) + \left(-\frac{KD + D^2 + 2}{2}\right) + D^2 + 1$$

$$= \underline{\frac{D(D-K)}{2} + \chi(\mathcal{O}_X)} \quad *$$

$$\boxed{\begin{aligned} &\chi(\mathcal{O}_X) \\ &= h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \\ &= 1 - h^{0,1} + h^{0,2} \end{aligned}}$$

* most nontrivial part :
(Noether's formula, special case of Hirzebruch R-R)

$$\underline{\chi(\mathcal{O}_X) = \frac{1}{12}\left(c_1^2(X) + c_2(X)\right)} \quad \text{topological information since } c_2(X) = \chi(X)$$

~~Main consequence~~ :

If $D^2 > 0$:

$$h^0(mD) - h^1(mD) + \overset{\displaystyle h^0(K - mD)}{\overset{\shortparallel}{h^2(mD)}} = \frac{mD(mD-K)}{2} + \chi(\mathcal{O}_X)$$

claim: $\underline{\underline{mD}}$ is ef for $m \to \infty$ or $m \to -\infty$

$$= \underset{\displaystyle\overset{\vee}{0}}{\frac{m^2}{2}D^2} - \frac{m}{2}D.K + \chi(\mathcal{O}_X)$$

pf: if not. then both $|K - mD|$, $|K + mD|$ has ∞ sections

but then $|K - mD| \otimes |K + mD| \to |2K|$ bounded. $\times$

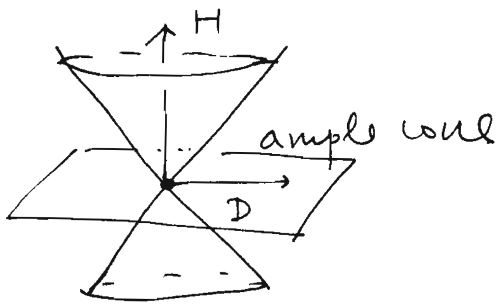
$\uparrow$ fixed one E.

$(S, E) \mapsto S + E$.

Hodge index theorem:
Let H be ample on X. D divisor $\not\equiv 0$ (numerical)
Then, $DH = 0 \Rightarrow D^2 < 0$.

Pf: if $D^2 \geq 0$.

$H^{1,1} \cap H^2(X, \mathbb{Z})$



<u>case 1. $D^2 > 0$:</u>

Let $H' = D + nH$ be ample $n >> 0$.

$\Rightarrow D.H' > 0$ and then mD is ef. $\not\equiv 0$

but then $mD.(kH) > 0$ $\times$

very ample

<u>case 2: $D^2 = 0$:</u>

$D \not\equiv 0 \Rightarrow \exists E$ st. $D.E \neq 0$

want to modify E <u>by H</u> st. <u>$E.H = 0$</u>

so use $E' = E - \dfrac{(E.H)}{(H.H)} H$

(ie. use $E' = (H^2)E - (E.H)H$)

then still <u>$D.E' \not\equiv 0$</u> and <u>$E'H = 0$</u>

But then we can modify D by E' st $D'^2 > 0$ & $D'.H = 0$.

use $D' = lD + E'$ then

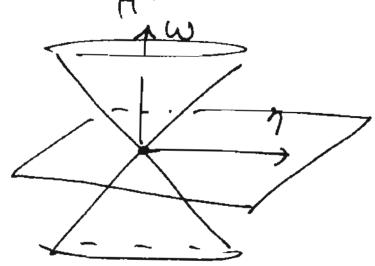
$$\begin{cases} D'^2 = \cancel{l^2 D^2} + 2lD.E' + E'^2 > 0 & \text{for some } l \in \mathbb{Z}. \\ D'H = 0 & \text{back to case 1.} \end{cases}$$
$\square$

17. Ex. (a) Thm is true more generally (Kähler case)

ω Kähler form, η (1,1) form $\neq 0$

$$\int_X \omega \wedge \eta = 0 \Rightarrow \int_X \eta^2 < 0.$$

(cf. G-H p.124-125)



(b) in fact show that: <u>Cauchy-Schwarz?</u>

$$\int \eta^2 \int \omega^2 \leq \left(\int \eta \wedge \omega\right)^2$$ for any (1,1) η.

Nakai - Moishezon : X proj. surface.

D ample $\Leftrightarrow$ $D^2 > 0$ &
$D.C > 0$ $\forall$ curve C.

pf: $\Rightarrow$ is trivial since $nD =$ v.a. $= \varphi^* H$ for
$\varphi = |nD| : X \longrightarrow \mathbb{P}^N \supset H = \mathcal{O}(1)$.

$\Leftarrow$: let H be an v.a. divisor (since X is proj.)
since $D^2 > 0$ & $D.H > 0$. $\exists$ mD is effective

So may assume D is eff. (but may be singular, reducible, non-reduced)

claim: $\mathcal{O}(D)|_D = \mathcal{O}_D(D)$ is ample (on D) $= \mathcal{L}|_D$

- may assume $\underline{D = \cup D_i}$, irreducible D_i
- may assume $\underline{D_i}$ is non-singular

lemma: $f : X \to Y$ finite morphism
$\mathcal{L} \to Y$ line bundle, $\mathcal{L}$ ample $\Leftrightarrow$ $f^*\mathcal{L}$ ample
(only need the case X smooth & $\Rightarrow$ part. then

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & & \\ Y & \xhookrightarrow[i]{} & \mathbb{P}^N \end{array}$$

$f^*\mathcal{L} = f^* i^* \mathcal{O}_{\mathbb{P}^N}(1)$
has positive curvature
since $i \circ f$ is finite.

(Rmk: This can be proved by using Serre's cohomological criterion for ampleness: Hart. III Ex. 5.7 d. Prop 3.3:

$\mathcal{L}$ ample $\Leftrightarrow$ $\forall$ coherent sheaves $\mathcal{F}$ on X, $\exists$ $n_0(\mathcal{F})$
s.t. $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ $\forall i > 0$ and $n \geq n_0$)

since D_i is a curve, normalization $\tilde{D}_i \to D_i$ is finite
and $\tilde{D}_i$ is already non-singular.

But then $\deg(\mathcal{L}|_{\tilde{D}_i}) = D.D_i > 0$ — This is in fact the def of int. number

and in a cpt R.S. $\deg > 0 \Leftrightarrow$ ample.
(~~irreducible~~) crucial assumption

$$0 \to \mathcal{O}_X((n-1)D) \to \mathcal{O}_X(nD) \to \mathcal{O}_D(nD) \to 0$$

$\Rightarrow$ $H^0(X, \mathcal{O}(nD)) \to H^0(D, \mathcal{O}_D(nD)) \to \underline{H^1(X, \mathcal{O}_X((n-1)D))}$

$\to \underline{H^1(X, \mathcal{O}_X(nD))} \to H^1(D, \mathcal{O}_D(nD)) \to \cdots$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\|}{0}$ for $n \gg 0$ since $\mathcal{O}_D(D)$ is ample.

$\Rightarrow$ $H^1(X, \mathcal{O}_X((n-1)D)) \twoheadrightarrow H^1(X, \mathcal{O}_X(nD))$

dim $\searrow$. finite dim $\Rightarrow$ hence $\cong$. must stops some where.

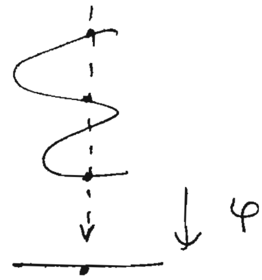
$\Rightarrow$ $H^0(X, \mathcal{O}_X(nD)) \xrightarrow{res|_D} H^0(D, \mathcal{O}_D(nD))$ $\quad n \gg 0$.

combine with $H^0(D, \mathcal{O}_D(nD))$ gen. sections at points.

$\Rightarrow$ $\underline{\mathcal{O}_X(nD) = \mathcal{L}^n \text{ is b.p.f}}$ for $n \gg 0$.

$\varphi = |\mathcal{L}^n| : X \longrightarrow \mathbb{P}^N$ i.e. $\mathcal{L}^n = \varphi^* \mathcal{O}(1)$.

but $D.C > 0 \ \forall C \Rightarrow \varphi$ is $\underline{\text{finite}}$ morphism



By previous lemma (Serre's thm)

$\mathcal{L}^n$ is ample. hence $\mathcal{L}$ is ample $\square$.

Ex18.(b) $\pi: \tilde{X} \to X$ blow up. $\dim X = 2$.

D v.a $\Rightarrow$ $2\pi^* D - E$ is ample on $\tilde{X}$

(Hart. p.394 . Ex V. 3.3)

effective estimate in Kodaira's vanishing thm.

How about general $\dim X = n$?

(a) For $\pi: \tilde{X} \to X$ blow up. at p. $E \cong \mathbb{P}^1$ exceptional div.

then $\operatorname{Div} \tilde{X} = \pi^* \operatorname{Div}(X) + \mathbb{Z}E$,

$E^2 = -1$ & as inner product space, i.e.

$\pi^* D . \pi^* D' = D.D'$

Castelnuovo's Criterion on <u>(-1) curve</u> :

$Y \subset X$ curve in a surface. $Y \cong \mathbb{P}^1$, $Y^2 = -1$
$\Rightarrow \exists f : X \longrightarrow X_0$ (nonsingular proj.) and $p \in X_0$
st. $f = Bl_p X_0$ with Y = exc. div.

pf: Pick H v.a. let $k = H.Y \geq 2$.
consider $L = H + kY$
may assume $H^1(X, H) = 0$

I. claim: $H^1(X, H + iY) = 0$ for $i = 0 \dots k$ (in fact to $k+1$)

$$0 \to \mathcal{O}_X(H + (i-1)Y) \to \mathcal{O}_X(H + iY) \to \mathcal{O}_Y(H + iY) \to 0$$

$$H^1(X, H + (i-1)Y) \to H^1(X, H + iY) \to H^1(\mathbb{P}^1, (H + iY)|_{\mathbb{P}^1})$$

$\uparrow$ deg ≥ 0 $(k - i)$

$H^1(\mathbb{P}^1, D) \cong H^0(\mathbb{P}^1, K - D)$ deg $K - D \leq -2$. $*$

II. $L = H + kY$ is bpf:
clearly L is very ample outside Y. Moreover

$$0 \to \mathcal{O}_X(H + (k-1)Y) \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L) \to 0$$

$$\Rightarrow H^0(X, L) \twoheadrightarrow H^0(Y, L|_Y) \quad \text{by I.}$$

now $\deg_Y(L) = 0$ ie. $L|_Y \cong \mathcal{O}_{\mathbb{P}^1}$ ($\Rightarrow$ any section $|_Y$ vanishes const. or)
$\Rightarrow$ L is gen. by global sections everywhere and $Y \mapsto P_1$ in pt.

$$f_1 : X \longrightarrow X_1 \subset \mathbb{P}^N \qquad L = f_1^* \mathcal{O}(1)$$

maybe 多餘.
($X - Y \cong X_1 - P_1$. Take $X_0 \to X_1$ normalization then

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_1"'] & X_0 \ni p \arrow[d] \\ & X_1 \ni P_1 \end{tikzcd}$$

(by univ. property)
still $X - Y \cong X_0 - P$.)

III. $H^1(X, H + (k-2)Y) = 0 \Rightarrow$

$$H^0(X, H + (k-1)Y) \twoheadrightarrow H^0(Y, (H + (k-1)Y)|_Y)$$

$$\cong H^0(\mathbb{P}^1, \mathcal{O}(1))$$

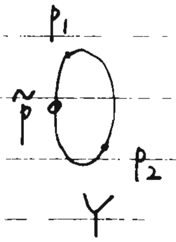
To show that $f = Bl_p X_0$, need to find coor system of nbd of Y: need 2 functions.

Let $p_1, p_2 \in Y$ $p_1 \neq p_2$ ($0, \infty$ of $\mathbb{P}^1$)

$\xi_i \in H^0(X, H+(k-1)Y) \subset H^0(X, L)$

st. $\xi_i|_Y$ vanishes exactly at p_i $i=1,2$

$\xi_0 \in H^0(X, L)$ nonzero everywhere on Y



$U_1 = Y - p_2$ $\tilde{U}_1$ nbd of U_1 in X (small enough)

$U_2 = Y - p_1$

$z_1 = \dfrac{\xi_1}{\xi_0}$; $z_2 = \dfrac{\xi_2}{\xi_0}$ holo. functions on X

* claim: $z = (z_1, z_2)$: nbd of $Y \xrightarrow{\sim}$ nbd of 0 in $\mathbb{C}^2$.

pf: show that z_1/z_2, z_2 is a local cor of $\tilde{U}_1$:

$d\left(\frac{z_1}{z_2}\right) \neq 0$ on $T_{\tilde{p}}(Y) \subset T_{\tilde{p}}(X)$ for $\tilde{p} \in U_1$

$dz_2 \neq 0$ on $T_{\tilde{p}}(X) / T_{\tilde{p}}(Y)$ * (basically bec. $H+(k-1)Y$ is still v.a outside Y)

Now its clear f near $Y = (z_1, z_2) = Bl_p X_0$.

The crucial part of Riemann-Roch for surfaces is :

$$\chi(\mathcal{O}_X) = \frac{c_1^2 + c_2}{12} \quad \text{(Noether's formula)}$$

$$\| $$

$$1 - h^{0,1} + h^{0,2}$$

$$\chi = 2 - 2b_1 + b_2 = \underline{-4h^{1,0} + 2 + b_2}$$

$$\sigma = b_2^+ - b_2^- = (2h^{2,0} + 1) - (b_2 - 2h^{2,0} - 1)$$

$$= \underline{4h^{2,0} + 2 - b_2} \quad \text{(Hodge index thm)}$$

$$\Rightarrow \quad \chi + \sigma = 4(1 - h^{1,0} + h^{2,0}) \underset{\substack{\uparrow \\ \text{expect}}}{=} \frac{c_1^2 + \chi}{3}$$

$\Rightarrow$ equiv. to $\boxed{c_1^2 = 2\chi + 3\sigma}$

This is again equiv. to $\quad \sigma = \dfrac{c_1^2(X) - 2c_2(X)}{3} \quad (*)$

Recall Pontryagin classes :

$$p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \; ; \quad p_i(X) := p_i(T_X)$$

Our case $\quad T_X \otimes \mathbb{C} \cong T_X \oplus \overline{T_X}$

$$c(T_X \otimes \mathbb{C}) = c(T_X) \cdot c(\overline{T_X})$$

$$= (1 + c_1 + c_2) \cdot (1 - c_1 + c_2)$$

$$\Rightarrow \quad p_1(T_X) = -c_2(T_X \otimes \mathbb{C}) = c_1^2(X) - 2c_2(X)$$

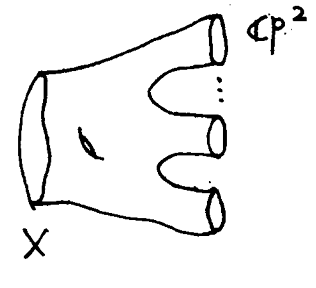
hence $(*) \Leftrightarrow \underline{\sigma(X) = \dfrac{p_1(X)}{3}}$ (Hirzebruch signature formula for $\dim_{\mathbb{R}} = 4$)

- Idea of Hirzebruch's proof :

(1) any orientable 4 manifold (C^∞) is cobordant to disjoint union of $\mathbb{C}P^2$'s (Thom's thm)

(2) $\sigma(X)$ and $p_i(X)$ are both cobordism invariants

(3) $\sigma(X) = 1 = \frac{1}{3} p_1(X)$ when $X = \mathbb{C}P^2$.



Basic structure of ruled surfaces

A • Thm: $X \to C$ geom. ruled $\Rightarrow$ $\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{P}_C(E) \arrow[dl] \\ & C & \end{tikzcd}$

E a rank 2 v.b over C, and

$\mathbb{P}_C(E) \cong \mathbb{P}_C(E') \iff E' \cong E \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on C.

pf: $$0 \to \mathbb{C}^\times \to GL(n,\mathbb{C}) \to PGL(n,\mathbb{C}) \to 0$$

$\Rightarrow$ exact sequence of non-abelian groups

$$\underset{\substack{\text{isom. class} \\ \text{of line bundles}}}{H^1(C, \mathcal{O}^\times)} \to \underset{\substack{\text{isom. class of} \\ \text{rk}=n \text{ v.b's}}}{H^1(C, GL(n,\mathcal{O}^\times))} \to \underset{\substack{\text{isom. class of} \\ \mathbb{P}^{n-1} \text{ bundles}}}{H^1(C, PGL(n,\mathcal{O}^\times))} \to H^2(C,\mathcal{O}^\times)$$

(the existence relies on the fact that $\mathbb{C}^\times$ is an ab. gp)

claim: $H^2(C,\mathcal{O}^\times) = 0$ if $\dim C = 1$:

$$0 \to \mathcal{O}_C^\times \to \mathcal{K}_C^\times \to \mathrm{Div}(C) \to 0$$

exact, by the definition of Cartier divisors.

but $\mathcal{K}_C^\times$ and $\mathrm{Div}(C)$ are flasque ($\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ if $U \supset V$)

hence a ~~flasque resolution~~ of $\mathcal{O}_C^\times$

$\Rightarrow H^2(C,\mathcal{O}^\times) = 0$ since has only 2 terms. $\square$

(Q: Are there and "Enrique-Noether" thm true for $\mathbb{P}^n$ bundle over C and geom. $\mathbb{P}^n$ fibration?

Enrique-Noether $\Rightarrow$ " $X \to C$ geom. ruled $\iff \mathbb{P}^1$-bundle ")

• Zariski Lemma: $p: X \to C$ with ~~conn. fiber~~

if $F = \sum n_i C_i$ is a reducible fiber, then $C_i^2 < 0$ $\forall i$

pf: $F.C_i = 0$ $\forall i$ since F moves, but then

$$n_i C_i^2 = C_i.(F - \textstyle\sum_{j\neq i} n_j C_j) = 0 - \sum_{j \neq i} n_j \underset{\vee\atop 0}{(C_i.C_j)} < 0 \quad \square$$

• Cor: X minimal and $p: X \to C$ generic fiber $\cong \mathbb{P}^1$

$\Rightarrow X \to C$ is geom. ruled.

pf: $F^2 = 0$, $F \cong \mathbb{P}^1 \Rightarrow F.K = -2$, for other reducible $F = \sum n_i C_i$

$\exists C_i$ s.t. $K.C_i < 0$ but $C_i^2 < 0 \Rightarrow C_i = (-1)$ curve ※

Last time have seen:

- ~~Thm (Severi)~~: for $C \ncong \mathbb{P}^1$, minimal models of $C \times \mathbb{P}^1$ are exactly geom. ruled surfaces, so are in fact $\mathbb{P}_C(E)$ for a rk 2 vector bundle.

Recall the pf:

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\varepsilon"'] \arrow[d] \arrow[drr, "\psi"] & & \\ X \arrow[r, dashed, "\varphi"'] & C \times \mathbb{P}^1 \arrow[rr, "pr_1"'] & & C \end{tikzcd}$$

$\varepsilon = \varepsilon_n \circ \varepsilon_{n-1} \circ \ldots$ sequence of blow-ups.

let $n =$ minimal number of all such resolution

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\varepsilon"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, dashed] & & C \end{tikzcd}$$

then $\psi(E_n) = pt$ since $E_n \cong \mathbb{P}^1$ and $C \ncong \mathbb{P}^1$

but then ε_n is unnecessary! $\square$.

For $C \cong \mathbb{P}^1$ the proof does not work.
but we have a even better classification of Castelnuovo.

B • ~~Thm: (Grothendieck lemma)~~: Any vector bundle over $\mathbb{P}^1$ splits as direct sum of line bundles.

- lemma: $\exists$ line sub-bundle of $E \to C$ (for any C):

pf: by taking $E \otimes \mathcal{O}_C(d)$ $d \gg 0$ may assume $h^0(E) > 0$.
(in fact. gen. by global secs)

a section of $E \longleftrightarrow$ " $\mathcal{O}_C \longrightarrow E$ "

~~take dual~~ get $E^* \longrightarrow \mathcal{O}_C$

but the image must be an ideal sheaf, so $= \mathcal{O}_C(-D)$.

i.e. get surjective morphism $E^* \longrightarrow \mathcal{O}_C(-D) \longrightarrow 0$

~~take dual~~ get $0 \longrightarrow \mathcal{O}_C(+D) \longrightarrow E \longrightarrow M \longrightarrow 0$ $\square$.

must be a vector sub bundle.

Corollary: Rieman-Roch for v.b. over C.

$$\chi(E) = c_1(E) + r(1-g)$$

pf: Induction + Whitney formula $\square$.
+ above lemma.

Rmk: $c_1(E) = \deg(E)$ in Beauville's book.

the pf of Grothendieck's lemma (G-H p.516-517)

Case $rk\, E = r = 2$: get, for a given section $\sigma \in H^0(E)$:

$$0 \to L_1 \to E \to L_2 \to 0$$

and in fact $L_1 = \mathcal{O}_{\mathbb{P}^1}(k)$ if σ has k zeros.

ie.

$$0 \to \mathcal{O}_{\mathbb{P}^1}(k) \to E \to \mathcal{O}_{\mathbb{P}^1}(d-k) \to 0 \quad \text{if } \deg E = d$$

In general, for an extension of sheaves

$$0 \to L \to E \to M \to 0 \quad \text{to be trivial}$$

$\Leftrightarrow$ the class $\delta(1) \in H^1(C, L \otimes M^*)$ is 0

$\delta(1)$ given by $0 \to L \otimes M^* \to E \otimes M^* \to \mathcal{O}_C \to 0$

and $\underset{\underset{\mathbb{C}}{\parallel}}{H^0(C, \mathcal{O}_C)} \xrightarrow[\delta]{} H^1(C, L \otimes M^*)$

Now $L \otimes M^* = \mathcal{O}_{\mathbb{P}^1}(2k-d)$ so

$H^1(C, L \otimes M^*) = H^0(\mathbb{P}^1, \mathcal{O}(d-2k-2)) \overset{\text{want}}{\underset{?}{=}} 0$

- Method 1: $\otimes E$ by $\mathcal{O}(\ell)$ st. $\deg E = 0$ or 1 (since $rk\, E = 2$).

  still have sections since $\chi(E) = \deg E + (1-\cancel{g}) \cdot 2$.

Case $rk\, E = r > 2$:

By induction, suppose that true for all $rk \leq r-1$

$\otimes E$ by $\mathcal{O}(\ell)$ to get sections and get

$$0 \to L_1 \to E \to \underset{\underset{\bigoplus_{i=2}^{r} L_i}{\parallel}}{M} \to 0 \qquad rk\, M = r-1$$

Let $L_i = \mathcal{O}_{\mathbb{P}^1}(k_i)$; $\Sigma k_i = d$

- Method 2: Pick L_1 st k_1 is maximal possible

  then $k_1 \geq k_i \quad \forall i$.

only need to consider rank 2 case. say $k_1 = n$, $k_2 = m$

If $m > n$ then $\tilde{\tau} \in \Gamma(L_2)$ with $(\tilde{\tau}) = p_1, \cdots, p_m \in \mathbb{P}^1$

but $H^0(\mathbb{P}^1, E) \to H^0(\mathbb{P}^1, L_2) \to \underset{\underset{0}{\parallel}}{H^1(\mathbb{P}^1, L_1)}$ since $n \geq 0$

$\Rightarrow \exists \tau$ st. $\tau \mapsto \tilde{\tau}$

but $0 \to L_1 \to E \to L_2 \to 0$ exact

$\Rightarrow \tau(p_i) \in L_1|_{p_i}$; now consider only $p_1, \dots, p_{n+1}$

consider $\tau_i \in \Gamma(L_1)$ s.t. $\begin{cases} \tau_i(p_i) = \tau(p_i) \\ \tau_i(p_j) = 0 \text{ for } j = 1, \dots, n+1, \ j \neq i \end{cases}$

the existence follows from

$$H^0(\mathbb{P}^1, L_1 - (q_1 + \dots + q_n)) \twoheadrightarrow \mathbb{C}_{q_{n+1}} \to H^1(\mathbb{P}^1, \mathcal{O}(n-(n+1))) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) \overset{\text{s.d.}}{=} H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$$

But then $\tau - \sum_{i=1}^{n+1} \tau_i \in \Gamma(\mathbb{P}^1, E)$

and τ vanishes at $p_1, \dots, p_{n+1} > n$ pts.

Now $H^1(\mathbb{P}^1, L_1 \otimes M^*) = H^1(\mathbb{P}^1, \oplus_{i=1}^r L_1 \otimes L_i^*)$

$= \oplus_i H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\underset{\geq 0}{k_1 - k_i})) = 0$

so $\delta(1) = 0$ and the sequence splits. □

So all geom. ruled surface over $\mathbb{P}^1$

$\cong \mathbb{P}_{\mathbb{P}^1}(E) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta)) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$, $n \geq 0$

this is called the Hirzebruch surface $\mathbb{F}_n$.

Later will see there are all the minimal rational surfaces with $\mathbb{P}^2$ and $\mathbb{F}_n$ $(n \neq 1)$

- Rmk: $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1)) \cong \mathbb{P}^2$ blow up 1 pt, not minimal (see later)

C. Trivial Formula for geom. ruled surfaces :

$X = \mathbb{P}_C(E)$:
$$\begin{array}{ccc} X & \leftarrow & p^*E \\ {\scriptstyle p}\downarrow & & \downarrow \\ C & \leftarrow & E \end{array}$$
Let N = universal subbundle (Hopf bundle)

$\Rightarrow \quad 0 \to N \to p^*E \to \mathcal{O}_{\mathbb{P}(E)}(1) \to 0$

$\mathcal{O}_{\mathbb{P}(E)}(1)$ = universal quotient bundle Q

$Q|_F = \mathcal{O}_F(1)$. hyperplane class of degree 1.

let $h = [\mathcal{O}_{\mathbb{P}(E)}(1)] \in \mathrm{Pic}(X)$ or $H^2(X,\mathbb{Z})$ Rule-p.5/6

($h^{2,0} = h^0(X,K_X) = P_1(X) = 0$

since $h^{i,0}$ and P_r all bi-rat'l inv.

and for ~~$\mathbb{P}^1\times C$~~ it's true)

Lemma: (1) $\mathrm{Pic}(X) = p^*\mathrm{Pic}(C) \oplus \mathbb{Z}h$

(h = class of fiber) (2) $H^2(X,\mathbb{Z}) = \mathbb{Z}f \oplus \mathbb{Z}h$

(3) $\deg E := c_1(E) = h^2$

(4) $[K] = -2h + (\deg E + 2g(C) - 2)f$ in $H^2(X,\mathbb{Z})$

pf: (1), (2) are obvious.

(3): $0 \to N \to p^*E \to \mathcal{O}_{\mathbb{P}(E)}(1) \to 0$

$\deg E = c_1(E) \longleftrightarrow$ pts in C

$= c_1(p^*E).h$

$= ([N]+h)h = h^2$ if $[N].h = 0$

Now $c_2(p^*E) = p^*c_2(E) = 0$ on C,

but ~~whitney formula~~ $\Rightarrow$ $\cancel{c_2(p^*E) = c_1(N).c_1(\mathcal{O}_{\mathbb{P}(E)}(1))} = [N].h$ ※

(4): Let $K = ah + bf$; since $f^2 = 0$

$-2 = Kf + f^2 = (ah+bf).f = a\,h.f = a$

so $K = -2h + bf$

Now $X \xrightarrow{p} C$ must admit sections $s: C \to X$

(take $U \subset C$ st $X|_{p^{-1}(U)} = p^{-1}(U) \xrightarrow{\sim} U \times \mathbb{P}^1$

take any trivial s then take ~~closure~~ s in X)

$C \cong s(C)$, so if $[s(C)] = h + rf$, then

$2g(C) - 2 = (h+rf)^2 + (h+rf).(-2h+bf)$

$= \deg E + \cancel{2r} - 2\deg E + b - \cancel{2r}$. done $\square$

Corollary: (1) $\mathrm{Pic}\,\mathbb{F}_n = \mathbb{Z}f \oplus \mathbb{Z}h$ since $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$

(2) $\exists!$ irreducible curve B on $\mathbb{F}_n$ st $B^2 < 0$

in fact, $[B] = h - nf$ and $[B]^2 = -n = -\deg E$

(3) $\mathbb{F}_n \cong \mathbb{F}_m \Leftrightarrow n = m$, $\mathbb{F}_n$ minimal $\Leftrightarrow n \neq 1$.

Pf: (2): $\pi: E = \mathcal{O} \oplus \mathcal{O}(n) \longrightarrow \mathcal{O}$

gives a section $\mathbb{P}(E) \xleftarrow{s} \mathbb{P}(\mathcal{O}) = C$ (via $x \mapsto [\ker \pi_x]$)

Let $B = s(C)$, then $[B] = h + rf$

why? but $s^* \mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}$ i.e. $\underset{[B].h}{s^* h} = 0$

$\Rightarrow \underset{n}{h^2} + rfh = 0$; $r = -n$

$[B]^2 = (h - nf)^2 = n - 2n = -n$.

Claim: Any irreducible curve $D \neq B \Rightarrow D^2 \geq 0$.

Let $D = \alpha h + \beta f$. $D.f \geq 0 \Rightarrow \alpha \geq 0$

$D.B \geq 0 \Rightarrow (\alpha h + \beta f)\cdot(h - nf)$

$= \cancel{\alpha n} - \cancel{\alpha n} + \beta \geq 0$

$\Rightarrow D^2 = \alpha^2 n + 2\alpha\beta \geq 0$. Now (2) $\Rightarrow$ (3) trivially $\square$.

$$\boxed{\begin{array}{l}\text{Rmk: Some important "Non-minimal rational surface" are} \\ \text{cubic surfaces, and more generally, del Pezzo surfaces} \\ \text{these are all non-ruled.} \\ \text{They = blow ups of } \mathbb{P}^2 \text{. (classical proj. geom)}\end{array}}$$

Structural Theorem for Rational Surfaces

A. Castelnuovo's criterion:

An alg. surf. X is rat'l $\iff q(X) = P_2(X) = 0$

pf: $\Rightarrow$ follows from birationality of q and P_r.

$\Leftarrow$: may assume X minimal, also $P_2(X) = 0 \Rightarrow P_1(X) = h^0(K) = 0$
(so $\chi(\mathcal{O}_X) = 1$)

Lemma I: X minimal alg. $q(X) = P_2(X) = 0$
D any div. $\Rightarrow |D + nK| = \phi$ for $n \gg 0$.

pf: Case $K^2 \geq 0$; (this step uses $q = P_2 = 0$ and projectivity)

$$R.R \Rightarrow \underset{\substack{\| \\ h^0(-K) - h^1(-K) + \underset{\substack{\| \\ \cancel{h^0(2K)} = P_2}}{h^2(-K)}}}{\chi(-K)} = \frac{(-K)(-2K)}{2} + \underset{\substack{\| \\ 1 - \cancel{q(X)} + h^0(K)}}{\chi(\mathcal{O}_X)}$$
$= 0$ too.

$\Rightarrow h^0(-K) > 0$

but $-K \not\sim 0$ (otherwise $h^0(2K) = h^0(0) = 1$)

so $-K \sim$ effective div.

$\Rightarrow |D + nK| = \phi$ for $n \gg 0$

eg. take H v.a. $D + nK \sim ef \Rightarrow D.H + n\underset{\substack{\wedge \\ 0}}{\underline{K.H}} > 0$ $\times$.

~~Case $K^2 < 0$~~ : (this step uses minimal only)

$(D + nK).K = D.K + nK^2 < 0$ for $n \geq n_0 > 0$ some n_0.

If $D + nK \sim ef$ for a large n

then $D + n_1 K \sim \sum a_i C_i \quad a_i > 0$

$0 > (D + n_1 K).K = \sum a_i C_i.K \Rightarrow$ say C_1, $\underline{K C_1 < 0}$

If $\underline{C_1^2 < 0}$ then C_1 is a (-1) curve $\times$

(adjunction formula $2g(C_1) - 2 = KC_1 + C_1^2$)

So $\underline{C_1^2 \geq 0}$; this $\Rightarrow C_1.S \geq 0$ for any ef. div S
(write S as irred. decomp)

$D + nK \sim ef \Rightarrow (D + nK).C_1 \geq 0$ this $\times$ to

$K.C_1 < 0$ for $n \gg 0$. $\square$.

Lemma II: X minimal alg. $q(X) = p_2(X) = 0$
$\Rightarrow \exists$ smooth $C \cong \mathbb{P}^1$ st $C^2 \geq 0$.

Pf: Take D v.a. then $\exists\ n = n(D)$ st
$$|D + nK| \neq \emptyset \text{ but } |D + (n+1)K| = \emptyset \text{ by lemma I.}$$

Case 1: $\exists\ D$ v.a st. $|D + (n+1)K| = \emptyset$ but $D + nK \sim \text{ef} \neq 0$:

say $D + nK \sim C = \sum_i a_i C_i$ ef. div.

$$D + (n+1)K \sim K + C = K + C_i + \Sigma \quad \text{not ef} \Rightarrow K + C_i \text{ not ef.}$$

- C_i ef $\Rightarrow h^0(-C_i) = 0$
- $K + C_i$ not ef $\Rightarrow h^0(K + C_i) = 0$, $h^0(K+C_i) = h^2(-C_i)$

$\Big\} \Rightarrow$ by R-R

(*) $$0 - h^1(-C_i) + 0 = \frac{(-C_i)(-C_i - K)}{2} + 1$$

ie. $2g(C_i) - 2 = (K + C_i).C_i = -2(1 + h^1(C_i)) \leq -2$

$\Rightarrow$ all C_i are smooth rational curves.

If $C_i^2 < 0$ for all i, then
$$X \text{ minimal} \Rightarrow K.C_i \geq 0 \quad \forall i$$

$$\Rightarrow K.C \geq 0 \text{ and } (K + C).C = (D + (n+1)K).C = \underbrace{D.C}_{>0} + \underbrace{(n+1)KC}_{\geq 0} > 0$$

But R.R for $\mathcal{O}_X(-C)$ get as before:

(**) $$\cancel{h^0(-C)} - h^1(-C) + \cancel{h^0(K+C)} = \frac{(-C)(-C-K)}{2} + 1$$

ie. $(K + C).C \leq -2$. $\times$

This case is impossible for any surface with $h^1(\mathcal{O}_X) = 0 = h^2(\mathcal{O}_X)$:

~~Case 2~~: Every ample div. $D = -nK$ for some $n \in \mathbb{N}$:

this $\Rightarrow$ every div. $D = mK$ for some $m \in \mathbb{Z}$.

ie. $\mathrm{Pic}(X) \cong \mathbb{Z}$, but $\underset{q}{h^1(\mathcal{O})} = \underset{p_g}{h^2(\mathcal{O})} = 0 \Rightarrow H^2(X, \mathbb{Z}) = \mathbb{Z}$.

with $c_1(X) = -K$ a generator, Poincaré duality $\Rightarrow c_1^2(X) = \pm 1$

$q = 0 \Rightarrow h^1 = 0 \Rightarrow c_2(X) = \chi(X) = 3$, $c_1^2 + c_2 \not\equiv 0 \,(12)$ $\times$ Noether's formula. $\square$

Pf of Castelnuovo's thm: (conti.)

Now $\exists\, C \cong \mathbb{P}^1$, $C^2 \geq 0$ get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$$

$\Rightarrow$

$$0 \to \mathbb{C} \to H^0(X, \mathcal{O}_X(C)) \twoheadrightarrow H^0(X, \mathcal{O}_C(C)) \to H^1(X, \mathcal{O}_X)$$

$\Rightarrow h^0(\mathcal{O}_X(C)) = \underline{2 + C^2} \geq 2$ $\qquad H^0(\mathbb{P}^1, \mathcal{L})$. $\deg \mathcal{L} = C^2 \geq 0$

Pick $D \in |C|$, $D \neq C$, then the linear system $\langle C, D \rangle$ spanned by $C, D = \{sC + tD\}$ has no base component, (bec. $C^2 \geq 0$)

but $\# Bs\langle C, D\rangle = C^2$!

$\Rightarrow$ get a rational map $X \dashrightarrow \mathbb{P}^1 = \{(s:t)\}$

Notice that <u>the linear system $\langle C, D\rangle$ maps $C \mapsto (1:0)$; $D \mapsto (0:1)$</u>

hence after blow ups points (may in C) get

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\varepsilon}\swarrow & & \searrow^{\varphi} \\ X & \dashrightarrow & \mathbb{P}^1 \end{array}$$

with $C' \cong \mathbb{P}^1 \subset \tilde{X}$ one of the fiber

<u>Enrique - Noether</u> thm $\Rightarrow$ fiberation $\tilde{X} \xrightarrow[\varphi]{} \mathbb{P}^1$ is loc. trivial near $\varphi(C) = pt$. hence $\tilde{X}$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$.

End.

In fact, more detailed analysis of lemma II shows the following classification of rat'l surfaces:

B. ⦿ THEOREM: X minimal rat'l $\Longleftrightarrow$ $X = \mathbb{P}^2$ or $\mathbb{F}_n$ for $n \neq 1$.

Rmk: X rat'l $\Rightarrow q(X) = 0 = P_2(X) \Rightarrow \exists\, C \cong \mathbb{P}^1$ st. $C^2 \geq 0$

$$\left(\begin{array}{l} \exists\, C \cong \mathbb{P}^1 \text{ st } C^2 \geq 0 \\ \text{and } q(X) = h^1(\mathcal{O}_X) = 0 \end{array} \right)$$

In fact if only have $C \cong \mathbb{P}^1$, $C^2 \geq 0$ then expect a map $X \to S$, ~~S curve of genus $= q(X)$~~ and will then have actually $C^2 = 0$.

pf: As in the pf of Castelnuovo's thm: consider $|C|$:

Case $C^2=0 \Rightarrow h^0(\mathcal{O}_X(C))=2$ and have
$|C|: X \xrightarrow{\varphi} \mathbb{P}^1$ with $C=$ one of the fiber
In this case no need to blow up X since $C^2=0 \Rightarrow$
no base point of $|C|$. So

fibers $\longleftrightarrow$ members of $|C|$

General Remark of such a linear system: (全未用到)

For any $D \in |C|$, has $(K+D).D=(K+C).C=-2$
since $C \cong \mathbb{P}^1$

- So D irreducible $\Rightarrow D \cong \mathbb{P}^1$ too

otherwise $D=\sum a_i C_i$, $a_i>0$

- CLAIM: All C_i are smooth rational curves:

but $h^0(K+D)=0$ bec. $(K+D).C=-2$ and $C^2 \geq 0$

$\Rightarrow h^0(K+C_i)=0 \ \forall i$

$$h^0(K+C_i) = h^2(-C_i)$$

$$\text{R.R} \Rightarrow \cancel{h^0(-C_i)} - h^1(-C_i) + \cancel{h^2(-C_i)} = \frac{(-C_i)(-C_i-K)}{2}+1$$

$$\Rightarrow (C_i+K).C_i \leq -2 \Rightarrow C_i \cong \mathbb{P}^1 \quad ※$$

So if $|C|$ has a non-smooth member $D=\sum a_i C_i$, then $C_i \cong \mathbb{P}^1 \ \forall i$

$$-2=(K+C).C \ (\text{since } C^2 \geq 0) \geq K.D=\sum a_i(K.C_i)$$

$\Rightarrow K.C_i<0$ for some i, then $C_i^2 \geq 0$ (X minimal)

$$C^2 = D^2 = \Big(a_i C_i + \sum_{j \neq i} a_j C_j\Big).C$$

$$= a_i C_i . D + \sum a_j C_j . C$$

$$= \underset{\geq 0}{a_i^2 C_i^2} + \underset{> 0}{a_i C_i . \sum a_j C_j} + \underset{\geq 0}{\Big(\sum a_j C_j\Big).C}$$

if more than 2 components. (C irr. $C^2 \geq 0$)

In case $C^2=0$ $\exists$ D reducible get $\times$. hence all
$D \in |C|$ are smooth $\mathbb{P}^1$'s, i.e.

$|C|: X \xrightarrow{\varphi} \mathbb{P}^1$ is a smooth $\mathbb{P}^1$ bundle $\xRightarrow{\text{Enrique-Noether}}$ Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$

We only need to deal with the case that

$\nexists$ smooth $\mathbb{P}^1 \cong C$ with $C^2 = 0$. (but do $\exists\, C \cong \mathbb{P}^1$, $C^2 > 0$)

Let $k = \min(B^2)$ with $B \cong \mathbb{P}^1$, $B^2 > 0$. Let $C^2 = k$.

未用到

but the above calculations shows that if $\exists\, D \in |C|$ non smooth

$$k = C^2 = \underset{\geq 0}{a_i^2 C_i^2} + \underset{> 0}{a_i C_i \cdot \sum a_j C_j} + \underset{\geq 0}{\left(\sum a_j C_j\right) \cdot C}$$

then C_i is a smooth $\mathbb{P}^1$ with $C_i^2 \geq 0$ but $< k$.

hence all $D \in |C|$ are smooth $\mathbb{P}^1$'s.

Pick $D \in |C|$, $D \neq C$, consider $\varphi = \langle C, D \rangle : X \dashrightarrow \mathbb{P}^1$

Blow up k points of C, then $C'^2 = 0$ and $C' \cong \mathbb{P}^1$

$$\begin{tikzcd} X' \arrow[d, "\varepsilon'"'] \arrow[dr, "\varphi' = |C'|"] & \\ X \arrow[r, dashed, "\varphi"'] & \mathbb{P}^1 \end{tikzcd}$$

$X' \longrightarrow \mathbb{P}^1$ a generic $\mathbb{P}^1$ bundle since $C' \longmapsto pt \in \mathbb{P}^1$.

- If all fibers of φ' irreducible then φ' is a $\mathbb{P}^1$ bundle and $X' \cong \mathbb{F}_n$. but this can happen only when $k = n = 1$. and so $X = \mathbb{P}^2$.

- If $\exists$ reducible fibers D,
  $D = \sum a_i C_i$, $KC_i < 0$ for some i, also
  $0 = C_i \cdot D = C_i\left(\sum a_j C_j\right) \Rightarrow C_i^2 < 0 \quad \forall i$ (need Zariski conn. thm)
  i.e. C_i is a (-1) curve of φ'. If C_i also ε'-exceptional,
  then can contract C_i to get

$$\begin{tikzcd} X'' \arrow[d, "\varepsilon''"'] \arrow[dr, "\varphi''"] & \\ X & \mathbb{P}^1 \end{tikzcd}$$

but the proper transform C'' of C is still a fiber of φ'', hence $C''^2 = 0$ $\times$.

So C_i is not ε'-exceptional, but then $\bar{C_i} = \varepsilon'(C_i) \subset X$ is a smooth $\mathbb{P}^1$ with $\bar{C_i}^2 \geq 0$ and $< k$ $\times$ again since $k = \min$.

$C_i^2 = -1$ and must meet some exc. divisor.

End

X minimal rat'l $\Rightarrow$ $X = \mathbb{P}^2$ or $\mathbb{F}_n$ $(n \neq 1)$.

pf: Know $\exists\, C \cong \mathbb{P}^1$ st. $C^2 \geq 0$

let $k = \min C^2$ among such $\mathbb{P}^1$'s.

Since $h^0(C) = 2 + C^2$

- Case $k = 0$ $\Rightarrow$ $|C| : X \xrightarrow{\varphi} \mathbb{P}^1$ base point free system

claim: all fibers are irreducible $\mathbb{P}^1$'s.

then φ is a $\mathbb{P}^1$ bundle $/\mathbb{P}^1$ i.e. $X = \mathbb{P}_{\mathbb{P}^1}(E)$

$= \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) = \mathbb{F}_n$ the Hirzebruch surface $(n \neq 1)$.

pf of claim: fiber = member $D \in |C|$

if $D = \sum a_i C_i$ then $-2 = K.C = K.D = \sum a_i (K.C_i)$

$\Rightarrow K.C_i < 0$ for some i

but $0 = C_i.C = C_i.D = a_i C_i^2 + \sum_{j \neq i} a_j (C_i.C_j) \Rightarrow C_i^2 < 0 \ \forall i$

(Here we need Zariski's conn. thm)

$\Rightarrow$ get a (-1) curve $\rightarrow\!\!\!\!\leftarrow$.

Notice also all irreducible D is $\mathbb{P}^1$ since $(K+D).D = (K+C).C = -2$.

- Case $k \geq 1$, We still look at $|C|$; $C^2 = k$.

此页以下部分全未用到]. if $\exists$ reducible $D = \sum a_i C_i \in |C|$ again

$K.D = K.C \leq K.C + C^2 = -2 \Rightarrow K.C_i < 0$ for some i

X minimal $\Rightarrow C_i^2 \geq 0$, then

$$
\begin{aligned}
k = C^2 = D^2 &= \Big(a_i C_i + \sum_{j \neq i} a_j C_j\Big).C \\
&= a_i C_i.D + \sum a_j C_j.C \\
&= \underbrace{a_i^2 C_i^2}_{\geq 0} + \underbrace{a_i C_i . \sum_{j \neq i} a_j C_j}_{> 0 \text{ if } \exists j \neq i} + \underbrace{\Big(\sum a_j C_j\Big).C}_{\geq 0 \text{ since } C \text{ irr and } C^2 \geq 0.}
\end{aligned}
$$

but then C_i is a $\mathbb{P}^1$ st. $C_i^2 \geq 0$ and $C_i^2 < k$ $\rightarrow\!\!\!\!\leftarrow$

General Rmk: Here all components C_i are $\mathbb{P}^1$'s is a general fact:

$C^2 \geq 0$ and $(K+D).C = -2 \Rightarrow K+D$ is not $\sim$ eff.

$\Rightarrow h^0(K + C_i) = 0 \ \forall i$

R-R $\Rightarrow$ $\cancel{h^0(-C_i)} - h^1(-C_i) + \underbrace{\cancel{h^2(-C_i)}}_{= h^0(K+C_i)} = \dfrac{(-C_i)(-C_i - K)}{2} + 1$

$\Rightarrow (K + C_i).C_i = -2$ ※.

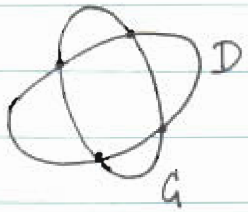
So all $D \in |C|$ are irreducible.

• Case $k \geq 1$: Again consider $\langle C, D \rangle$:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^1 \\ & X' \arrow[ul, "\varepsilon'"] \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

$\varepsilon' = k$-times blow-ups

φ' is well-defined since $C'^2 = 0$, $|C'|$ b.p.f.



(C_i' = proper transf of C_i)

claim: φ' has no reducible fibers: by applying the $k=0$ case to the map $\varphi' : X' \to \mathbb{P}^1$: any reducible fiber $\supset (-1)$ curve C_i, hence can be blow down to X''

$$\begin{tikzcd} X \arrow[rr, dashed] & & \mathbb{P}^1 \\ & X'' \arrow[ul, "\varepsilon''"] \arrow[ur, "\varphi''"'] & \end{tikzcd}$$

if C_i is ε'-exceptional then get $*$ since $C'' \subset X''$ still $C''^2 = 0$ i.e. C only requires $(k-1)$ blow ups to drop C^2 to 0!

So C_i is not ε'-exceptional, but then $\varepsilon'(C_i) = \bar{C}_i \subset X$ is a smooth $\mathbb{P}^1$ with $\bar{C}_i^2 \geq 0$ and $\bar{C}_i^2 < k$

($C_i^2 = -1$, go back a blow up one time ↑ at most 1) $*$

Finally so, $\varphi' : X' \to \mathbb{P}^1$ has irred. fibers $\cong \mathbb{P}^1$,

$\Rightarrow$ φ' is a non-minimal $\mathbb{P}^1$-bundle $= \mathbb{F}_n$

$\Rightarrow$ $X' = \mathbb{F}_1$ and ε' = single blow up, i.e. $k = 1$.

i.e. $X = \mathbb{P}^2$.

the proof is now complete. □

Classical rational Surfaces : Projective Geometry

(cf. Hartshorne V.4 ; Beauville ch IV .)

In general, for a smooth proj. variety X

$|D|$ complete linear system asso. to a div D

$|D - P_1 - \cdots - P_r| =$ sublinear system of $|D|$

st. passes through P_i $\forall i$

$\Rightarrow$ 1-1 correspondence to $|\varphi^* D - E_1 - \cdots - E_r|$

$$\begin{tikzcd} X' \arrow[d, "\varphi = \text{blow up } P_1, \cdots, P_r"] \\ X \end{tikzcd}$$

call P_i ~~assigned base points~~

can always blow up to resolve base points.

I. eg. (simplest linear system of lines on $\mathbb{P}^2$)

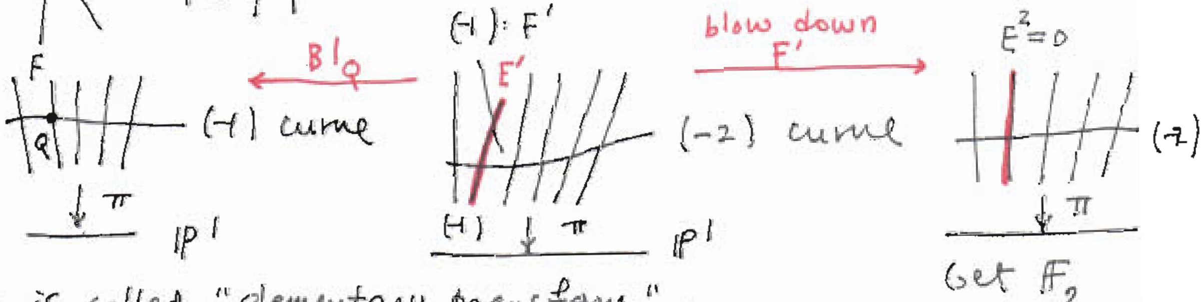
(1).

$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

$|l - p|$

$$\begin{tikzcd} & Bl_p\, \mathbb{F}_1 \arrow[dl] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$

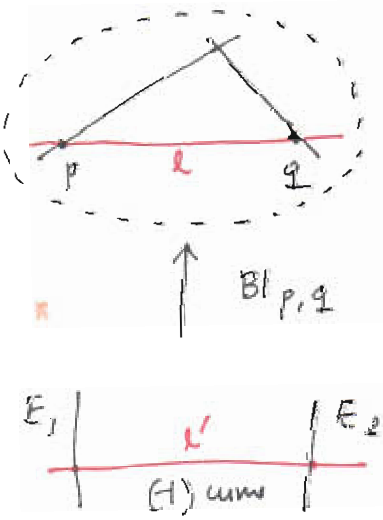
(2).



This is called "elementary transform".

In this way may get all $\mathbb{F}_n$. (include $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$)

II. eg. (Product linear system of lines)



$|L - p| \otimes |L - q| \hookrightarrow |2L - p - q|$

get

$$\begin{tikzcd} \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0 \\ & X' \arrow[ul, "Bl_{p,q}"] \arrow[ur, "\text{contracts } l'"'] & \end{tikzcd}$$

This is a very special sub-li sys. of conics.

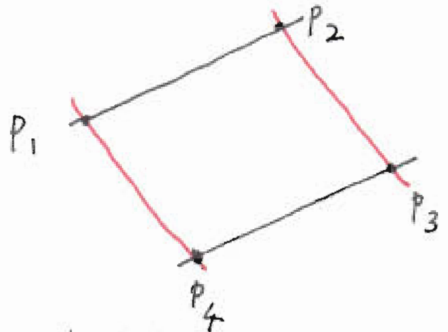
III. Linear system of conics :

Fact: Let $\mathcal{L}$ be the li. sys. of conics in $\mathbb{P}^2$

$|2L - p_1 - \cdots - p_r|$ with no 3 pts on a line.

If $r \leq 4$ then $\mathcal{L}$ has no other base points.

pf: May assume $r=4$.



$(l_{12}+l_{34}) \cap (l_{14}+l_{23})$

$= p_1+p_2+p_3+p_4$ with mult 1.

So no other base point in $\mathcal{L}$ *

Rmk: May allow p_2 infinitely near p_1, ie. $p_2 \in E$ of Bl_{p_1}.

Since $|2L|$ is very ample of dim $= 5$, the above $\Rightarrow$ if $r \leq 5$, then $\dim \mathcal{L} = 5-r$

eg. $r=0$. Veronesse embedding: $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$r=1$. $\varphi = Bl_p$

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl, "\varphi"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^4 \end{tikzcd}$$

$|\varphi^*(2L)-E|$ is v.a.

(need the above Rmk)

Here degree of X in $\mathbb{P}^N = X.H^* = (H|_X)^*$

hence $=$ $\cap$ number of elements in $\mathcal{L}$ outside the assigned base points.

so $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$ $\deg = 4$

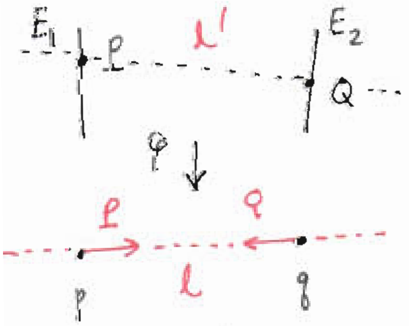
$\mathbb{F}_1 \longrightarrow \mathbb{P}^4$ $\deg = 3$ (cubic scroll)

$r=2$.

$$\begin{tikzcd} & X' \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^3 \end{tikzcd}$$

$|\varphi^* 2L - E_1 - E_2|$ is bpf

only trouble for this to be not v.a. is when $P \in E_1$, $Q \in E_2$ in the following special way:



bec. then p, q, P, Q are not in general position.

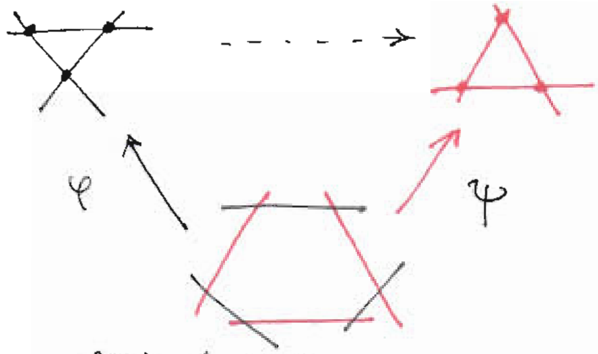
In fact ψ contract l' to get $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$.

the Segre embedding, of degree 2. (This recovers II).

~~r=3 : quadratic transform of $\mathbb{P}^2$~~ :

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^2 \\ & X' \arrow[u, "\text{Bl. 3 pts}"] \arrow[ur, "\text{blow down } 3\ (-1)"'] & \end{tikzcd}$$

This is useful in the theory of plane curves (eg. Walker)



get 6 (-1) curves.

IV <u>Linear system of cubics</u> :

$$h^0(\mathbb{P}^2, \mathcal{O}(d)) = H^3_d = C^{3+d-1}_d = C^{2+d}_d = C^{2+d}_2 = \frac{(d+2)(d+1)}{2}$$

so $\dim|2L| = \frac{4\cdot 3}{2} - 1 = \underline{5}$

$\dim|3L| = \frac{5\cdot 4}{2} - 1 = \underline{9}$

<u>THEOREM</u>: $\mathcal{L} = |3L - P_1 - \cdots - P_r|$ has no unassigned base point if : $r \leq 7$, no 4 of P_i are colinear
no 7 of P_i are on a conic.

<u>Corollary</u>: $r \leq 8 \Rightarrow \dim \mathcal{L} = 9 - r$
$r = 8$, $\dim = 1$, almost all curves in $\mathcal{L}$ is irreducible

Rmk: May allow $P_2 \longrightarrow P_1$.

• <u>THEOREM</u>:
$$\begin{tikzcd} & X' \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & \mathbb{P}^{9-r} \end{tikzcd}$$
$|\varphi^* 3L - E_1 - \cdots - E_r|$ is very ample if $r \leq 6$
and no 3 P_i colinear, no 6 P_i on a conic.

<u>Corollary</u>: For $r = 0, 1, \cdots, 6$ get $X' \hookrightarrow \mathbb{P}^{9-r}$
a surface of degree $9-r$, $X' = \mathbb{P}^2$ blow up r pts.
(For $r=6$, get ~~cubic surface in $\mathbb{P}^3$~~.) Moreover,
$K_{X'} \cong \mathcal{O}_{X'}(-1)$, the negative hyperplane section in $\mathbb{P}^{9-r}$.

pf: $\deg$ of $X' = D^2 - r = 3^2 - r = 9 - r$.

$$K_{X'} = \varphi^* K_{\mathbb{P}^2} + E_1 + \cdots + E_r$$
$$= -\varphi^* 3L + E_1 + \cdots + E_r = -D'. \quad \square$$

Def: X is a Del Pezzo surface if $-K_X$ is very ample.

Ex. Every cubic surface = $\mathbb{P}^2$ blow up 6 pts.
(Need Castelnuovo's thm)

V. 27 lines on cubics :

Thm : Let $X_d \subset \mathbb{P}^d$ be the del Pezzo surface of degree $d = 9 - r$ then X_d has finite lines :

r	0	1	2	3	4	5	6
# E_i	0	1	2	3	4	5	6
# $\langle p_i p_j \rangle$	0	0	1	3	6	10	15
# conics thr. 5 of p_i	0	0	0	0	0	1	6
Total	0	1	3	6	10	16	$\boxed{27}$

pf: Since $K_X = -H$ (H hyperplane section in $\mathbb{P}^d$)
ℓ a line $\Leftrightarrow H.\ell = 1 \Leftrightarrow K.\ell = -1$
but $\ell \cong \mathbb{P}^1$, $g(\ell) = 0$ hence $\ell^2 = -1$ too
i.e. ℓ is a (-1) curve. May assume that $\ell \neq E_i$
Let E_i be those φ - exc. curves.

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^d \end{tikzcd}$$

$\begin{cases} \ell . H = 1 \\ \ell . E_i = 0 \text{ or } 1 \end{cases} \Rightarrow \ell \sim mL - \sum m_i E_i$

(L = pull back of a line in $\mathbb{P}^2$ in X)

$\begin{cases} 3m - \sum m_i = 1 \\ m_i = 0 \text{ or } 1 \end{cases}$

$\Rightarrow$ $m = 1$, 2 of $m_i = 1$ or $\longleftrightarrow$ $\ell \in |L - E_i - E_j|$
$m = 2$, 5 of $m_i = 1$ $\qquad \ell \in |2L - E_{i_1} - \cdots - E_{i_5}|$.

the proof is complete. $\square$

Modern Powerful method :

Let $G(2,4) = \{ \mathbb{C}^2 \subset \mathbb{C}^4 \}$ = all lines in $\mathbb{P}^3$

$$\begin{tikzcd} E = \mathrm{Sym}^3 S^* \arrow[d] \\ G(2,4) \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

let S = universal rk 2 subbundle
S^* = linear forms in $\{x, y\}$

$\dim G(2,4) = 2 \cdot 2 = 4$; $\mathrm{rk}\, \mathrm{Sym}^3 S^* = 4$

f: a cubic poly $\longmapsto$ a section $\sigma \longmapsto (\sigma)$ = lines in (f).

So need to calculate $c_4(E)$. use Schubert !

End

Albanese variety :

X compact Kähler manifold

$i : H_1(X, \mathbb{Z}) \longrightarrow H^0(X, \Omega^1_X)^*$

γ defines a functional $\omega \longmapsto \int_\gamma \omega$

since $H^1(X, \mathbb{Z}) \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ by Hodge Theory

where $H^{1,0}(X) \cong H^0(X, \Omega^1_X)$ and $H^{0,1} = \overline{H^{1,0}}$,

thus i is injective on torsion-free part, $\operatorname{Im} i$ is a full rk lattice $= \Lambda$ by Poincaré duality

let $\mathrm{Alb}(X) = H^0(\Omega^1_X)^* / \Lambda = \mathbb{C}^q/\Lambda$

Called the Albanese torus. $\exists$ natural map, fix $p \in X$

$$\alpha : X \longrightarrow \mathrm{Alb}(X) \qquad q = h^0(\Omega^1_X)$$

$$x \longmapsto \int_p^x \vec{\omega} \ ; \quad \vec{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_q \end{bmatrix}; \ \omega_i \text{ basis of } H^0(\Omega^1_X)$$

Rmk 1: If $\dim X = 1$, ie. X a Riemann Surface

$\mathrm{Alb}(X) = \mathrm{Jac}(X)$ the Jacobian variety of X

$X \to \mathrm{Jac}(X)$ is called the Abel-Jacobi map.

Abel-Jacobi thm asserts that $X \hookrightarrow \mathrm{Jac}(X)$ and induces

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X).$$

Rmk 2: For any Kähler X, $\mathrm{Alb}(X)$ is an abelian variety (why?) but we do not need this here. In fact, if X is projective over k then $X \xrightarrow{\alpha} \mathrm{Alb}(X)$ is too!

Thm: $\alpha^* : H^0(\mathrm{Alb}(X), \Omega^1) \xrightarrow{\sim} H^0(X, \Omega^1)$ and $\mathrm{Alb}(X)$ is the unique complex tori A satisfies the functorial property: Any $X \longrightarrow T$, T cpx tori factors through α:

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\varphi"] & \\ A \arrow[rr, "\psi"] & & T \end{tikzcd} \qquad \exists!\, \psi \text{ st } \varphi = \psi \circ \alpha.$$

pf: α^* is an isom. is simply the ~~definition~~ of $Alb(X)$.

existence:

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\varphi"] & \\ Alb(X) & & T \end{tikzcd}$$

To find $\psi : Alb(X) \longrightarrow T$, where $Alb(X) = H^0(\Omega^1_X)^*/\Lambda$ and $T = V/L$,

is equiv to find $V^* \longrightarrow H^0(\Omega^1_X)$, with $V^* \cong H^0(\Omega^1_T)$.

This step is trivially true via φ^*

Such that under this map (φ^*)

$$\begin{array}{ccc} L^* & \longrightarrow & \Lambda^* \\ \wr\wr & & \wr\wr \\ H^1(T, \mathbb{Z}) & & H^1(X, \mathbb{Z})/tor \end{array}$$

which is again trivial.

uniqueness of $Alb(X) \longrightarrow T$ is also obvious. $\square$

- Corollary:

(1) $\alpha(X) \not\subset$ any translates of sub-tori of $Alb(X)$

i.e. $\alpha(X)$ generates $Alb(X)$

(2) $X \to Y$ induces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Alb(X) \arrow[r] & Alb(Y) \end{tikzcd}$$

and $X \twoheadrightarrow Y \Rightarrow Alb(X) \twoheadrightarrow Alb(Y)$

simply bec. $\Gamma(Y, \Omega^1) \hookrightarrow \Gamma(X, \Omega^1)$, dual get $\twoheadrightarrow$.

- Theorem: X cpt Kähler, $\alpha : X \to Alb(X)$. If $\alpha(X)$ is a curve C, then C is a smooth curve of genus $q(X)$ and $\alpha : X \to C$ has connected fibers.

pf:

$$\begin{tikzcd} & & X \arrow[dll, "\tilde{\alpha}"'] \arrow[dl, "\alpha'"] \arrow[dr, "\alpha"] & \\ \tilde{C} \arrow[r, "\varepsilon"'] & C \arrow[rr, hook, "j"'] & & Alb(X) \end{tikzcd}$$

let $\tilde{C} \xrightarrow{\varepsilon} C$ be the normalization

X normal ($\Leftarrow$ smooth)

$\Rightarrow \alpha'$ lifts to $\tilde{\alpha} : X \to \tilde{C}$.

$$\Rightarrow \begin{tikzcd} Alb(\tilde{C}) \arrow[r] & Alb(Alb(X)) \\ \tilde{C} \arrow[u, "\text{Abel-Jacobi thm}"] \arrow[r, "j\circ\varepsilon"] & Alb(X) \arrow[u, equal] \arrow[ul] \\ & X \arrow[u, "\alpha"'] \arrow[ul, "\tilde{\alpha}"] \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} Alb(\tilde{C}) \arrow[r, "\cong", "\text{by Cor(1)}"'] & Alb(X) \arrow[dl, leftrightarrow] \\ \tilde{C} \arrow[u, hook] & \end{tikzcd}$$

$\Rightarrow C = \tilde{C}$ and $g(C) = q(X)$.

Now stein factorization thm (cf. Hartshorne)

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\alpha'"] & Y \arrow[dl, "g"] \\ C & \end{tikzcd}$$

p surj. connected fiber
g : finite morphism with Y normal
(hence Y smooth since $\dim C = 1$)

Previous argument shows again $Y \hookrightarrow Alb(X)$, $Y \to C \hookrightarrow Alb(X)$
hence $Y = C$. $\square$.

Thm: In case X a surface with $P_1 (= P_g) = 0$ and $q \geq 1$
$\alpha(X) \subset Alb(X)$ is a curve. $\quad \overset{\shortparallel}{h^0(X,K)}$

pf: For if $\alpha(X)$ is 2-dim'l, $\exists$ holo 2 form ω on $Alb(X)$ such that $\omega[\alpha(X)] \neq 0$ (This part is not trivial):

take $x \in \alpha(X)$ smooth pt, then $\alpha(X)$ near x has local equation $u_3 = \cdots = u_q = 0$ (u_i loc. coor.)
But $Alb(X)$ is a tori $\Rightarrow \exists\ \omega \in \Gamma(\Omega^2)$ st

$$\omega(x) \equiv du_1 \wedge du_2 (x)$$

this is the desired ω.
But $\alpha^* \omega \in \Gamma(X, \Omega^2) = \Gamma(X, K)$ which is not possible if $P_g = 0$. #

Rmk: Peternell has recently shown that if $\dim X = 3$ the existence of holomorphic form usually implies existence of map $X \longrightarrow A$ (tori) st. the form is a pull back.

☆ We already know all minimal models for ruled surfaces

- for irrational ruled, all are $\mathbb{P}_C(E)$, E rk 2 v.b $/C$, $g(C) \geq 1$
- for rational surfaces, $\mathbb{P}^2$, $\mathbb{F}_n$ $(n \neq 1)$ $n = 0, 2, 3, \cdots$

The remaining cases are Non-ruled surfaces, will show uniqueness of minimal model!

Fundamental Theorem of minimal model (dim 2).

Every non-ruled surface X admits unique minimal model. up to isomorphisms ($Aut(X)$).

pf: let

$$\begin{tikzcd} X' \arrow[rr, dashed, "\varphi"] & & X \\ \cdot \arrow[u, "\varepsilon_1"] & & \\ & \tilde{X} \arrow[ul, "\varepsilon_n"] \arrow[uur, "f"'] & \end{tikzcd}$$

birat'l map between two minimal surfaces

let ε_i blow-ups resolving indeterminacy. with n smallest

let E_n = exc. curve of ε_n. then $f(E_n)$ is a curve $C \cong \mathbb{P}^1$

otherwise ε_n is not necessary.

Now $K_{\tilde{X}} . E_n = -1 = E_n^2$

$\dim X = 2 \Rightarrow f$ is also composition of blow ups

E_n must touch f-exc. divisors, otherwise $C^2 = -1$ $\times$

CLAIM: $K_X . C \leq -2$, hence $C^2 \geq 0$. (bec. $g(C) = 0$)

For a single blow up

$$\begin{array}{ccc} Y & \supset & C' \\ \downarrow \varphi & & \downarrow \\ X & \supset & C \end{array}$$

$K_Y . C'$
$= (\varphi^* K_X + E) . C'$
$= K_X . C + E . C'$

$K_Y . C' \geq K_X . C$ and $>$ if C' touch E

So $\Rightarrow$ $-1 = K_{\tilde{X}} . E_n > K_X . C$. $*$

- $P_r(X) = 0$ $\forall r \in \mathbb{N}$ since $C^2 \geq 0$, $rK \sim ef \Rightarrow r.C.K \geq 0$

(later will see this is equiv. to ruled)

case $q(X) = 0$: Castelnuovo $\Rightarrow X$ rat'l $\times$

case $q(X) \geq 1$: Albanese map $\alpha : X \twoheadrightarrow Z$ curve $g(Z) = q \geq 1$

$\Rightarrow$ C is in a fiber F

$C^2 \geq 0$ + Zariski lemma $\Rightarrow F = mC$, $m \in \mathbb{N}$

$\Rightarrow C^2 = 0$; $K_X . C = -2$

$2g(F) - 2 = (F + K).F = -2m \Rightarrow m = 1$, $g(F) = 0$

Noether - Enrique thm $\Rightarrow X$ ruled. $\times$ End

Surfaces with $p_g = 0$ and $q \geq 1$

(and surfaces with $P_n = 0$ $\forall n \in \mathbb{N}$ $\Leftrightarrow$ ruled)

Main Result will be

Thm (Enrique) : TFAE :

1) X is ruled
2) $\exists C \neq (-1)$ curve st $K.C < 0$
3) Adjunction terminates : Any div D, $|D+nK| = \phi$ for $n \gg 0$.
4) $P_n = 0$ $\forall n \in \mathbb{N}$
5) $P_{12} = 0$

1) $\Rightarrow$ 2) : classification of minimal ruled surfaces $\Rightarrow$
$f: X \to M$, M = geom. ruled (or $\mathbb{P}^2$)
$\exists$ fiber F (or line L in $\mathbb{P}^2$) st f = isom there,
then $f^*F^2 = 0$; $K_X . f^*F = K_M . F = -2 < 0$
(or $f^*L^2 = 1$; $K_X . f^*L = K_{\mathbb{P}^2} . L = -3 < 0$).

2) $\Rightarrow$ 3) : $K.C < 0$ and $C \neq$ exc. curve $\Rightarrow$ $C^2 \geq 0$
(adjunction formula), then
$(D+nK).C = D.C + n(K.C) < 0$ for $n \gg 0$
$\Rightarrow$ $D + nK$ $\not\sim$ effective.

3) $\Rightarrow$ 4) : take $D = 0$ and notice that $P_n = 0$ $\forall n \gg 0 \Rightarrow P_n = 0$ $\forall n$.

(4) $\Rightarrow$ 5) : trivial.)

Need only the part : $P_{12}(X) = 0 \Rightarrow X$ ruled.

Lemma : X minimal with $K^2 < 0$ then
$P_n(X) = 0$ $\forall n \in \mathbb{N}$ and $q(X) \geq 1$.

pf : $|nK| \neq 0 \Rightarrow nK \sim \sum n_i C_i$, $n_i > 0$
$nK^2 < 0 \Rightarrow (\sum n_i C_i).K < 0 \Rightarrow \exists i$ $C_i.K < 0$
since $C_i . C_j \geq 0$ for $i \neq j$, $\Rightarrow$ $C_i^2 < 0$
adjunction formula $\Rightarrow$ $C_i = (-1)$ curve $\times$

Now $q \neq 0$, otherwise X = rational by Castelnuovo's thm
but then $K_{\mathbb{F}_n}^2 = 8 > 0$, $K_{\mathbb{P}^2}^2 = 9 > 0$. $\times$. $\square$

Prop: X minimal surface $K^2<0 \Rightarrow X$ ruled (irrationally ruled) $K^2<0$ - p. 2/5

Pf: lemma $\Rightarrow$ $p_g(x)=0$; $q(x)\geq 1$

in fact the method $\Rightarrow P_n(x)=0 \ \forall n$

$\Rightarrow$ Albanese map $p: X \to B$ smooth curve $g(B)=q(X)\geq 1$

- If apply ~~Iitaka's conjecture $C_{2,1}$~~ i.e.

$$\underset{\overset{\vee\!\vee}{0}}{\kappa(X_s)} + \kappa(B) \leq \underset{\overset{\shortparallel}{-\infty}}{\kappa(X)} \quad \text{since } P_n(x)=0 \ \forall n$$

then $\kappa(X_s)=-\infty$, $\Rightarrow X_s \cong \mathbb{P}^1$ for general s. done by Noether-Enrique lemma.

- Now we use elementary method. Assume that X is not ruled:

step 1. $\exists$ irred. C st $|K+C|=\phi$, $K.C\leq -2$ (can be any $-k$)

$\frac{1}{2}\to$ step 2. $p|_C \to B$ is étale, and is an isom if $q\geq 2$.

step 3. show that the fiber $\cong \mathbb{P}^1$

Recall for X minimal

I. $\exists n, |nK+H|\neq\phi$ but $|(n+1)K+H|=\phi$ (H any ef. div)

know $(nK+H).K = nK^2+HK<0$ for $n>>0$

This is to show $\exists$ such an ef. C

$nK+H \sim ef \Rightarrow nK+H \sim \sum a_i C_i$, $a_i>0$

$\Rightarrow (nK+H).K = (\sum a_i C_i).K<0$

$\exists C_i$ st. $C_i.K<0$

But how to see $K.C\leq -2$?

but X min $\Rightarrow C_i^2\geq 0$

Now use $H=2C$, repeat the process $2C+nK$ ... get divisor $D\sim 2C+mK \Rightarrow D.K\leq 2CK\leq -2$.

$$mK+H \sim ef \Rightarrow \underset{\shortparallel}{(mK+H).C_i} \geq 0$$

$$\underset{\overset{\wedge}{0}}{mK.C_i} + H.C_i \quad \cancel{\longrightarrow}$$

- Now we assume that X is not ruled:

Since $K^2<0$, can't have $nK+H\sim 0$ ($n^2K^2=(-H)^2>0$)

So $nK+H \sim ef \underset{\ni D}{\neq} 0$ Want D irreducible:

$D=\sum_{i=1}^{r} n_i C_i$, for the statement of step 1, may choose only C_i with $K.C_i<0$.

- D is reduced: (i.e. $n_i=1, \forall i$), otherwise $\exists i$

$|K+2C_i|=\phi$; know $h^2(K+2C_i)=h^0(-2C_i)=0$

$$0=h^0(K+2C_i)\geq \frac{(K+2C_i)\cancel{2}C_i}{\cancel{2}}+1-q$$

step 2 须用 X not ruled，故须用反证法.

$$2\,(K+C_i).C_i - K.C_i + (1-q) \le 0 \quad \text{*}$$

$\underset{2\cdot d\cdot(2q-2)}{\|}$ with step 2. ie. $p|_{C_i}: C_i \to B$ is étale of deg $= d$.

• ~~Now let $D = C_1+\cdots+C_r$~~ $(r\geq 2)$:

$$|K+C_1+C_2| = \phi \quad \text{and also} \quad h^2(K+C_1+C_2) = 0$$

$$\text{R.R} \Rightarrow \quad 0 - h^1(K+C_1+C_2) + 0 = \frac{(K+C_1+C_2)(C_1+C_2)}{2} + 1-q$$

$$\tfrac{1}{2}\left(K(C_1+C_2) + (C_1+C_2)^2 \right)$$

$$\tfrac{1}{2}\left[\underset{\substack{\vee \\ 2(q-1)}}{\underline{(K+C_1).C_1}} + \underline{(K+C_2).C_2} + \underline{2C_1.C_2} \right]$$

$\Rightarrow$ must $\underline{q(X)=1}$ and $\underline{C_1.C_2 = h^1(K+C_1+C_2)} = \underline{\underline{0}}$!

But $C_1 \neq C_2$, $C_1.C_2 = 0 \Leftrightarrow C_1 \cap C_2 = \phi$, get

$$0 \to \mathcal{O}(-C_1-C_2) \to \mathcal{O}_X \to \underset{\mathcal{O}_{C_1}\oplus\mathcal{O}_{C_2}}{\mathcal{O}_{C_1\cup C_2}} \to 0$$

since $q=1$:

$$0 \to \mathbb{C} \to \mathbb{C}\oplus\mathbb{C} \to H^1(X, \mathcal{O}(-C_1-C_2)) \to \underset{\mathbb{C}}{H^1(\mathcal{O}_X)} \to \underset{\mathbb{C}}{H^1(\mathcal{O}_{C_1})} \oplus \underset{\mathbb{C}}{H^1(\mathcal{O}_{C_2})} \to 0$$

$\Rightarrow h^1(\mathcal{O}(-C_1-C_2)) \neq 0 \Rightarrow h^1(K+C_1+C_2) \neq 0$ *

?? 没用到 $q(X)=1$.

So D is irreducible $|K+D| = \phi$, $K.D < 0$.

II. $\left(\begin{array}{l} \text{If } X \text{ not ruled, and minimal} \\ C \text{ irred. } |K+C| = \phi,\ K.C<0 \end{array} \right) \Rightarrow p|_C$ étale

isom. if $q\geq 2$.

Pf: R-R :

$$0 - h^1(K+C) - 0 = \frac{(K+C).C}{2} + 1-q$$

ie. $\underline{2g(C)-2} = \underline{2q-2} - 2h^1(K+C)$

ie. $g(C) \le q(X)$. Arithm. genus

$p(C) = pt \Rightarrow C$ is a fiber, but if inside an reducible fiber then $C^2<0$ (Zariski lemma), + "$K.C<0$" * to X minimal.

But then $C^2 = 0$ and $K.C < 0 \Rightarrow K.C = -2 \Rightarrow C \cong \mathbb{P}^1$, ruled *

$2g(C)-2 = K.C + C^2$.

Hence $p(C) = B$ and $p: C \longrightarrow B$ a (ramified) covering,
say of degree d.

Let

$$\begin{tikzcd} C \arrow[d, "p"'] & N \arrow[l] \arrow[dl, "\tilde{p}"] \\ B & \end{tikzcd}$$

N normalization of C

let r = ramification index

Riemann – Hurwitz $\Rightarrow$ $2g(N)-2 = d\cdot\left(2g(B)-2\right) + r$

i.e. $g(N) = 1 + d\cdot(g(B)-1) + \frac{r}{2}$

$\Rightarrow$ $q \geq g(C) \geq g(N) = d(q-1) + 1 + \frac{r}{2}$

↑ general fact about arithmetic genus: ↘ under resolution (eg. Exercise)

$q = 1 \Rightarrow r=0$ and $g(C) = g(N)$
i.e. $p: C \longrightarrow B$ étale.

$q \geq 2 \Rightarrow d = 1$, $r = 0$ and $g(C) = g(N)$. i.e. $p: C \xrightarrow{\sim} B$. ✱

III. X minimal with $K^2<0$, want to show that X ruled.
If X is not ruled.

I $\Rightarrow$ $\exists$ irred. C, $|K+C| = \emptyset$, $K.C<0$

II $\Rightarrow$ (If X not ruled, then) $p: C \to B$ is étale.

Case: $p: C \xrightarrow{\sim} B$ isom (i.e. C a section):

$$h^0(C) - h^1(C) + h^2(C) = \tfrac{1}{2}C(C-K) + 1 - q = -C.K \geq 2 \ !$$

$h^2(C) = h^0(K-C) = 0$, $(K-C)+2C = (K+C)$

$\Rightarrow$ C moves in linear equivalence class C_K

$\Rightarrow$ $C_K \cap F$ (= one point in F) linearly equiv.

But 2 pts can be equiv. $p \sim q$ only when $F \cong \mathbb{P}^1$.

$\Rightarrow$ (by Noether – Enriques) X is ruled ✱.

Case: $p: C \longrightarrow B$ étale: (so may assume $q(X) = 1$):

$i: C \hookrightarrow X$ gives a section

$e: C \hookrightarrow X \times_B C$; let X' = conn. component $\supset e(C) = C'$

$\pi: X' \to X$ still étale

By definition of fiber-product, $X'_u = X_{p(u)}$

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d, "p"] \\ C \arrow[u, dashed, bend left, "e"] \arrow[ur, "i"] \arrow[r, "\text{etale}"'] & B \end{tikzcd}$$

$K_{X'} = \pi^* K_X$

So $\underline{X \text{ not ruled} \Rightarrow X' \text{ not ruled}}$.

$K_{X'}.C' = \deg_{C'}(e^* K_{X'}) = \deg_C(e^* \pi^* K_X) = \deg_C i^* K_X = \underline{K.C < -1}$

and $C^2 \geq 0 \Rightarrow C'^2 \geq 0$

So $|nK_{X'}| = \emptyset$ $\forall n$. in particular $\underline{P_1(X') = 0}$

$\Gamma(X', nK_{X'}) = \Gamma(X', \pi^*(nK_X)) = \Gamma(X, \pi_* \pi^*(nK_X))$

So the conclusion is not trivial.

Now $\chi(\mathcal{O}_X) = 0 \Rightarrow \chi(\mathcal{O}_{X'}) = 0$ (see below), so

(ie. with $p_g(X') = 0$, we also want $q(X') = 1$ unchanged, $\underline{\text{but}}$...) (see Rmk)

$$h^0(C') - h^1(C') + h^2(C') = \frac{1}{2} C'(C' - K_{X'}) + \cancel{\chi(\mathcal{O}_{X'})}$$

$$h^2(C') = h^0(K_{X'} - C') = 0$$

$$\frac{1}{2} C'(C' - K_{X'}) = \underbrace{g(C') - 1}_{0} - C'.K_{X'} = -C'.K_{X'} \geq 2$$

$g(C') = g(C) = g(B) = 1$ since p étale

As before this $\Rightarrow X'$ ruled $\times$

- $\underline{\text{Lemma}}$: let $\pi: X' \to X$ be étale of surfaces of deg $= d$ then $\underline{K_{X'}^2 = nK_X^2}$, $\underline{\chi(X') = d \cdot \chi(X)}$ and hence $\underline{\chi(\mathcal{O}_{X'}) = d \cdot \chi(\mathcal{O}_X)}$.

pf: $K_{X'}^2 = K_{X'}.\pi^* K_X = (\pi_* K_{X'}).K_X = d \cdot K_X^2$

$\chi(X') = d \cdot \chi(X)$ follows from topology.

Finally: $$\chi(\mathcal{O}_{X'}) = \frac{c_1^2(X') + c_2(X')}{12} = \frac{1}{12}\left(K_{X'}^2 + \chi(X')\right) = d \cdot \chi(\mathcal{O}_X). \square$$

$\underline{\text{Rmk}}$: In case $q(X) \neq 1$ then $q(X')$ varies irregularly.

Algebraic Surfaces with $K=0$

i.e. $P_n(X) \leq 1$ $\forall n$ and $=1$ for some n. Assume X minimal

$\boxed{\text{Fact I. } K^2=0}$

let $D \in |nK|$, then $D.K = nK^2 \geq 0$

for if $D = \sum n_i D_i$, $D.K<0 \Rightarrow D_i.K<0$ for some i

but $D_i^2 \geq 0$ (otherwise $D_i = (-1)$ curve)

$\Rightarrow D_i.Z \geq 0$ for any ef div Z, eg. $Z = D \sim nK$, get $D_i.K \geq 0$ $\times$.

claim: $K^2=0$,

if $K^2>0$, R-R $\Rightarrow$

$$h^0(\ell K) - h^1(\ell K) + h^2(\ell K) = \frac{\ell(\ell-1)}{2} \cdot \underset{\overset{\vee}{0}}{K^2} + \chi(\mathcal{O}_X) \nearrow \infty$$

$h^2(\ell K) = h^0(-(\ell-1)K) = 0$ for $\ell_i \gg 0$ since nK ef, $-nK$ ef (Main trick)

$\Rightarrow nK=0 \Rightarrow K^2=0$ $\times$.

So $P_{\ell_i}(X) \nearrow \infty$, again get $\times$.

$\boxed{\text{Fact II. } \chi(\mathcal{O}_X) \geq 0}$

Noether's formula $1-q+p_g = \chi(\mathcal{O}_X) = \dfrac{\cancel{K^2}+\chi(X)}{12}$

$\Rightarrow 12 - 12q + 12p_g = 2 - 4q + b_2$

$\Rightarrow 10 - 8q + 12p_g = b_2$

$\Rightarrow 8\chi(\mathcal{O}_X) = 8(1-q+p_g) = b_2 - 2 - 4p_g \geq -6$ ($p_g \leq 1$)

$\Rightarrow \chi(\mathcal{O}_X) \geq 0$.

THEOREM: let X minimal with $K(X)=0$, then $X \in$

$p_g = 0$ $\begin{cases} q=0 \Rightarrow \text{Enrique Surface}, & 2K=0 \\ q=1 \Rightarrow \text{bielliptic Surface}, & 4K=0 \text{ or } 6K=0 \end{cases}$

$p_g = 1$ $\begin{cases} q=0 \Rightarrow \text{K3 surface}, & K=0 \\ q=2 \Rightarrow \text{Abelian Surface}, & K=0 \end{cases}$

Rmk: existence of Enriques Surface $\Rightarrow$ Castelnuovo's thm is sharp: $q=0$, $P_2=0$.

pf: $\boxed{p_g = 0}$ then $\chi(\mathcal{O}_X) = 1 - q \geq 0 \Rightarrow q = 0, 1$:

- $q=0$:
  $P_2(X) \neq 0$ (otherwise Castelnuovo $\Rightarrow$ X rational)
  so $P_2(X) = 1$, $\Rightarrow P_3(X) = 0$

$*$ [(in fact $P_n(X) = P_m(X) = 1 \Rightarrow P_d(X) = 1$ for $d = (n, m)$:
  our case: $\sigma \in \Gamma(X, K^{\otimes 2})$, $\tau \in \Gamma(X, K^{\otimes 3})$
  $P_6(X) \leq 1 \Rightarrow \sigma^3 = \lambda \tau^2 \Rightarrow \sigma = \lambda \cdot (\tau/\sigma)^2$
  i.e. τ/σ is a global holomorphic section of K .)]

  By the main trick, want to see $h^0(-2K) \neq 0$, hence $\underline{2K = 0}$.

  R-R : $h^0(-2K) - h^1(-2K) + \cancel{h^2(-2K)} = 1$ done !

  $h^2(-2K) = h^0(3K) = 0$, $\uparrow K^2 = 0$

- $q = 1$:

  $p_g = 0$ & $q = 1$ get Albanese map $X \longrightarrow B$ and in fact
  $X = (B \times F)/G$. By the classification $K=0 \Leftrightarrow$ bielliptic
  and also $\underline{4K = 0 \text{ or } 6K = 0}$.

$\boxed{p_g = 1}$ then $\chi(\mathcal{O}_X) = 1 - q + 1 \geq 0 \Rightarrow q = 0, 1, 2$:

- $q = 0$:
  Since $h^0(K) = 1$, want to look at $h^0(-K)$:
  R-R: $h^0(-K) - h^1(-K) + h^2(-K) = 0 + \chi(\mathcal{O}_X) = 2$
  $\Rightarrow h^0(-K) \neq 0 \Rightarrow \underline{K = 0}$.

  $h^2(-K) = h^0(2K) \geq 1$

- $q = 1$: $\exists$ div $S \neq 0$ st. $2S \sim 0$
  $p_g(X) = 1$ in fact $\Rightarrow P_n(X) = 1 \ \forall n$; $\Rightarrow S.D = 0 \ \forall D$ & $h^0(S) = h^0(-S) = 0$.
  R-R : $h^0(S) - h^1(S) + h^2(S) = \cancel{\frac{S.(S-K)}{2}} + (1 - 1 + 1) = 1$

  $h^2(S) = h^0(K-S)$

  ($H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z})$
  $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2q} \hookrightarrow \mathbb{C}^q \cong H^1(X,\mathcal{O})$ pick $S \in \mathbb{C}^q/\mathbb{Z}^{2q}[2]$
  but $\tau^2 = \lambda \sigma^2$ (in $\mathcal{O}(2K)$) .)

  $D \in |K - S| \neq \phi$, sect σ
  $L \in |K|$, sect τ
  then τ/σ mero sect of $\mathcal{O}(-S)$
  $* \Rightarrow S = 0$ ~~$*$~~ .

- $q=2$: (under $P_g=1$)

The above argument leads to no contradiction since $\underline{\chi(\mathcal{O}_X)=0}$.

$$0\to \mathbb{C}^2/\Lambda \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O})$$

The actual pf is to show ~~the Albanese map is an isom.~~ $p: X \xrightarrow{\sim} \mathrm{Alb}(X)$.
(due to Kawamata 1979 : any $\dim X = n$, $\kappa(X)=0 \Rightarrow$
$X \twoheadrightarrow \mathrm{Alb}(X)$ as an algebraic fiber space, hence if $q=n$,
then get an birational morphism $X \longrightarrow \mathrm{Alb}(X)$)
Here we only show that p is étale in dim 2 case. (G-H p.583)
First notice that $\dim X=2$, $\kappa=0$, $q=2$ and minimal $\Rightarrow P_g=1$:
and hence $\chi(\mathcal{O}_X)=0$ and also $\chi(X)=0$. (bec. $\chi(\mathcal{O}_X)\geq 0$)

- $\underline{\text{Step 1. } p: X \twoheadrightarrow A \text{ and non-étale locus} \sim K}$:

If not, then $\mathrm{Im}(p)=B \subset A=\mathrm{Alb}(X)$, $g(B)=q=2$.
but $0=\chi(X)\geq \chi(B)\cdot\chi(F) \Rightarrow g(F)=1$ (X not ruled, $g(F)\geq 1$)
but this $\Rightarrow \kappa(X)=1$ $\times$.
This $\Rightarrow$ for $\langle \eta, \zeta \rangle = H^0(X,\Omega^1)$, $\eta\wedge\zeta$ generates $H^0(X,\Omega^2)$.
In fact $p: X \twoheadrightarrow A$ fails to be a local isom (étale)
precisely on the canonical div $D=\mathrm{div}(\eta\wedge\zeta)\sim K$.
because $A := H^0(\Omega^1)^*/H_1(\mathbb{Z})$.

- $\underline{\text{Step 2. Case } p(D) = \text{pts} \in A \text{ : (} D \text{ may } = 0 \text{)}}$

$\Rightarrow p: X-D \longrightarrow A-\text{pts}$ is an unbranched cover
but $\pi_1(X-D) \longrightarrow \pi_1(A-\text{pts}) = \pi_1(A) = \mathbb{Z}^4$ abelian
$\downarrow\!\!\downarrow$ and $H_1(X,\mathbb{Z})/\text{tor} \cong H_1(A,\mathbb{Z})$
$\pi_1(X)$ $\Rightarrow p$ is a degree 1 covering
$\Rightarrow p: X \twoheadrightarrow A$ is binat'l $\Rightarrow p: X \xrightarrow{\sim} A$ since X minimal.
(and so in fact $K=0$)

• Step 3. Case $D \neq 0$ and $p(D)$ a curve:

If $D=0$ then p is étale. but étale cover of Ab. V. is again an Ab. V. ※. So may assume $D \neq 0$:

claim: $D = \sum n_i D_i$, $g(D_i) = 0$ or 1:

$$D \sim K,\ D^2 = K^2 = 0 \Rightarrow D_i . D \geq 0 \quad (\text{i.e. } D_i . K \geq 0)\ \forall i$$

$$0 = D.K = (\sum n_i D_i).K \Rightarrow D_i . K = 0 \quad \forall i$$

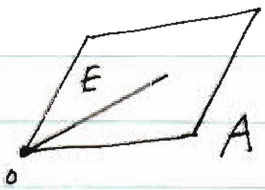
$$0 = D_i . (\sum n_i D_i) = n_i D_i^2 + (\geq 0) \Rightarrow D_i^2 \leq 0$$

hence $2g(D_i) - 2 = (D_i + K).D_i = D_i^2 \leq 0$ ※

So D_i may be rat'l, singular elliptic, elliptic.

But $A \not\supset$ rat'l curve $\Rightarrow \exists D_i$ elliptic and $p(D_i)$ elliptic $\equiv E$.

(May assume $0 \in E$ and $E \subset A$ sub-gp) consider $\mu: A \longrightarrow Pic^0(A)$ via $a \longmapsto [(a+E) - E]$.



$\Rightarrow$ • $(a+E) \neq E$ for general $a \notin E$.

$\Rightarrow \mu: A \twoheadrightarrow \mu(A) = B$ a curve

but $\mu^{-1}(0) =$ sub gp, smooth

So $E =$ one component of $\mu^{-1}(0)$.

Main technique point: want $E = \mu^{-1}(0)$ exactly!

$$\begin{tikzcd} X \arrow[r, "p"] & A \arrow[r, "\tilde{\mu}"] \arrow[dr, "\mu"'] & \tilde{B} \arrow[d] \\ & & B \subset Pic^0(A) \end{tikzcd}$$

← maybe a branched cover ☆

then $\tilde{\mu} \circ p : X \longrightarrow \tilde{B}$ is a fiber space with D_i a fiber $g(D_i) = 1$ i.e. X is elliptic. But since $K \sim ef \neq 0$, By the classification of elliptic surface, this $\Rightarrow \kappa(X) = 1$. ※

(End of proof).

2 problems:

(1) ☆ need to be proved in detail

(2) Not yet discussed the canonical bundle formula for elliptic surfaces.

$\boxed{\text{K3 surface}}$ X minimal $\kappa=0$, $p_g=1$, $q=0$

equiv. to X with $K_X=0$ and $q=0$.

$$2 = 1-q+p_g = \chi(\mathcal{O}_X) = \frac{\cancel{K^2}+\chi(X)}{12} \Rightarrow \chi(X)=24$$

Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

So all different K3's all have the same coh. groups

In fact, will see that $\exists!$ universal family $\begin{matrix}\mathcal{X}\\ \downarrow \\ M\end{matrix}$ of all Kähler K3 surfaces with $\dim M = H^1(X, T_X) = h^1(\Omega^1) = 20$.

hence $\Rightarrow$ all K3 are diffeomorphic.

- 2 basic constructions of K3's:

I. Kummer construction:

let $T^4 = \mathbb{C}^2/\mathbb{Z}^4$ with involution $\iota(z_1,z_2) := (-z_1,-z_2)$

fixed pts $\Leftrightarrow$ $(2z_1, 2z_2) \in \mathbb{Z}^4$, 16 pts, let $X = T^4/\langle\iota\rangle$

each pt has a local coor. model $\mathbb{C}^2/\langle -1\rangle$, ie. A_1 singularity

$\tilde{X}$ = blow up 16 pts

$p \downarrow$

$X = T^4/\langle\iota\rangle$

$H^0(T^4, K) = \mathbb{C}\cdot dz_1\wedge dz_2$ inv. under ι

so $K_X = 0$ (X Gorenstein)

A_1 sing is crepant: $K_{\tilde{X}} = p^* K_X = 0$

$$\begin{matrix} T^4 \\ \downarrow \pi \\ X \end{matrix}$$

(eg. $\chi(\tilde{X}) = \chi(X) - 16\,(\text{pts}) + 16\,(\mathbb{P}^1\text{'s}) = \chi(X)+16$

$= \frac{1}{2}\cdot(\cancel{\chi(T^4)}+16) + 16 = 8+16 = 24$)

$H^0(\tilde{X}, \Omega^1) = 0$ (hence $q(\tilde{X})=0$):

$\alpha \in H^0(\tilde{X}, \Omega^1) \Rightarrow \alpha$ is a rat'l 1-form on $X - 16$ pts

$\Rightarrow \pi^*\alpha$ is a rat'l 1-form on $T^4 - 16$ pts, ι-inv.

T^4 nonsingular $\Rightarrow \pi^*\alpha$ holo 1-form, ι-inv

$\Rightarrow \pi^*\alpha = 0$ hence $\alpha = 0$.

ie. $\tilde{X}$ is a K3 surface.

II. Projectively normal K3's.

- $X \subset \mathbb{P}^{n+2}$ K3 surface, if ~~complete intersection~~ $(d_1, \cdots, d_n)$
  ie. $K_X = K_{\mathbb{P}} + \sum d_i = \sum d_i - (n+3) = 0$; $d_i \geq 2$
  ie $\sum (d_i - 1) = 3 \Rightarrow X = (4) \subset \mathbb{P}^3, (2,3) \subset \mathbb{P}^4, (2,2,2) \subset \mathbb{P}^5.$
  $h^1(\mathcal{O}_X) = 0$ by Lefschetz hyperplane thm. $\Rightarrow$ really K3's.
- In general, $X \hookrightarrow \mathbb{P}^g$, $C = X \cap H$, if ~~proj. normal~~
  $$K_C = (K_X + C)|_C = C|_C \qquad \text{so } g(C) = g \text{ !}$$
  ie. $C \hookrightarrow \mathbb{P}^g$ is the embedding via $|K_C|$, ie. canonical curves
  (know C canonical $\Longleftrightarrow$ C is not hyperelliptic :
  $|K_C| : C \longrightarrow \mathbb{P}^g$ always free, just 1–1 or 2–1)
  deg X in $\mathbb{P}^g$ $= H^2 \cdot X = C^2 = \deg K_C = 2g - 2$.
- Conversely, any K3 X, C smooth curve in X, $g(C) = g$
  $\Rightarrow C^2 = 2g - 2$ and $K_C = C|_C$.
  if $\mathcal{O}_C(C)$ bpf (v.a.) on C then $\mathcal{O}_X(C)$ bpf (v.a) on nbd of C in X

  In particular if $|C|$ not hyperelliptic for general member (eg. C is v.a.) $\Rightarrow$ birat'l (isom.) $|C| : X \longrightarrow \mathbb{P}^g$:
  $$h^0(C) - \cancel{h^1(C)} + \cancel{h^2(C)} = \frac{C(C-K)}{2} + 2 = (g-1) + 2 = g+1$$
  by Kodaira vanishing since $K = 0$.

- Fact: for any $g \geq 3$, $\exists$ (proj. normal) K3 surface
  $X \hookrightarrow \mathbb{P}^g$ and $\deg X = 2g - 2$ (must be),
  they form a 19-dim'l family.

$g = 3, 4, 5$ done by above.
$(4) \subset \mathbb{P}^3$: $\quad H^4_4 - \dim GL(4) = C^7_4 - 16 = 35 - 16 = 19$
$(2,3) \subset \mathbb{P}^4$
$(2,2,2) \subset \mathbb{P}^5$

But $H^1(X, T) = H^1(X, \Omega^1) = 20$ dim'l, what happens ?!

Enrique Surface: X minimal, $K=0$, $p_g=0$, $q=0$

equiv. to X with $2K=0$, $q=0$ but $K\neq 0$.

$$1 = 1-q+p_g = \chi(\mathcal{O}_X) = \frac{K^2+\chi(X)}{12} \Rightarrow \chi(X)=12$$

Hodge numbers:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 12 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\Rightarrow$ all Kähler Enrique are projective (not like K3's).

Existence $\Rightarrow$ Castelnuovo's thm is sharp: $q=0=P_2 \Leftrightarrow$ rat'l.

- Fact: X has a double cover $\tilde{X} \xrightarrow{\pi} X$ with $\tilde{X}$ a K3

  conversely $K3/\iota$ = Enrique if ι is fixed pt free invo.

pf: let L line bundle $\cong \mathcal{O}(K)$.

$$\begin{array}{c} L \\ \downarrow \\ X \end{array}$$

let $\tilde{X} = \{ v \in L \mid \text{"}v^2=1\text{"} \text{ in } L^{\otimes 2} \cong \mathcal{O}_X \}$

- since $K\neq$ trivial $\Rightarrow \tilde{X}$ is connected.

$K_{\tilde{X}} = \pi^* K_X = 0$ since $\pi^* L$ has a trivial section "v".

$\chi(\mathcal{O}_{\tilde{X}}) = 2\cdot\chi(\mathcal{O}_X) = 2 = 1-q+p_g \Rightarrow q=0 \Rightarrow X$ is K3.

by same pf, $K3/\iota$ = Enrique. $\square$

- construction:

let $X = (2,2,2) \subset \mathbb{P}^5$ via 3 quadrics:

$$Q_i(x_0,x_1,x_2) + Q_i'(x_3,x_4,x_5) = 0 \qquad i=1,2,3$$

$\iota(x_0,\dots,x_5) := (x_0,x_1,x_2,-x_3,-x_4,-x_5)$

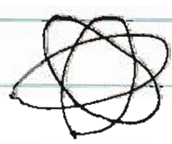
then $\mathrm{Fix}(\iota)$ = 2 plans $= \{x_0=x_1=x_2=0\} \cup \{x_3=x_4=x_5=0\}$

may choose Q_i st. $\cap\{Q_i=0\} = \phi$ in $\{x_3=x_4=x_5=0\}$

(trivial: 3 general conics in $\mathbb{P}^2$ $\cap = \phi$).

resp. for Q_i'. $\Rightarrow \mathrm{Fix}(\iota) = \phi$ on X

$\Rightarrow X/\iota$ an Enrique Surface. ※



X Surface with $p_g = 0$, $q \geq 1$

Lemma: $K^2 < 0$ or

$K^2 = 0$ and $q = 1$, $b_2 = 2$.

Pf: Noether's formula

$$1 - q + \cancel{p_g} = \chi(\mathcal{O}_X) = \frac{K^2 + \chi(X)}{12}$$

$$\Rightarrow \quad 12 - 12q = K^2 + 2 - 4q + b_2$$

$$10 - 8q = K^2 + b_2$$

We need to exclude the case that $q = 1$, $K^2 = 1 = b_2$

but $p_g = 0 \Rightarrow$ Albanese map

$$X \twoheadrightarrow B, \quad g(B) = q = 1$$

then generic fiber $f \in H^2(X, \mathbb{Z})$
and hyperplane class $h \in H^2(X, \mathbb{Z})$ linearly indep.
(since $f^2 = 0$, $fh \neq 0$, $h^2 > 0$) i.e. $b_2(X) \geq 2$. $\square$

The 1st case is studied:

~~X minimal with $K^2 < 0$~~

($\Rightarrow q \geq 1$ and $P_n(X) = 0 \ \forall n$) $\Rightarrow$ irrational ruled.

Now we study the 2nd case:

~~X minimal, $K^2 = 0$, $q = 1$, $b_2 = 2$.~~

Lemma A. $p : X \to B$, F a (smooth) general fiber, then

all χ means χ_{top}:
$$\chi(X) = \chi(B) \cdot \chi(F) + \sum_{p|_s \text{ not smooth}} (\chi(X_s) - \chi(F))$$

Lemma B. let C be an reduced curve, then

$$\chi_{top}(C) \geq 2\chi(\mathcal{O}_C); \text{ equality} \Leftrightarrow C \text{ is smooth}$$

Theorem I. let X minimal with $K^2 = 0$, $p_g = 0$, $q = 1$ (so $b_2 = 2$), then the Albanese map $p : X \to B$ has 3 possibilities:

1) fiber genus $= 0 \Rightarrow$ ruled (elliptic ruled)
2) fiber genus $= 1 \Rightarrow$ all singular fiber are like nE, $g(E) = 1$
3) fiber genus $\geq 2 \Rightarrow p$ is a smooth fibration.

lemma A is a simple topological fact via triangulations

* ~~pf of lemma B~~: may let C be connected:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}_C & \to & \varphi_*\mathbb{C}_{\tilde C} & \to & \mathcal{F} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi & & \\ 0 & \to & \mathcal{O}_C & \to & \varphi_*\mathcal{O}_{\tilde C} & \to & \mathcal{G} & \to & 0 \end{array}$$

let $\varphi: \tilde C \to C$ be the normalization

$\mathcal{F}$ = top'l branch

$\mathcal{G}$ = analytic branch

ϕ is injective:

$$\begin{array}{ccccc} \exists\gamma? & \dashrightarrow & \alpha & \overset{1}{\longmapsto} & a \\ \vdots & & \downarrow 2 & & \downarrow \text{if} \\ \beta & \xrightarrow[4]{} & b & \underset{3}{\longmapsto} & 0 \end{array}$$

but b can be regarded as $\mathcal{O}_C$-sections which is locally constant (in some sheaf $\varphi_*\mathbb{C}_C$)
$\Rightarrow \exists\gamma$ locally const in $\mathcal{O}_C$.

$\Rightarrow \chi_{top}(\tilde C) = (\varphi_*\mathbb{C}_{\tilde C}) = \chi(\mathbb{C}_C) + \chi(\mathcal{F})$

$\quad\ \ \|$

$2\cdot\chi(\mathcal{O}_{\tilde C}) = 2\chi(\varphi_*\mathcal{O}_{\tilde C}) = 2\chi(\mathcal{O}_C) + 2\chi(\mathcal{G})$

but $\mathcal{F}, \mathcal{G}$ has 0-dim'l support $\Rightarrow \chi = h^0$

$\Rightarrow \quad \chi(\mathbb{C}_C) - 2\chi(\mathcal{O}_C) = \underline{2h^0(\mathcal{G}) - h^0(\mathcal{F}) \geq 0}$

$\quad\ \|$

$\chi_{top}(C)$ and $= 0 \iff h^0(\mathcal{G}) = 0$ (i.e. $\mathcal{G} = 0$)

(then $\mathcal{F} = 0$ automatically) ※

* ~~pf of thm I.~~ let $p: X \to B$ Albanese map: $g(B) = 1$.

p has irreducible fibers: if $F_1 \neq F_2$ all in a fiber F:

$b_2 = 2 \Rightarrow \exists\, \alpha, \beta, \gamma \neq 0$ st. $\alpha H + \beta F_1 + \gamma F_2 = 0$

hyp. class

$\Rightarrow \alpha H.F + (\beta F_1 + \gamma \cancel{F_2}).F = 0$

$\Rightarrow \alpha = 0$. So $F_2 = k F_1$. $\cdot H \Rightarrow k > 0$.

but $\cdot F_1$ get $0 < F_2.F_1 = k F_1^2 < 0$ by Zariski lemma ※.

For a multiple fiber $F_0 = nC$:

$\chi_{top}(F_0) = \chi_{top}(C) \geq 2\chi(\mathcal{O}_C) = 2 - 2g(C) = -(K+C).C$

$\qquad = -K.C = \frac{1}{n}(-K.nC) = \frac{1}{n}(-K.F) = \frac{1}{n}\chi_{top}(F)$

but $g(F)$ may assume ≥ 1 (otherwise ruled)

$\Rightarrow \underline{\chi_{top}(F_0) \geq \chi_{top}(F)}$ with "=" $\iff$ $\underline{n=1 \text{ or } g(F)=1}$.

Finally: $\chi_{top}(X) = \cancel{\chi_{top}}(B)\cdot\chi_{top}(F) + \sum_s \left(\chi_{top}(F_s) - \chi_{top}(F)\right)$

$\chi_{top}(X) = 2-4+2 = 0$, $\quad \chi_{top}(B) = 0$, $\quad \chi_{top}(F_s) - \chi_{top}(F) \geq 0$

$\Rightarrow$ "=" holds. done. $\square$.

* Question: What is a "Galois Cover"?

We want to further use branch covering trick (base change) $z \mapsto z^n$ in case 2).

Main Fact: One can do this globally for all sing. fiber, get a ramified Galois cover with gp $= G$ st. $\tilde{p}$ is smooth and $\tilde{X}/G = X$. We will assume this.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{p}"] & X \arrow[d, "p"] \\ \tilde{B} \arrow[r, "\varphi"] & B \end{tikzcd}$$

◉ Smooth morphisms are very special, usually can be classified:

Theorem II. $p: X \to B$ smooth morphism, then (let $F =$ a fiber)

case (1) $F = \mathbb{P}^1 \Rightarrow X$ is ruled, $= \mathbb{P}_B(E)$ ...

case (2) $g(F) = 1$, B arbitrary,

case (3) $g(F) \geq 1$ but $g(B) = 1$

In (2) and (3), $\exists$ étale cover $\varphi: B' \to B$ st. $X' \to B'$ is trivial ie. $X' \cong B' \times F \cong X \times_B B'$ and may take φ to be Galois with gp G, st. $X = (B' \times F)/G$.

$$\begin{tikzcd} X \times_B B' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ B' \arrow[r, "\varphi"] & B \end{tikzcd}$$

Rmk: such $p: X \to B$ is usually called "iso-trivial".

Corollary:

X minimal with $p_g = 0$, $q = 1$, $K^2 = 0$ (so $b_2 = 2$), then

(1) X is ruled with elliptic base or

(2) $X \cong (B \times F)/G$, $G \subset \mathrm{Aut}(B)$ finite sub group st $g(B/G) = 1$ with $g(B), g(F) \geq 1$ and G acts on $B \times F$ compati with B.

If $g(F) \geq 2$ then $g(B) = 1$ and G is a gp of translations.

Much more is true for case (2):

◉ THEOREM III. $X = (B \times F)/G$ $G \subset \mathrm{Aut}(B)$, (i) $\boxed{G \subset \mathrm{Aut}(F)}$

B, F are irrational, $g(B/G) = 1$, (ii) $\boxed{g(F/G) = 0}$ and either

(I). $g(B) = 1$, and G is a gp of translations, or

(II). $g(F) = 1$ and G acts freely on $B \times F$

(iii) Conversely, every surface as above is minimal, non-ruled with $p_g = 0$, $q = 1$, $K^2 = 0$.

Idea of pf of thm II. construct $M_{g,n}$ moduli spaces (of F)
with ~~universal family~~ $\pi: \mathcal{X} \to M_{g,n}$
where "n" is the level n-structure on $H^1(F, \mathbb{Z}/n)$: n-torsion part
to kill automorphisms of F :

Lemma (Serre) : let $A \in GL(N,\mathbb{Z})$, if $\bar{A} = 1 \in GL(N, \mathbb{Z}/n)$
for $n \geq 3$, then $A = 1$.

so $M_{g,n} = \{(F,H) \mid H \cong H^1(F,\mathbb{Z}/n)\} / \text{isom}$.
a given isom.

can be ~~glued together~~ to form an analytic space. (in fact, q-proj. v.)

- For a smooth morphism $p: X \to B$ with fiber genus ≥ 1
want to get a map $\varphi: B \to M_{g,n}$:
to define it, fix a fiber F over $s \in B$
$\pi_1(B,s)$ acts on $H^1(F,\mathbb{Z})$, ~~the monodromy~~
we need this action to be trivial on the
group: $H^1(F,\mathbb{Z}/n)$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ B \arrow[r, "\varphi"] & M_{g,n} \end{tikzcd}$$

But since $\pi_1(B,S) \xrightarrow{\Phi} \text{Aut}(H^1(F,\mathbb{Z}/n))$ a finite group
theory of covering space $\Rightarrow \exists\ B' \to B$ finite covering
with $\pi_1(B',s') = \text{Ker}\,\Phi$.

Hence the family $X' \xrightarrow{p'} B'$.

Case $g(F) = 1$:

get $B' \xrightarrow[\varphi']{} M_{1,n} \xrightarrow[j]{} \mathbb{C}$

ie. $j \circ \varphi'$ is a holo. fcn on cpt B', hence constant !
in this case B would be arbitrary.

Case $g(F) \geq 2$:

get $B' \xrightarrow[\varphi']{} M_{g,n}$, we know $M_{g,n}$ has universal cover
a bounded domain. If $g(B') = 1$,
lifting get $\mathbb{C} \to$ bounded, hence pt.

- Method 1 : Teichmüller theory : $\widetilde{M}_{g,n} \subset \mathbb{C}^{3g-3}$.
- Method 2 : Via Abelian variety : (principally polarized)

$B' \xrightarrow[\varphi']{} M_{g,n} \xrightarrow[\alpha]{} A_{g,n}$, but $\widetilde{A}_{g,n}$ is the Siegel upper space
abel-Jacobi map
α is finite (Torelli)

$\{Z = Z^t,\ \text{Im} Z > 0\} \subset \mathbb{C}^{g^2}$
which is bounded. $\square$

(Part)

Pf of thm III. only (i)(ii)(iii) needs proofs:

(i) $G \subset \mathrm{Aut}(F)$:

write $g(b,f) = (g(b), \phi_g(b) f)$

with $\phi_g : B \longrightarrow \mathrm{Aut}(F)$ a continuous function.

but $\underline{g(F) \geq 2 \Rightarrow \mathrm{Aut}(F) \text{ is finite}}$ (look at $|K| : F \hookrightarrow P^N$ canonical curve or hyperelliptic curve.)

hence $\phi_g \in \mathrm{Aut}(F)$

* ~~the case $g(F)=1$ will be omitted here~~ interesting Exercise cf *p.7/7.

(ii) Let $X = \tilde{X}/G = (B \times F)/G$, $g(B), g(F) \geq 1$ and $q = 1$:

Trivial fact for todd genus.

$$\begin{tikzcd} & B \times F = \tilde{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ B & & F \end{tikzcd}$$

$$\Omega^1_{\tilde{X}} \cong p_1^* \Omega^1_B \oplus p_2^* \Omega^1_F$$

$\Rightarrow H^0(\tilde{X}, \Omega^1_{\tilde{X}}) = H^0(B, \Omega^1) + H^0(F, \Omega^1)$ and

$H^0(\tilde{X}, \Omega^2) = H^0(B, \Omega^1) \otimes H^0(F, \Omega^1)$

i.e. $\underline{\chi(O_{\tilde{X}})} = 1 - q + p_g = 1 - (g_B + g_F) + g_B g_F$

$= (1 - g_B) \cdot (1 - g_F) = \underline{\chi(O_B) \cdot \chi(O_F)}$

$\Rightarrow \boxed{\chi(O_{\tilde{X}}) = 0}$

Also if $g(B) = 1$ say, then $\Omega^2_{\tilde{X}} \cong p_2^* \Omega^1_F$, i.e. $K_{\tilde{X}} \sim$ sum of p_2 fibers. hence $\Rightarrow \boxed{K^2_{\tilde{X}} = 0}$

Now $\tilde{X} \longrightarrow X$ étale $\Rightarrow \underline{\chi(O_X) = 0 = K^2_X}$. (via c_2 + Noether's formula)

Also

$$H^0(X, \Omega^1) = H^0(\tilde{X}, \Omega^1)^G = [H^0(B, \Omega^1) \oplus H^0(F, \Omega^1)]^G$$

$$= H^0(B/G, \Omega^1) \oplus H^0(F/G, \Omega^1)$$

Since $g(B/G) = 1$, get $\underline{q(X) = 1 \Leftrightarrow g(F/G) = 0}$

(iii) $0 = \chi(O_X) = 1 - q(X) + p_g(X)$ so $p_g(X) = 0$. done

X is minimal, non-ruled bec. in fact

X contains no $\underline{\text{rat'l curves}}$:

$$\begin{tikzcd} & \tilde{X} \arrow[r, "\pi"] \arrow[dl] \arrow[dr] & X \\ B & & F \end{tikzcd}$$

If $C \subset X$ rat'l then $\pi^{-1}(C)$ is a union of rat'l's. hence proj to B or F. ※. □

Final Step: $X = (B\times F)/G$ is minimal, non-ruled
we want numerical condition to distinguish them.

◉ <u>Theorem IV</u>: $P_4(X)\neq 0$ or $P_6(X)\neq 0$ ($\Rightarrow P_{12}(X)\neq 0$),
Moreover if X is not bi-elliptic then $P_{n_i}(X)\to\infty$ some n_i.

<u>Lemma</u>: let $\tilde{C}\xrightarrow{\pi} C = \tilde{C}/G$ be a Galois cover of curves,
then $\boxed{\pi^*: H^0\left(C, K_C^{\otimes k}\left(\sum_p \left[k\left(1-\frac{1}{e_p}\right)\right] p\right)\right) \cong H^0(\tilde{C}, K_{\tilde{C}}^{\otimes k})^G}$

Pf: if $\pi^{-1}(p) = \{Q_1, \cdots, Q_s\}$
then G acts on it with $\ker = I$ (inertial gp)
$\Rightarrow$ all Q_i have the same ramification order e_p.
So near Q_i, π has the form $\pi^* z = w^e$

for $\alpha = z^{-r}(dz)^{\otimes k}$ $\qquad$ $\tilde{C}\ni w$
$= w^{-re}\cdot e^k\cdot w^{k(e-1)}(dw)^k$ $\qquad$ $\downarrow$ $\qquad$ $C\ni z$

regular $\Leftrightarrow$ $\underline{-re + k(e-1)\geq 0}$
i.e. $r\leq\left[k\left(1-\frac{1}{e_p}\right)\right]$. ※

Rmk: for $k=1$. this reduces to $H^0(C,K)\cong H^0(\tilde{C},K)^G$.

<u>Pf of thm IV</u>:

- Case I. $g(B)=1$:

G-inv ($g(B)=1$)

$$H^0(X, K^{\otimes k}) \underset{\uparrow}{=} H^0(\tilde{X}, K^{\otimes k})^G = \left(\underline{H^0(B, K^{\otimes k})}\otimes H^0(F, K^{\otimes k})\right)^G$$

$\tilde{X}\to X$ étale

$$= H^0(F, K^{\otimes k})^G$$

$\Rightarrow P_k(X) = h^0(F, k\cdot K)^G = h^0(\underline{\underline{F/G}}, \mathcal{L}_k)\geq \deg\mathcal{L}_k + 1$

"$\mathbb{P}^1$"

(※) $\mathcal{L}_k = K_{\mathbb{P}^1}^{\otimes k}\left(\sum_p \left[k\left(1-\frac{1}{e_p}\right)\right]\cdot p\right)$

<u>Riemann-Hurwitz</u>: $2g_F - 2 = -2\cdot n + \sum_Q (e_Q - 1)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\sum_Q e_Q\left(1-\frac{1}{e_Q}\right)}$

$\Rightarrow \deg\mathcal{L}_k \geq -2k + \sum_p k\left(1-\frac{1}{e_p}\right) \underline{- r_F}$ $\qquad$ $\overset{\shortparallel}{\sum_p n\left(1-\frac{1}{e_p}\right)}$

$= \frac{k}{n}(2g_F - 2) - r_F$ $\qquad$ A very bad estimate.

r_F = # of ramified pts Q's.

So $g_F\geq 2 \Rightarrow P_k(X)\to\infty$ as $k\to\infty$ ※.

Now write $e_1 \le e_2 \le \cdots \le e_r$

Riemann-Hurwicz: $0 \le 2g_F - 2 = -2n + \sum_p n\left(1 - \frac{1}{e_p}\right)$

$\Rightarrow r \ge 3$ ($r = \#\, p$'s)

$r \ge 4 \Rightarrow \deg L_2 \ge -2\cdot 2 + 2\cdot 4\left(1-\frac{1}{2}\right) \ge 0$, ie. $P_2 \ne 0$

↑ estimated by (*) directly.

$r \le 3$ may assume $r = 3$:

R-H $\Rightarrow$ $0 \le n\left(-2 + 3 - \sum \frac{1}{e_p}\right)$

ie $\boxed{\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \le 1}$

$\frac{3}{e_3} \le$

if $e_1 \ge 3$ then $\deg L_3 \ge -2\cdot 3 + 3\cdot 3\left(1-\frac{1}{3}\right) \ge 0$, i.e. $P_3 \ne 0$

Case $e_1 = 2$. $\Rightarrow$ $\boxed{\frac{1}{e_2} + \frac{1}{e_3} \le \frac{1}{2}}$

if $e_2 \ge 4$ then $\deg L_4 \ge -2\cdot 4 + 4\cdot\frac{1}{2} + 2\cdot 4\left(1-\frac{1}{4}\right) \ge 0$; $P_4 \ne 0$

case $e_1 = 2$, $e_2 = 3 \Rightarrow e_3 \ge 6$

$\deg L_6 \ge -2\cdot 6 + 6\cdot\frac{1}{2} + 6\cdot\frac{3}{4} + 6\left(1-\frac{1}{6}\right) \ge 0$; $P_6 \ne 0$

- Case II. $g(F) = 1$:

every auto of F = translation $\circ$ gp automorphism

$\Rightarrow$ Lie algebra isom $\mathbb{C} \to \mathbb{C}$, $\Lambda \to \Lambda$ ie. mult. by some cpx number $\alpha = a\tau + b$; $\Lambda = \mathbb{Z}\tau \oplus \mathbb{Z}$.

but $\alpha^n = 1 = (a\tau+b)^n = a^n\tau^n + \cdots$

$\Rightarrow$ (since G is finite) must $a = \pm 1$ and $|\tau| = 1$. ie. $\begin{cases} i \\ \rho \end{cases}$

Then $H^0(X, K_X^{\otimes k}) \cong \left[H^0(B, K_B^{\otimes k}) \otimes H^0(F, K_F^{\otimes k}) \right]^G$

always 1-dim'l

$\cong H^0(B, K^{\otimes k})^G$

let $k = 4l \longleftrightarrow i$

or $k = 6l \longleftrightarrow \rho$

$\cong H^0\left(\underset{\text{tori}}{\underset{\parallel}{B/G}}, \underset{\mathcal{O}}{\underset{\parallel}{K}}\left(\sum\left[k\left(1-\frac{1}{e_p}\right)\right]p\right)\right) \ne 0$

"effective div.

eg. Hartshorne P.321. IV. Cor 4.7

If $g(B) \ge 2$ then $\exists$ at least one ramified pt p $\Rightarrow P_k(X) \to \infty$ for such k's $\to \infty$.

END

Rmk: Enriques' thm follows: X ruled $\Leftrightarrow P_4 = P_6 = 0$.

Elliptic Surfaces

First of all, we need the Refined Zariski lemma :
(BPV, P.90 Lemma (8.2) ; Zariski Lemma ;
Beauville, p.91 Prop. VIII.3)

⦿ $C_i \subset X$ irred. curves, $F = \sum n_i C_i$ st. $F.C_i \leq 0$ $(n_i \in \mathbb{N})$
for any $D = \sum r_i C_i$, $r_i \in \mathbb{Q}$
(a) $D^2 \leq 0$ (ie. (C_i, C_j) negative semi-definite)
(b) If F connected and $D^2 = 0$ then $D = rF$, $r \in \mathbb{Q}$ and also $F.C_i = 0$ $\forall i$. (ie. 1-dim'l kernel at most)

This can be used in 2 cases :
I. contraction $X \to X'$, X' may have isolated sing.
II. $X \to S$ a fibration.

pf : write $F = \sum n_i C_i = \sum G_i$
$$D = \sum r_i C_i = \sum \frac{r_i}{n_i} \cdot n_i C_i =: \sum s_i G_i$$
$$D^2 = \sum_i s_i^2 G_i^2 + 2 \sum_{i<j} s_i s_j G_i G_j$$
$\|$ since $G_i^2 = G_i (F - \sum_{j \neq i} G_j)$, get
$$\sum_i s_i^2 G_i.F - \sum_{i<j} (s_i^2 + s_j^2 - 2 s_i s_j) G_i G_j$$
$$= \underline{\sum_i s_i^2 G_i . F} - \sum_{i<j} \underline{(s_i - s_j)^2 G_i . G_j} \leq 0 \text{ . get (a)}$$
If $D^2 = 0$, then must $s_i = s_j$ whenever $C_i \cap C_j \neq \phi$
so if F is connected then all s_i equal.
Moreover $\sum_i s_i^2 G_i.F = 0$, hence $C_i.F = 0$ $\forall i$ (b) $\square$
all the same $s \neq 0$ $\Rightarrow D = \sum s G_i = sF$, $s \in \mathbb{Q}$.

Rmk : This is in fact a local thm. rel. to F.

Also for $f: X \to S$; $X_s := f^{-1}[s]$, X_s is called a multiple fiber if $X_s = \sum n_i C_i$ with $\gcd(n_i) = n > 1$
ie. $X_s = nF$:

⦿ Fact : $\mathcal{O}_X(F)$ and $\mathcal{O}_F(F)$ are torsion bundles of order exactly n. (Exercise)

Elliptic Surfaces (Basic Version)

先記這件事.

Fact:

X non-ruled minimal $\Rightarrow$ K is nef, and

a) If $K^2 > 0$ then $\kappa(X) = 2$ (in fact $\Leftrightarrow$)

b) If ($K^2 = 0$ and) $\kappa(X) = 1$, for $|rK| = M + Z$, have
auto. true

$K.Z = K.M = Z^2 = Z.M = M^2 = 0$ mobile fixed part

pf: a). take H very ample div. R.R $\Rightarrow$

$h^0(nK - H) + h^2(nK - H) \longrightarrow \infty$ as $n \to \infty$

$h^2(nK-H) = h^0(H - (n-1)K) = 0$ for $n \gg 0$

(or use $K.H > 0$: $(H - (n-1)K).H < 0$ since $nK \sim \text{eff} \neq 0$) v.s. (since $(H - (n-1)K).K < 0$, eff. nef $\ast$.)

$\Rightarrow \exists E \in |nK - H|$ i.e. $nK \sim H + E$

$\Rightarrow |nK|$ is v.a. outside E. #

b). If $\kappa(X) = 1$, by a) know must $K^2 \leq 0$

but $K^2 < 0$ is impossible ($K^2 < 0$, min $\Rightarrow$ ruled)

Now $K^2 = 0 \Rightarrow 0 = K.(M + Z) = KM + KZ$ ← use K nef
$\hphantom{Now K^2 = 0 \Rightarrow 0 = K.(M + Z) = } \geq 0 \quad \geq 0$

so $K.M = 0 = K.Z$

$0 = M.rK = M.(M + Z) = M^2 + M.Z$
$\hphantom{0 = M.rK = M.(M + Z) = } \geq 0 \quad \geq 0$

so $M^2 = 0 = M.Z$

finally $Z^2 = (rK - M)^2 = 0$. #

! How can one has such fixed part $Z^2 = 0$? Can one then conclude here that $Z = 0$? (I think so)

• Prop: let X min. $\kappa(X) = 1$, then $K^2 = 0$ and
$\exists$ elliptic fibration $p : X \longrightarrow B$ (smooth B).

pf: $|M|$ is b.p.f since $M^2 = 0$: $X \longrightarrow C$

get via Stein factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & C \\ & B \arrow[ur] & \end{tikzcd}$$

$M^2 = 0 \Rightarrow M = $ sum of fibers

$K.M = 0 \Rightarrow K.F = 0 \Rightarrow g(F) = 1$

for F general smooth fiber $\square$.

我們這樣就証到了這個結果了嗎? $Z^2 = 0$ 為何 $\Rightarrow$ Z moves?

Answer: show directly that $nK \sim$ sum of fibers
By adjunction, this is easy.

Rmk: $f = \Phi_{rK} : X \twoheadrightarrow B$ is the pluri-can. map (Abundance conj.), but the statement is really K nef $\Rightarrow$ $|rK|$ is free for $r \gg 0$.

Abundance theorem:

Prop: $f: X \to B$ minimal elliptic, let $X_s := f^*[s]$.
Then $K^2 = 0$ and $\kappa(X) \le 1$; if $\kappa = -\infty$ then X is ruled over an elliptic curve.

* if $\kappa = 1$ then $\exists\, d$ st $dK \sim \sum n_i F_{s_i}$
moreover $|rdK|$ is free and factors through $f: X \to B$.

pf: If X ruled over C, then elliptic curve $X_s \twoheadrightarrow C$
so C must be rat'l or elliptic
but X minimal ruled $\Rightarrow$ $K^2 = 8(1-g(C)) \ge 0$
(exp $\mathbb{P}^2$ case $K^2 > 0$)

claim: X minimal elliptic $\Rightarrow K^2 = 0$
with this then the case $\kappa = -\infty$ is done. ($g = 1$)

pf: If X ruled then above $K^2 \ge 0$.
If X not ruled, then K is nef hence also $K^2 \ge 0$
Now if $K^2 > 0$ then $h^0(nK) + h^0((1-n)K) \to \infty$
i.e. $\exists\, l \in \mathbb{Z}$ st $h^0(lK) \ne 0$. say $D \in |lK|$
But X elliptic $\Rightarrow$ $0 = (K + F_s).F_s = K.F_s$
$\Rightarrow DF_s = 0$ (for one smooth fiber then for all s)

$\Rightarrow D$ = sum of fibers (maybe $\mathbb{Q}$ welf at multi. fiber)
for this, we need to use refined Zariski lemma
$\Rightarrow K^2 = D^2 = 0$ $\divideontimes$.

Now $K^2 = 0 \Rightarrow \kappa(X) \le 1$ by fact.

case $\kappa(X) = 1$:
$D \in |rK| \Rightarrow D = \sum r_i F_{s_i}$ $\quad r_i \in \mathbb{Q}^+$, eg. $r_i = \frac{m_i}{m}$
$mrK \sim \sum m_i F_{s_i} = f^*\big(\underbrace{\sum m_i [S_i]}_{= A}\big)$

for $l \gg 0$: $|lA| : B \hookrightarrow \mathbb{P}^N$ v.a.
hence $|lmrK| = f^*|lA|$ is b.p.f
and define the morphism $j \circ f : X \to B \hookrightarrow \mathbb{P}^N$. $\square$

Remark: the only key point for the abundance is simply to show that, for elliptic fibrations, $rK \sim$ sum of fibers (hence = pull back).

Kodaira's table for singular fibers :

X $\quad X_s$ smooth elliptic for $s \neq 0$, assume relative minimal

$f \downarrow$

Δ unit disk

this case if fact $K_X = 0$.

a) X_0 irreducible : $2g(X_0) - 2 = (K_X + X_0) . X_0$

i.e. $g(X_0) = 1$ $\quad = K_X . X_0 = K_X . X_s$

arithmetic genus $\quad = (K_X + X_s) . X_s = 0$

but $g(\tilde{X}_0) = g(X_0) - \frac{1}{2} \sum_p r_p(r_p - 1) \leq 1$

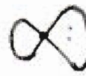
($g(X_0) = 1$; counted with all infinitesimal points)

hence must

- $g(\tilde{X}_0) = 1 \Rightarrow X_0 = \tilde{X}_0$ smooth elliptic curve $\bigcirc : \mathrm{I}_0$
- $g(\tilde{X}_0) = 0 \Rightarrow$ only one ordinary double pt

smooth after one blow up $\qquad r_p = 2$

only 2 possibilities $\left\langle \begin{array}{l} \text{node} : \ y^2 = x^2 : \\ \text{cusp} : \ y^2 = x^3 : \end{array} \right.$


$: \mathrm{I}_1$


$: \mathrm{II}$

(Hint : Can show all double point are of the form $y^2 = x^{n+1}$, then use $s = \frac{y}{x}$

$\Rightarrow s^2 x^2 = x^{n+1} \Rightarrow s^2 = x^{n-1}$ smooth $\Leftrightarrow n \leq 2$)

b) X_0 reducible $= \sum n_i C_i$: $(n_i \in \mathbb{N})$ and ≥ 2 components :

Zariski lemma $\Rightarrow C_i^2 < 0$ (and $\neq -1$ by rel. min.)

but $\quad 0 = K_X . X_0 = \sum n_i K . C_i$

$= \sum n_i (\underbrace{2g(C_i) - 2 - C_i^2}_{\geq 0})$

$\Rightarrow g(C_i) = 0 \ \forall i$ (so $C_i \cong \mathbb{P}^1$)

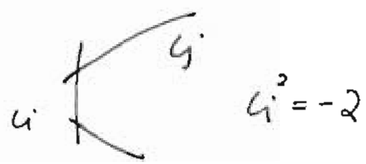
and all C_i are (-2) curves. ($K . C_i = 0 \ \forall i$)

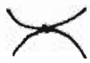
From $\quad 0 = C_i . X_0 = \sum n_j (C_i . C_j)$

$\Rightarrow C_i . C_j = 0, 1,$ or 2 $(i \neq j)$

$C_i . C_j = 2$ occurs only when

$X_0 = n(C_i + C_j)$ some $n \in \mathbb{N}$



- If X_0 is not multiple, then get  : III

For other non-multiple case: the intersection graph Γ has the properties
  - 2 vertexes joined by at most one edge (simply)
  - $Q(\Gamma) \leq 0$ with 1-dim'l annihilator $= X_0$
    (refined Zariski lemma)

Γ : each $C_i \longmapsto$ a vertex
$C_i \cdot C_j = 1 \longmapsto$ join by an edge

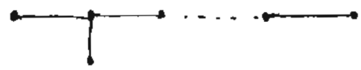
$Q(\Gamma) =$ quadratic form determined by $(C_i, C_j)_{i,j}$

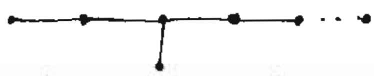
Algebraic classification of $Q(\Gamma)$:

If $Q(\Gamma) < 0$ then have

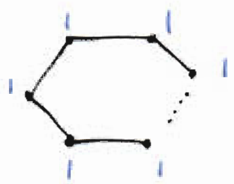
$n = \#$ vertexes

A_n :  $(n \geq 1)$

D_n :  $(n \geq 4)$

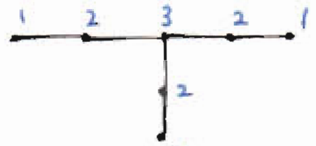
E_n :  $(n = 6, 7, 8)$

If $Q(\Gamma) \leq 0$ with 1-dim'l annihilator, then have $\#$ vertexes $= n+1$:

$I_n = \tilde{A}_n$: 

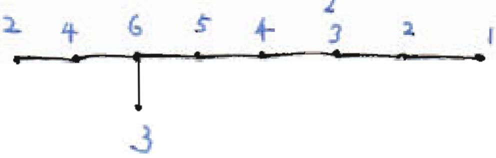
$n \geq 2$

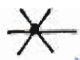
↑ for $n \geq$

$\mathrm{IV}^* = \tilde{E}_6$: 

$n \geq 4$ $\mathrm{III}^* = \tilde{E}_7$: 

$I_{n-4}^* = \tilde{D}_n$: 

$\mathrm{II}^* = \tilde{E}_8$: 

Rmk: $\tilde{A}_2$ has 2 cases :  or   .
"I_2" "IV"

- If $X_0 = m X_0'$ a multiple fiber, then X_0' also has the same structure as above.

But $\mathcal{O}_{X_0'}(X_0')$ is a torsion bundle of degree $m \neq 1$

$\Rightarrow$ X_0' can't be simply connected (topologically)

hence left with $X_0' = I_0, I_1,$ or I_b $(b \geq 2)$

i.e. $X_0 = m I_b$ $(b \geq 0)$ ※

Canonical Bundle formula:

$f: X \to S$ rel. min. elliptic fibration

$X_{s_i} = m_i F_i \quad (i=1 \dots k)$ multiple fibers

then $K_X = f^*\left[\underline{K_S \otimes (R^1 f_* \mathcal{O}_X)^\vee}\right] \otimes \mathcal{O}_X\left(\sum_i (m_i - 1) F_i\right)$

(and $\deg \mathcal{L} = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_S)$ for $\mathcal{L} = K_S \otimes (R^1 f_* \mathcal{O}_X)^\vee$.)

* Corollary: $\kappa(X) \leq 1$ and $\kappa(X) = 1$ if one of the

- $\exists$ pluri. can. div > 0 (i.e. $P_f \neq 0$)
- $g(S) \geq 2$ or $g(S) = 1$ and f not loc. trivial.

Pf: $K_X^{\otimes \mu m} = f^*[K_S \otimes (R^1 f_* \mathcal{O}_X)^\vee]^{\otimes \mu m} \otimes \mathcal{O}_X\left(\sum_i (m_i - 1) \frac{m}{m_i} X_{s_i}\right)$

$= f^*\left[\left(\underline{(K_S \otimes (R^1 f_* \mathcal{O}_X))^{\otimes -m} \otimes \mathcal{O}_S\left(\sum_i (m_i - 1) \frac{m}{m_i} s_i\right)}\right)^{\otimes \mu}\right]$

$\Rightarrow$ grows at most linearly in μ — a line bundle on S say $= D$

clearly $\kappa(X) = 1 \iff \deg D > 0$

**: Important fact is that $\deg (R^1 f_* \mathcal{O}_X)^\vee = \deg f_* \omega_{X/S} \geq 0$

and $\deg = 0 \iff$ loc. trivial outside s_i — $\overset{\shortparallel}{\chi(\mathcal{O}_X)}$ ($\chi(\mathcal{O}_X) \geq 0$ can be got for all non-ruled)

for $g(S) \geq 2$, $\deg K_S = 2g - 2 > 0$ hence $\deg D > 0$.

for $g(S) = 1$, $\deg D = 0 \iff$ above: loc. trivial outside s_i

$m_i = 1 \ \forall i$: no mult. fibers. $\square$

- start with relative Duality thm:

for $f: X \to S$ smooth surf. to smooth curve

$\mathcal{F}$ loc. free $\mathcal{O}_X$-module, then

$$f_*(\omega_{X/S} \otimes \mathcal{F}^\vee) \xrightarrow{\sim} (R^1 f_* \mathcal{F})^\vee$$

where $\omega_{X/S} := K_X \otimes f^* K_S^{-1}$ is the rel. dualizing sheaf.

$\boxed{\text{Important Facts:}}$

- $R^i f_* \omega_{X/S}$ is loc. free and satisfy base change thm (not very difficult)
- f rel. min $\Rightarrow \deg f_* \omega_{X/S} \geq 0$, and $= 0$ iff f is loc. trivial or $g(F) = 1$ and all sing. fiber $= m \cdot E$: smooth elliptic

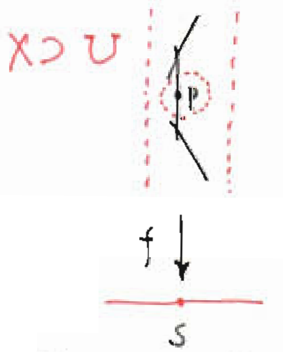
pf of can. bdl. formula:

by def: $K_X = \omega_{X/S} \otimes f^* K_S$

$\Rightarrow \underbrace{f_* K_X = f_* \omega_{X/S} \otimes K_S}_{\text{projection formula}} = \underbrace{(R^1 f_* \mathcal{O}_X)^\vee}_{\text{rel. duality theorem}} \otimes K_S$

$\Rightarrow \quad K_X \xleftarrow{\lambda} f^* f_* K_X = f^*(K_S \otimes (R^1 f_* \mathcal{O}_X)^\vee)$

meaning of λ: restriction from stripe to nbd of p:



- λ = isom. on smooth fibers since X_s elliptic
- λ may vanish on a part of a singular fiber but not the whole fiber: for $\omega \not\equiv 0$, $\omega|_{X_s} \equiv 0 \Rightarrow \omega/g^\ell|_{X_s} \not\equiv 0$ some ℓ, $(g) = X_s$

(This is true for any $f^* f_* \mathcal{L} \to \mathcal{L}$)

If D = the divisor where $\lambda = 0$ (must be a divisor!)

over a pt s, let D_s be the part $\subset X_s$ (sing. fiber)

By def, D_s is a canonical divisor on a nbd U of X_s.

By the analysis in Kodaira's table: (locally in U)

$D_s . C = K . C = 0$ for all irred. comp. $C \subset X_s$

$\Rightarrow D_s^2 = 0 \xRightarrow{\text{refined Zariski lemma}} D_s = r X_s$

but D_s in an integral div and can't $= 0$ on "X_s"

hence X_s must be a multiple fiber and $r < 1$.

i.e. $D = \sum n_i F_i$ with $n_i < m_i$

Now $\omega_{F_i} = K_X \otimes \mathcal{O}_{F_i}(F_i) = \mathcal{O}_{F_i}(F_i)^{\otimes(n_i+1)}$

but for mult. X_{s_i}, F_i are of type I_b ($b \geq 0$)

where we still have $\omega_{F_i} \cong \mathcal{O}_{F_i}$

But $\mathcal{O}_{F_i}(F_i)$ is torsion of order exactly m_i, hence

$m_i \mid n_i + 1 \Rightarrow m_i = n_i + 1$, i.e. $n_i = m_i - 1$. $\square$

Appendix :

Elementary Theory of singular curves :

— curves on a smooth surface —

C : General curve = 1-dim'l projective variety /$\mathbb{C}$
(may be reducible, but at least reduced)

$$\begin{array}{c} \tilde{C} \\ \downarrow p \ \text{normalization} \\ C \end{array} \qquad 0 \to \mathcal{O}_C \to p_* \mathcal{O}_{\tilde{C}} \to \delta \to 0$$

supported on singular pts

• ~~Definition~~: (1) The ~~(Arithmetic) genus~~ $g(C) \equiv P_a(C)$

$:= (-1)^n [\chi(\mathcal{O}_C) - 1] = h^1(\mathcal{O}_C)$

if C is a curve (n=1) and connected (so $h^0(\mathcal{O}_C) = 1$)

(2) The ~~geometric genus~~ $P_g(C)$ can be defined only for ~~Gorenstein~~ varieties : i.e. K_C is a line bundle

$$P_g(C) \equiv P_1(C) := h^0(K_C)$$

Fact : P_a is a (flat) deformation invariant
P_g is a birational invariant for ~~"normal" Gor.~~ variety.

(3) For curve, also define geom. genus $P_g(C) := P_g(\tilde{C}) \equiv g(\tilde{C})$

duality for smooth curves.

• For irreducible curve :

$$\underset{1 - g(\tilde{C})}{\chi(\mathcal{O}_{\tilde{C}})} = \underset{1 - g(C)}{\chi(\mathcal{O}_C)} + \underset{\sum_p \dim(\delta_p)}{\chi(\delta)}$$

arithm. geom.

$$\Rightarrow \boxed{g(C) = g(\tilde{C}) + \sum_p \dim(\delta_p)}$$

C is called rational if $g(\tilde{C}) = 0$ i.e. C is birational to $\mathbb{P}^1$
elliptic " $g(\tilde{C}) = 1$ " elliptic

Rmk: So a curve with high $g(C)$ may still be rational.

$$\boxed{\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + \sum_p \dim(\delta_p)}$$

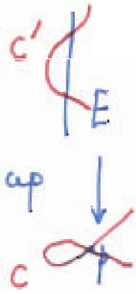
although tautological, is very fundamental
In general δ_p is very difficult to compute.

Let $G \hookrightarrow X$ an effective divisor $\overset{\Longleftrightarrow}{\text{def}}$ G is a curve in X
i.e. 1-dim'l sub scheme
G always be Gorenstein
- non-reduced
- reducible
- non-connected

In case G reduced, irreducible (i.e. sub. variety)

Fact : $\boxed{g(C) = g(\tilde{C}) + \frac{1}{2}\sum_p r_p(r_p - 1)}$

r_p = multiplicity of G at p = $C'.E$ on a blow up
Hartshorne § 5.3



$$0 \to \underset{\mathcal{J}_G}{\mathcal{O}_X(-G)} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

$$\Rightarrow \cancel{\chi(\mathcal{O}_X)} = \chi(\mathcal{O}_C) + \chi(\mathcal{O}_X(-G))$$

$$= \chi(\mathcal{O}_C) + \frac{-G(-K-G)}{2} + \cancel{\chi(\mathcal{O}_X)}$$

$$\Rightarrow \underset{h^0(\mathcal{O}_G) - h^1(\mathcal{O}_G)}{\chi(\mathcal{O}_G)} = -\frac{(K+G).G}{2}$$

(for connected reduced G get $2g-2 = (K+G).G$)

i.e. Adjunction formula : $(K+G).G = -2\cdot\chi(\mathcal{O}_G)$
for any effective divisor G in X

For a blow up at p:
$$\begin{array}{ccc} X' & \supset & G' \\ \pi\downarrow & & \downarrow \\ X & \supset & G \end{array}$$

$$K'.G' + G'.G' = (\pi^* K + E).G' + (\pi^* G - rE).G'$$

$$= r_p - r_p^2 \left(= -r_p(r_p - 1)\right) + K.C + C^2$$

$\Rightarrow$ For $\begin{array}{c}\tilde{X}\\ \downarrow \\ X\end{array}$ get $-2\chi(\mathcal{O}_{\tilde{G}}) = \underset{-2\chi(\mathcal{O}_C)}{(K+C).C} - \sum_p r_p(r_p - 1)$

hence : $\boxed{\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + \frac{1}{2}\sum_p r_p(r_p - 1)}$

unfortunately these points count also the infinitesimal ones, i.e. p is on some successive blow ups

more fundamental

$\Rightarrow$ corollary for special cases :
e.g. reduced. irred. (connected)

Miyaoka-Yau Inequality for non-ruled Surfaces

X minimal surface (non-ruled)

Fact : $K^2 \geq 0$

pf, by assumption, $D \in |nK| \neq \emptyset$

$K^2 < 0 \Rightarrow \quad 0 > D.K = \sum n_i D_i . K \Rightarrow D_i . K < 0$ some i

but then $D_i^2 \geq 0$, hence $D_i . D \geq 0$
$\qquad\qquad \overset{\shortparallel}{n . D_i . K}$ $\times$.

Fact : K is nef, i.e. $K.C \geq 0$

— this is simply nK nef and $(nK)^2 \geq 0$.

Rmk : Converse K nef $\Rightarrow$ minimal is the very first Abundance thm.

stated later

- Prop II: For X minimal,
  - (i) $h^0(Hom(\mathcal{O}(D), \Omega_X^1)) \neq 0 \Rightarrow K.D \leq \max(c_2(X), 0)$

    just $c_2(X)$. since ≥ 0
  - (ii) $h^0(Hom(\mathcal{O}(D), Sym^n \Omega_X^1)) \neq 0 \Rightarrow K.D \leq n \cdot c_2(X)$.

(WHAT'S THE MEANNING ??) Something more than stability

Fact : X non-ruled $\Rightarrow c_2(X) \geq 0$ (not nec. minimal)

Pf : If $\quad 0 > \chi(X) = 2 - 4q + b_2$

then $q \neq 0$. so $H^1(X, \mathbb{Z}) \neq 0$

Pick an (say deg 6) covering space $Y \longrightarrow X$

then $-6 \geq \chi(Y) > 2 - 4q(Y) + 2p_g(Y)$

$\Rightarrow \quad p_g \leq 2q - 4 = 2(q-2) < (2q-3)$

$\Rightarrow \exists$ 1-forms ω_1, ω_2 on Y st. $\omega_1 \wedge \omega_2 = 0$

(bec. $G(2,q) \longrightarrow \mathbb{P}(\Lambda^2 \mathbb{C}^q)$ is an embedding) with image of decomp forms dim $= 2q-3$ in $\Lambda^2 \mathbb{C}^q$.

next page. Lemma A $\Rightarrow \exists$ curve B, $Y \overset{f}{\twoheadrightarrow} B$ and ω_i are pull backs (so $g(B) \geq 2$)

If F is a general fiber

then $\quad \underset{\geq 0}{\chi(Y)} \geq \underset{\geq 0}{\chi(B)} \cdot \underset{\geq 0}{\chi(F)} > 0 \quad \times$ since Y is also not ruled

This Euler inequality is similar to the one used before. $\square$

$\boxed{\text{Lemma B}}$: $f: Y \twoheadrightarrow B \Rightarrow \chi(Y) \geq \chi(B) \cdot \chi(F)$.

Thm: X surface of general type $\Rightarrow$ $3c_2(X) \geq c_1^2(X)$.

- Prop I: if $h^0(L^* \otimes \Omega^1_X) \neq 0$ then $\exists\, c$ st.
$$h^0(L^{\otimes k}) \leq ck \quad \forall k \in \mathbb{N}$$

pf: May assume $h^0(L^{\otimes k_0}) \geq 2$ some k_0.

<u>case $k_0 = 1$:</u> $s_1, s_2 \in \Gamma(L)$

$\exists\, h : \mathrm{Hom}(L, \Omega^1_X)$ i.e. $L \xrightarrow{h} \Omega^1_X$

$\Rightarrow$ $h(s_1)$, $h(s_2)$ linearly indep. 1-forms on X st.
$h(s_1) \wedge h(s_2) = 0$.

$\boxed{\text{lemma A}}$ $\exists$ map (connected fiber) $f: X \to Y$ st
Y is a smooth curve and $h(s_i) = f^*\omega_i$, ω_i 1-forms.

pf: Write $h(s_i) = \alpha_i(z_1, z_2)\, dz_1 + \beta_i(z_1, z_2)\, dz_2$
in a local coor. system (z). $i = 1, 2$
say $\alpha_2 \not\equiv 0$ (or $\alpha_1, \beta_1, \beta_2$ any one)

$\Rightarrow g = \dfrac{\alpha_1(z_1, z_2)}{\alpha_2(z_1, z_2)}$ is a global meromorphic function

$\Rightarrow$ get a map: $\varphi : X \longrightarrow \mathbb{P}^1$

Now use Stein factorization to get connected fibers

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi"'] & Y \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

$\leftarrow$ finite map, Y a normal v. hence smooth.

Exercise: Show (2) $h(s_i) = f^*\omega_i$ for some holo. 1-form ω_i on Y.
and (1) $g = \alpha_1/\alpha_2$ is meromorphic.

lemma A $\Rightarrow$ $\operatorname{div}(s_1) \subset$ sum of fibers of f, i.e. $L = \mathcal{O}(D)$

Let F be a fiber of f; A ample

$(D - nF) \cdot A < 0$ for $n \gg 0$. $\Rightarrow$ $k(D - nF) \not\geq$ ef.

Pick such a large n, then

$$0 \to \mathcal{O}_X(k(D-nF)) \to \mathcal{O}_X(kD) \to \mathcal{O}_{knF}(kD) \to 0$$

$h^0 = 0$

this may be taken to be kn disjoint smooth general fibers of f.

$$\Rightarrow h^0(L^{\otimes k}) = h^0(kD) \leq h^0(knF, kD) = kn. \quad \square$$

trivial bundle on "knF"

Case $k_0 \geq 2$:

This requires the "Branche Covering Trick"

$\boxed{\text{lemma}}$ X cpt cpx mfd, L holo line bundle.
$h^0(L^{\otimes k}) \geq 2 \Rightarrow \exists\ Y \xrightarrow{f} X$ generic finite st
$h^0(Y, f^*L) \geq 2$.

With this, also via pull back:

$$h^0(Hom(L, \Omega_X^1)) \neq 0 \Rightarrow h^0(Hom(f^*L, \Omega_Y^1)) \neq 0$$

hence $\quad h^0(L^{\otimes k}) \leq h^0(f^*L^{\otimes k}) \leq nk$. (some n). $\square$

pf of covering lemma: First: "BCT": (used before)

Z : $\mathbb{P}^1$ bundle $/X \supset S$ divisor st. $S.F = n$ pts
$\downarrow p$ $\quad$ then $\exists\ f: Y \twoheadrightarrow X$ gener. finite $\quad$ general fiber
X

any normal cpx space.

st.

$$\begin{tikzcd} Z \arrow[d, "p"'] & Z\times_X Y =: Z' \arrow[l, "g"'] \arrow[d] \\ X & Y \arrow[l, "f"'] \end{tikzcd} \supset S_1, \cdots, S_n$$

S_k meets general fiber in one pt.

and $g^*S = S_1 + \cdots + S_n \quad \square$.

- Exercise: Prove this BCT. Hint: consider $\bar{S} = \text{normaliz}(S)$ and take $Z_1 = \bar{S} \times_X Z$, do induction on n.

pf (conti.):

proof of Prop II:

(i) $h^0(\Omega^1_X \otimes \mathcal{O}(-D)) \neq 0 \Rightarrow \Omega^1_X \otimes \mathcal{O}(-D-S)$ has a section with isolated zeros. (ef)

i.e. $0 \leq c_2(\Omega^1_X \otimes \mathcal{O}(-D-S))$

$$= c_2(X) - K.(D+S) + (D+S)^2$$

so $K.D \leq c_2(X) - K.S + (D+S)^2$

Only need to consider the case $\underline{(D+S)^2 > 0}$: ($=: \tilde{D}$)

R.R. $\Rightarrow$

$h^0(n\tilde{D}) + h^0(K - n\tilde{D}) > cn^2$ for some $c > 0$ and $n \gg 0$.

so

$$\begin{cases} h^0(n\tilde{D}) > \frac{1}{2}cn^2 & \text{or} \\ h^0(K-n\tilde{D}) > \frac{1}{2}cn^2 \end{cases}$$

for ∞ many n's.

1st $\ast$ by Prop I.

eg. $ch(E \otimes \mathcal{L}) = \sum e^{a_i + \ell}$

$$= (\sum e^{a_i}) \cdot e^{\ell}$$

$$= \left(r + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2}\right) \cdot \left(1 + c_1(L) + \frac{c_1^2(L)}{2} \dots\right)$$

$$\parallel$$

$$r + [c_1(E) + rc_1(L)] + \left[\frac{c_1^2(E)}{2} - c_2(E) + c_1(E)c_1(L) + \frac{c_1^2(L)}{2} \cdot r\right]$$

$$= r + c_1(E \otimes L) + \frac{c_1^2(E \otimes L)}{2} - c_2(E \otimes L)$$

$$c_2(E \otimes L) = c_2(E) - \frac{c_1^2(E)}{2} - c_1(E)c_1(L) - c_1^2(L) \cdot \frac{r}{2} + \frac{c_1^2(E) + 2rc_1(E)c_1(L) + r^2c_1^2(L)}{2}$$

$$\boxed{c_2(E \otimes L) = c_2(E) + (r-1)c_1(E).c_1(L) + \frac{r(r-1)}{2}c_1^2(L)}$$

2nd case, since K is nef, $\Rightarrow$

$$0 \leq (K - n\tilde{D}).K = K^2 - n\tilde{D}.K$$

i.e. $(D+S).K \leq \frac{1}{n}K^2$ for $n_i \to \infty$ i.e. ≤ 0

so $\underline{D.K \leq -S.K \leq 0}$ (S ef. K nef). $\square$

(ii). let $Z = \mathbb{P}(\Omega^1_X)$ (in fact $\cong \mathbb{P}(T_X)$)

$$\begin{array}{c} Z \\ \downarrow p \\ X \end{array}$$

$H := \mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$: relative $\mathcal{O}(1)$. take to be a divisor H_p

then $\Gamma(X, \mathrm{Sym}^n \Omega^1_X \otimes \mathcal{O}(-D)) \cong \Gamma(Z, \mathcal{O}_Z(nH) \otimes p^*\mathcal{O}(-D))$

via projection formula

$\Rightarrow \exists$ ef. div $G \subset Z$, $G \sim nH - p^*D$

By BCT:

$$\begin{tikzcd} \mathbb{P}(f^*\Omega^1_X) \arrow[r, "\tilde f"] \arrow[d, "q"'] & \mathbb{P}(\Omega^1_X) \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \quad \mathbb{P}(\Omega^1_X) \supset G \sim nH - p^*D$$

say of degree k

$$\Rightarrow \tilde f^* G \sim \coprod \big(\overset{\text{eff.}}{H_q} - \underbrace{\tilde f^* p^*}_{= q^* f^*} D_i\big)$$

and $\sum D_i = D$

Now $f^*\Omega^1_X \hookrightarrow \Omega^1_Y$ as a rank 2 subsheaf but not nec. $= \Omega^1_Y$ unless f is unbranched.

* In fact, Prop II (i), (ii) both true for $\mathcal{F} \subset \Omega^1$ a rk 2 subsheaf, st.
- $c_1(\mathcal{F})$. is nef
- $\Gamma(\mathcal{H}om(\mathcal{O}_X(D), \mathcal{F})) \neq 0$

$\Rightarrow c_1(\mathcal{F}).D \leq \max(c_2(\mathcal{F}), 0)$.

with this in (i): get

by construction, $\underline{\Gamma(\mathcal{H}om(\mathcal{O}_Y(f^*D_i), f^*\Omega^1_X)) \neq 0}$ **

$c_1(f^*\Omega^1_X).G = c_1(\Omega^1_X).f_*G \geq 0$ still nef

So (i) $\Rightarrow$

$$c_1(f^*\Omega^1_X).f^*D_i \leq \max(c_2(f^*\Omega^1_X), 0)$$

$$\Rightarrow f^*(c_1(\Omega^1_X).D) \leq \max(n\, c_2(f^*\Omega^1_X), 0)$$

i.e. $\cancel{k}\cdot c_1(\Omega^1_X).D \leq \cancel{k}\cdot n\, c_2(\Omega^1_X)$

done. □.

**: the reason is, again on $Z' = \mathbb{P}(f^*\Omega^1_X) \xrightarrow[q]{} Y$:

$$\Gamma(Y, \mathrm{Sym}^\ell(f^*\Omega^1_X) \otimes \mathcal{O}(S)) \cong \Gamma(Z', \underbrace{\mathcal{O}_{Z'}(\ell H_q) \otimes q^*\mathcal{O}_Y(S)}_{= \mathcal{O}_{Z'}(\underline{\ell H_q + q^*S})})$$

take $\ell = 1$, and use $S = -f^*D_i$

and $\underline{H_q - q^*f^*D_i}$ is effective. ※.

pf of $3c_2(X) \geq c_1^2(X)$:

may assume that X is minimal. also that $K^2 > 0$.

Assume $\quad \alpha = \dfrac{c_2}{c_1^2} < \dfrac{1}{3}$

$\quad$ let $\beta = \frac{1}{4}(1-3\alpha) \in (0, \frac{1}{4}]$, $\quad n(\alpha+\beta) \in \mathbb{Z}$

consider v.b. $\quad V_n := \mathrm{Sym}^n \Omega^1_X \otimes \mathcal{O}_X(-n(\alpha+\beta)K)$

<u>claim: $h^0(V_n) = h^2(V_n) = 0$ for $n \gg 0$.</u>

Apply Prop II. (ii) to $D = n(\alpha+\beta)K$

$$h^0(V_n) \neq 0 \Rightarrow K.D = n(\alpha+\beta)K^2 \leq n \cdot c_2(X)$$

$$\Rightarrow \alpha = \frac{c_2}{c_1^2} \geq \alpha+\beta \quad \times$$

for $h^2(V_n)$: Notice that $\Omega^1 \otimes \Omega^1 \longrightarrow \Omega^2 = K$

hence $\Omega^1 \cong (\Omega^1)^* \otimes K$ or $T \cong \Omega^1 \otimes \mathcal{O}(-K)$

$$\Rightarrow h^2(V_n) \underset{\text{Serre duality}}{=} h^0(\mathrm{Sym}^n T \otimes \mathcal{O}_X((n(\alpha+\beta)+1)K))$$

$$\| $$

$$h^0(\mathrm{Sym}^n \Omega^1 \otimes \mathcal{O}_X((n(\alpha+\beta-1)+1)K))$$

if $\neq 0 \Rightarrow$ for $D = -(n(\alpha+\beta-1)+1)K$

$$K.D = -(n(\alpha+\beta-1)+1)K^2 \leq n \cdot c_2(X)$$

i.e. $\quad n\alpha \geq -n(\alpha+\beta) + (n-1)$

$$\alpha \geq -(\alpha + \tfrac{1}{4}(1-3\alpha)) + 1 - \tfrac{1}{n}$$

$$\alpha + \tfrac{1}{4}(1+\alpha) \geq 1 - \tfrac{1}{n}$$

$$\tfrac{1}{3} > \alpha \geq \tfrac{1}{2}\left(\tfrac{3}{4} - \tfrac{1}{n}\right) \sim \tfrac{3}{8} > \tfrac{3}{9} = \tfrac{1}{3} \quad \text{for } n \gg 0 \quad \times$$

So $\quad 0 \geq \chi(X, \mathrm{Sym}^n \Omega^1_X \otimes \mathcal{O}(-n(\alpha+\beta)K))$

$\quad \| \leftarrow$ Leray spectral sequence

$Z = \mathbb{P}(\Omega^1_X) \supset H_p \qquad \chi(Z, H^n \otimes p^* \mathcal{O}_X(-n(\alpha+\beta)K))$

$\quad \| \leftarrow$ R.R for 3 folds Z :

$$\begin{array}{c} Z \\ \downarrow p \\ X \end{array} \qquad \frac{n^3}{3!}(H + p^*\underset{\in \mathbb{Q}}{(\alpha+\beta)}K)^3 + O(n^2)$$

want this > 0

Leray - Hirsh thm again, get

$$H^2 + p^* c_1(X) \cdot H + p^* c_2(X) = 0 \quad \text{on } Z.$$

(Relation of chern class for bundle Ω_X^1)

Since $H \cdot p^*(pt) = pt$ — in Z as $H^6(Z;\mathbb{Z})$

in X as $H^4(X;\mathbb{Z})$

$$\Rightarrow H^3 + p^* c_1(X) \cdot \underbrace{H^2}_{= -p^* c_1(X) \cdot H - p^* c_2(X)} + \underbrace{p^* c_2(X) \cdot H}_{= c_2(X)} = 0$$

(so $p^* c_1(X) \cdot H^2 = -c_1^2(X)$.)

$$\Rightarrow H^3 = c_1^2(X) - c_2(X) \, ;$$

notice that $c_1 = -K$

so $\left(H + p^*(\alpha+\beta)K\right)^3$

$$= H^3 + 3(\alpha+\beta) H^2 \cdot p^*K + 3(\alpha+\beta)^2 H \cdot p^*K^2$$

$$= c_1^2 - c_2 + 3(\alpha+\beta) c_1^2 + 3(\alpha+\beta)^2 c_1^2$$

$$= c_1^2\left(1 - \alpha + \tfrac{3}{4}(1+\alpha) + \tfrac{3}{16}(1+\alpha)^2\right)$$

$$= \frac{c_1^2}{16} \cdot \left(3\alpha^2 + 6\alpha + 3 + 12\alpha + 12 - 16\alpha + 16\right)$$

$$= \frac{c_1^2}{16}\left(3\alpha^2 + 2\alpha + 31\right) > 0$$

$\alpha + \beta = \alpha + \frac{1}{4}(1 - 3\alpha)$
$= \frac{1}{4}(1+\alpha)$

ie. $0 \geq \chi(\cdots) > 0$ $\times$.

so $\alpha \geq \frac{1}{3}$ ie. $\boxed{3c_2(X) \geq c_1^2(X)}$. $\square$

Rmk: Yau proved "=" $\Leftrightarrow$ $X = B^2/\Gamma$

- Exercise: Carry out the latter computation using R.R. for vector bundles on surfaces:
  $\chi(V) = ch(V) \cdot td(X)$.

$\boxed{\text{End}}$.

Geography of surfaces of general type : $\kappa=2$

- <u>Fact</u>: X minimal of general type then $K^2 = c_1^2(X) > 0$

pf: let H hyperplane $\subset X$. (already know that $K^2 \geq 0$ and K nef)

$$0 \to \mathcal{O}(nK - H) \to \mathcal{O}(nK) \to \mathcal{O}_H(nK) \to 0$$

since $h^0(nK) > cn^2$, and $h^0(H, \mathcal{O}_H(nK)) \leq c' \cdot n$

so $nK - H \sim D$ eff.

$$\Rightarrow n^2K^2 = (H+D)^2 = H^2 + 2H.D + D^2$$
$$= H^2 + H.D + \underbrace{(H+D)}_{nK}.D > 0 \quad \square$$

(in general, X with $\kappa(X) = n = \dim X$ & K nef $\Rightarrow K^n > 0$)

- <u>Fact</u>: C irred. $K.C = 0 \Leftrightarrow C = (-2)$ curve ($\mathbb{P}^1$)

moreover, $\exists$ only finite # of (-2) curves

and are indep. $/\mathbb{Q}$, $\# \leq \rho(X) - 1$.

pf: a relation is given by $\underbrace{\sum \lambda_i C_i = \sum \lambda_j C_j}_{D}$; $\lambda_i, \lambda_j \geq 0$

but $K^2 > 0$, $K.D = 0 \Rightarrow D^2 \leq 0$

Hodge index thm

but $D^2 = (\sum \lambda_i C_i).(\sum \lambda_j C_j) \geq 0$ and > 0 if $D \not\sim 0$

thus $D \sim 0$, but D is effective, hence $D = 0$ in the homology classes. $\square$

- <u>Fact</u>: X any surface of general type then $c_2(X) > 0$

(already know $c_2(X) \geq 0$ for non-ruled X)

※ this certainly follows from $3c_2(X) \geq c_1^2(X)$. but we want an <u>easy pf</u>:

pf: If $\exists X \twoheadrightarrow B$, $g(B) \geq 2$ then $g(F) \geq 2$ too and

$$\chi(X) \geq \chi(F) \cdot \chi(B) \geq (2-2g_F) \cdot (2-2g_B) \geq 4.$$

If $\nexists$ such map

<u>claim</u>:
$$h^{2,0} \geq 2h^{1,0} - 3$$
$$h^{1,1} \geq 2h^{1,0} - 1$$
$(q = h^{1,0})$

then $\chi = 2 - 4q + (2h^{2,0} + h^{1,1}) \geq p_g - 2$ (claim need " $\geq$ " otherwise here $>$)

保留-4

$K=2$

so $\chi > 0$ unless : ($q > 0$ otherwise $\chi > 0$ always)

$p_g = 1$, then $q = 2$
$p_g = 0$, then $q = 1$ $\Rightarrow$ both sat. $\chi(\mathcal{O}) = 0$, $q > 0$
($p_g = 2$ must $\chi > 0$) $\quad$ $\overset{\shortparallel}{1 - q + p_g}$

but now $\chi \geq 0$ (for any non-ruled)

$$0 = \chi(\mathcal{O}) = \frac{\overset{0}{\ulcorner} K^2 + \chi(X)}{12} \Rightarrow \chi(X) < 0 \quad \text{*}$$

(Recall the pf of the fact : X non-ruled $\Rightarrow \chi \geq 0$. we use

$$p_g \leq \underset{\underset{2(q-2)}{\shortparallel}}{2q-4} \Rightarrow \exists\, \omega_1, \omega_2 \text{ st } \omega_1 \wedge \omega_2 = 0$$

$$\underset{\text{lemma A}}{\Rightarrow} \exists \text{ map } X \twoheadrightarrow B \ldots)$$

For claim : $\nexists\, X \twoheadrightarrow B \Rightarrow \nexists\, \omega_1, \omega_2$ st. $\omega_1 \wedge \omega_2 = 0$

$\Rightarrow \Lambda^2 \Gamma(\Omega^1) \longrightarrow \Gamma(\Omega^2)$ has kernel $\cap$ decomposable $= 0$

From $Gr(2,q) \hookrightarrow \mathbb{P}(\Lambda^2 \mathbb{C}^q)$, get dim (decomp) in $\Lambda^2 \mathbb{C}^q = 2(q-2)+1$
$= 2q-3$

$$\Rightarrow p_g \geq 2q - 3.$$

For $h^{1,1} \geq 2h^{1,0} - 1$: use $\wedge : H^{1,0} \times H^{0,1} \longrightarrow H^{1,1}$

get $\mathbb{P}(H^{1,0}) \times \mathbb{P}(H^{0,1}) \longrightarrow \mathbb{P}(H^{1,1})$

Q: Now any $\mathbb{P}^r \times \mathbb{P}^s \longrightarrow M$ with dim $< r+s$ will factor through $\mathbb{P}^r$ or $\mathbb{P}^s$, hence * to non deg. of each factor

hence $h^{1,1} - 1 \geq 2(h^{1,0} - 1)$.

$$\alpha \wedge \bar\beta \text{ in } H^{1,1} = 0 \Rightarrow \alpha \wedge \bar\beta = d\gamma$$

$$\Rightarrow \alpha \wedge \beta \wedge \bar\alpha \wedge \bar\beta = d\gamma \wedge d\bar\gamma = d(\gamma \wedge d\bar\gamma)$$

$$\Rightarrow \int_X (\alpha \wedge \beta) \wedge (\overline{\alpha \wedge \beta}) = 0$$

$$\Rightarrow \alpha \wedge \beta \text{ holo 2-form} = 0 \text{ ptwise.}$$

$$\Rightarrow \alpha \,/\!/\, \beta. \text{ (by assumption)}$$

but then $\alpha \wedge \bar\alpha = 0$ in $H^{1,1}$ * to Hodge-Riemann relation $\int \alpha \wedge \bar\alpha \wedge \omega \neq 0$ for some Kahler form ω $\square$.

Noether inequality:

$$p_g \le \frac{1}{2} K^2 + 2 \quad \text{for } X \text{ min. general type}$$

pf: may assume $p_g \ge 3$ since $K^2 > 0$.

write $|K| = \underbrace{|C|}_{\text{mobile part}} + \underbrace{V}_{\text{fixed part}}$

$\Rightarrow$ $C^2 \ge 0$ and $C.V \ge 0$

Adjunction: $2 - 2g(C) = (K+C).C \ge 0$

$$g(C) := h^1(O_C) \overset{*}{=} h^0(K_C) = h^0(C, K+C) = h^0(C, 2C+V)$$

$$\ge h^0(C, 2C) \overset{A}{\ge} 2h^0(O_C(C)) - 1$$

$$\overset{B}{\ge} 2h^0(O_X(C)) - 3$$

$$= 2h^0(K) - 3 = 2p_g - 3$$

$//$

$\frac{KC + C^2}{2} + 1$ (use K nef)

$$\le \frac{K(C+V) + (C+V).K}{2} + 1 = K^2 + 1$$

$\Rightarrow$ $p_g \le \frac{1}{2} K^2 + 2$ $\square$

$*$ Grothendieck-Serre duality for Gorenstein scheme

A. trivial fact: $\left\{ \begin{array}{llll} 1 & s_2 & \cdots & s_{n-1} \\ 1 & s_1 & & \end{array} \right.$ & s_1^2, $\Rightarrow$ $2(n-1) + 1 = 2n-1$

B. $0 \to O_X \to O_X(C) \to O_C(C) \to 0 \Rightarrow h^0(O_C(C)) \ge h^0(O_X(C)) - 1$.

Rmk: X minimal 3 fold, H ample smooth divisor

$K_H = (K_X + H)|_H$ is again nef (in fact, general type)

hence $(3c_2(H) - c_1^2(H)) \ge 0$

Adjunction:

$$c_1(X)|_D = c_1(D) + D|_D$$

$$c_2(X)|_D = c_2(D) + c_1(D).D|_D$$

ie. $c_1^2(H) = (c_1(X)| - H|_H)^2 = c_1^2.H - 2c_1.H^2 + H^3$

$c_2(H) = c_2 H - c_1 H^2$

hence $3c_2 H - 3c_1 H^2 + 2c_1 H^2 - c_1^2 H - H^3 \ge 0$

$$(3c_2 - c_1^2)H \ge (3c_1 + H).H^2 > 0$$

For higher dim, such inequalities hold for ample H_i's:

$(3c_2 - c_1^2) H_1 \cdots H_{n-2} \ge 0$ by Miyaoka.

Cor: K^2 even: $5K^2 - c_2 + 36 \geq 0$

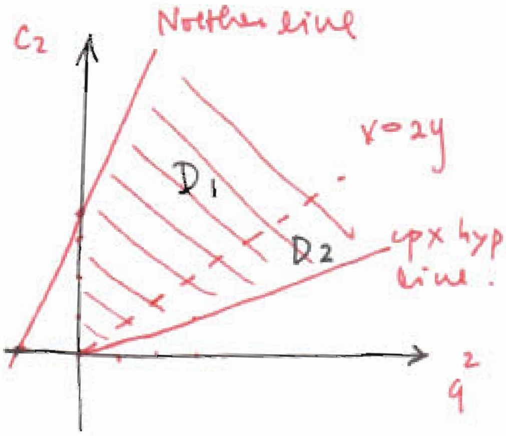
K^2 odd: $5K^2 - c_2 + 30 \geq 0$; $=0 \Rightarrow q = 0$

Pf: $1 - q + p_g = \chi(\mathcal{O}_X) = \dfrac{K^2 + c_2}{12}$

$p_g \leq \frac{1}{2}K^2 + 2$ (for odd use $\frac{1}{2}(K^2-1)$)

$12 - 12q + 6K^2 + 24 \geq K^2 + c_2$ ok. $\square$

Rmk: let X minimal surface, general type $(x, y) = (K^2, c_2)$:



$K^2 > 0$	$x > 0$
$c_2 > 0$	$y > 0$
$K^2 \leq 3c_2$	$x \leq 3y$
$5K^2 - c_2 + 36 \geq 0$	$5x - y + 36 \geq 0$
$\dfrac{K^2+c_2}{12} \in \mathbb{N}$	$12 \mid x+y$

- Thm (Persson 1981): Most pairs $(x, y) \in D_1$ is realizable by a surface of general type.

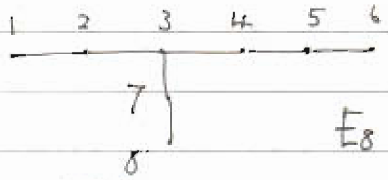
X K3 Surface := cpx surface, $K=0$ and $b_1(X)=0$
let's assume that X is Kähler.

- ~~Theorem A~~: All Kähler K3's form an 20 dim'l irreducible families. In particular, they are all diffeomorphic and simply connected ($\pi_1=0$).
- ~~Theorem B~~: (Torelli theorem) let X, X' be 2 K3's
$\exists$ Hodge isometry $\phi: H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z}) \Leftrightarrow X \cong X'$.
- ~~Theorem C~~: (Surjectivity of period map)
- ~~Theorem D~~: Every cpx K3 is automatically Kähler

I. Let $V \cong \mathbb{Z}^{22}$ equipped with quad. form $3H \oplus 2(+E_8)$
K3 lattice: $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{Z}^2$ of signature $(3,19)$:

combine this part with IV

$$E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & 1 & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & & \\ & & 1 & & & & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}$$



on $\mathbb{Z}^8$
and even.

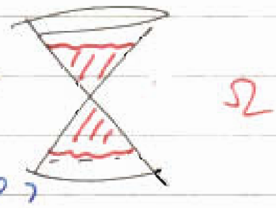
classification of $\mathbb{Z}$-quad form $\Rightarrow$ $\sigma=(3,19)$, must be $3H \oplus 2E_8$.

II. A marked K3 $X = (X,\phi)$; $\phi: H^2(X,\mathbb{Z}) \xrightarrow{\sim} V$

(Naive) Period map: $(X,\phi) \xmapsto[p]{} [H^{2,0}(X)] \in P(V_{\mathbb{C}})$
combine with local Torelli
with image lies in $\Omega := \{ v \in V_{\mathbb{C}},\ v^2=0,\ v\cdot\bar{v}>0 \}/\sim\ \subset P(V_{\mathbb{C}})$
$\dim_{\mathbb{C}} \Omega = 22-1-1 = 20$.

$H^1(X,T) = h^{1,1} = 20$, $H^2(T) = h^{2,1} = 0$
$\Rightarrow$ local moduli = smooth 20-dim'l.

$M \xrightarrow{p}$



injective?
Surjective?

III. Special points: X = Kummer surfaces.

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/\langle \iota \rangle \end{array}$$

: degree 2 cover
branched over 16 curves $\tilde{C}_i \xrightarrow{\sim} C_i$
with $\tilde{C}_i^2 = -1$, $C_i^2 = -2$

Main idea: use T to study X

ι lifts to $\tilde\iota$ by fixing $\tilde C_i$ pointwise.

(*)
$$\begin{tikzcd} \tilde T \arrow[r, "p"] \arrow[d, "\pi"'] & X = \tilde X/\tilde\iota \arrow[d, "\pi"] \\ T \arrow[r, "p"'] \arrow[ur, dashed, "\sigma"] & T/\iota \end{tikzcd}$$

$\alpha \in H^1(T,\mathbb{Z})$ or $H^1(\tilde T,\mathbb{Z})$ is never ι or $(\tilde\iota)$-inv.

$\Rightarrow b_1(T/\iota) = 0 = b_1(X)$.

but $\alpha \in H^2(T,\mathbb{Z})$ or $H^2(\tilde T,\mathbb{Z})$ is always ι or $(\tilde\iota)$-inv

i.e. "α" $\in H^{\cdot}(X)$. eg. $\alpha = \tilde C_i \Rightarrow 2\tilde C_i = p^* C_i$

but not. nec. pull backs : $\alpha \neq \tilde C_i \Rightarrow \alpha = p^*\beta$ some β

case 1). $2\tilde C_i^{\,2} = 2\tilde C_i\cdot\tilde C_i = (p^* C_i)\cdot\tilde C_i = C_i\cdot p_*\tilde C_i = C_i^{\,2}$

this explains $(-1) \longmapsto (-2)$

case 2). for $\alpha_i \in H^2(T,\mathbb{Z})$, define $\sigma(\alpha_i) := p_*\pi^*\alpha_i$:

$\alpha_1.\alpha_2 = p_*(\tilde\alpha_1.\tilde\alpha_2) = p_*(p^*\beta_1 . p^*\beta_2)$

$= \beta_1 . p_* p^*\beta_2 = 2\beta_1.\beta_2$; but $\pi^*\alpha_i = p^*\beta_i$

$\Rightarrow p_*\pi^*\alpha_i = p_*p^*\beta_i = 2\beta_i$

$\parallel$

$\sigma(\alpha_i)$

Basic Fact (1) $\sigma : H^2(T,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})$,

with $2(\alpha_1,\alpha_2) = (\sigma(\alpha_1),\sigma(\alpha_2))$

so σ is injective.

Combine 1), 2) this formula is also true in the level $\tilde T$.

Basic Fact (2) $\mathrm{Image}(\sigma) = \mathbb{Z}\langle C_1,\cdots,C_{16}\rangle^{\perp}$.

Non-Basic Fact (3) If a K3 surface X contains 16 disjoint (-2) curves $C_1,\cdots,C_{16}$ st. $\sum_{i=1}^{16} C_i$ is 2-divisible in $\mathrm{Pic}(X)$ $(= H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ since $q=0)$, then X is a Kummer Surface.

pf. First the converse : $C = \sum_{i=1}^{16} C_i$ is 2-divisible :

this follows from $C.\,\mathrm{Image}(\sigma) = 0$

$C.C_i = -2 \quad \forall i$

and from Poincaré duality on $H^2(X,\mathbb{Z})$.

for $\Rightarrow$: see (*)

C is 2-divisible $\Rightarrow \exists$ degree 2 branched cover $\tilde T \to X$ over C $\Rightarrow \tilde C_i := p^* C_i$ are all (-1) curves.

let π be the contraction (Grauert them in the non-alg. case) $\pi : \tilde T \longrightarrow T$

$\chi(T) = \chi(\tilde T) - 16 = (2\chi(X) - 16\cdot 2) - 16 = 48 - 48 = 0$

Moreover $K_T = 0$ (why?), hence T is a 2 torus. (why?)

$\square$.

IV. Picard lattice of K3's.

$h^1(\mathcal{O})=0 \Rightarrow \mathrm{Pic}(X) \cong H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$.

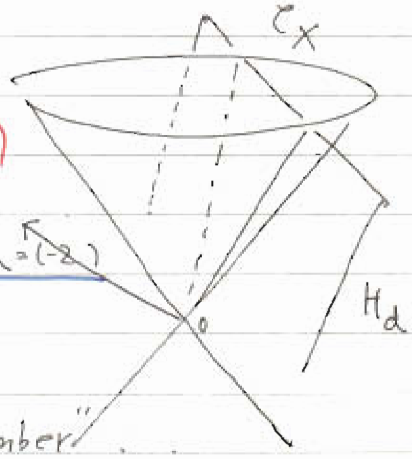
$\mathcal{C}_X \cup \mathcal{C}'_X = \{ x \in H^{1,1}(X,\mathbb{R}) \mid x^2 > 0 \}$, $\mathcal{C}_X \supset$ Kähler classes

- Fact(1). Ef_X: the effective cone $= \mathbb{Z}^+ \langle \overline{\mathcal{C}}_X \cap H^2(X,\mathbb{Z});\ (-2)$ curves $\rangle$

pf: let C be a curve of indecomposable class (so C irred.)
then $2g_C - 2 = KC + C^2 = C^2$
so $C^2 \geq 0$ or $C = (-2)$ rational curve.
Conversely, $h^0(C) + h^2(C) \geq \frac{1}{2}C(C-K) + \chi(\mathcal{O}_X) = \frac{1}{2}C^2 + 2$
$h^2(C) = h^0(-C) = 0$ since $H.C > 0$ (bec. $C^2 \geq 0$)
for the case $C^2 = -2$, know $h^0(C)$ or $h^0(-C) \neq 0$
ie for C a (-2) class, C or $-C$ is a (-2) curve. ※

⊙ Let $\Delta :=$ classes of (-2) curves. $d \in \Delta$, not nec. finite!
Picard-Lefschetz reflection (Weyl transform):
$s_d : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$ is a Hodge isometry $\quad s_d(x) := x + (x.d)\,d$, $\quad W_X = \langle s_d \rangle_d$
conn. comp (= chambers) of $\mathcal{C}_X - \cup_d H_d$

** $\mathcal{C}_X^+ := \{ y \in \mathcal{C}_X \mid y.d > 0 \ \forall d \in \Delta \}$
~~the pre-Kähler cone~~ (is a chamber)



- Fact(2). $s_d : \mathcal{C}_X \to \mathcal{C}_X$ and $\overline{\mathcal{C}}_X^+$ is a fundamental domain of W_X $d^2 = (-2)$

pf: s_d maps $\mathcal{C}_X$ to $\mathcal{C}_X$ or $\mathcal{C}'_X$, but
$s_d = id$ on H_d. hence $\mathcal{C}_X \to \mathcal{C}_X$.
Now only need to use "$\mathcal{C}_X^+$ is a chamber".
and W_X acts properly and discontinuously on $\mathcal{C}_X$. □

- Fact(3): let $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ a Hodge isometry between K3 surfaces, then
ϕ is effective $\Leftrightarrow \phi : \mathcal{C}_X^+ \to \mathcal{C}_{X'}^+ \Leftrightarrow \exists\, \alpha \in \mathcal{C}_X^+$ st. $\phi(\alpha) \in \mathcal{C}_{X'}^+$.
Moreover, can always arrange ϕ to be effective by composing s_d's. (in X or X')

pf: $\Rightarrow$: $\Rightarrow$ is trivial, for $\Leftarrow$: since $\mathcal{C}_X \to \mathcal{C}_{X'}$, only need to show (-2) ef $\longmapsto (-2)$ ef. but this follows from $(\phi(\alpha), \phi(d)) = (\alpha, d) > 0$ □

• V. Torelli for Kummer

Prop: let X, X' be Kummer of T, T' and $\phi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ ef. isometry and $T[2] \xrightarrow{\text{maps}} T'[2]$ 1-1, If $T \ni$ ef. div. then $\phi = f^*$ for some isom $f: X' \xrightarrow{\sim} X$.

pf: ϕ isometry $\Rightarrow$ $H^2(T,\mathbb{Z}) \to H^2(T',\mathbb{Z})$ isometry — as $\perp$ space of (-2)'s.
together with $T[2] \to T'[2]$, get
$H^1(T,\mathbb{Z}) \to H^1(T',\mathbb{Z})$ isometry (nontrivial step)
Torelli for cpx tori $\Rightarrow$ $T' \xrightarrow{\sim} T$, hence $X' \xrightarrow{\sim} X$ $\square$.
? where do we use the ef. div on T!

Thm: let X' K3, X proj. Kummer, $\phi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ an effective Hodge isometry, then $\phi = f^*$, $f: X' \xrightarrow{\sim} X$.

pf: let $C_1, \dots, C_{16}$ (-2) curves on X, ϕ^{-1} is also ef.
let $C_i' = \phi(C_i)$, $\sum_{i=1}^{16} C_i$ 2-divisible $\Rightarrow$ $\sum_{i=1}^{16} C_i'$ too.
$\Rightarrow X'$ is a Kummer surface.
X proj $\Rightarrow$ $\exists$ ef class $\perp C_1, \dots, C_{16}$ i.e. $\exists$ ef div on T where X = Kummer of T. Apply above prop. $\square$

Corollary: without "ef" of ϕ, still $\exists$ $X \cong X'$. — notice $T[2] \mapsto T'[2]$

pf: Apply Weyl reflections to make ϕ ef. $\star$.

• VI. Density of projective Kummer

X: exceptional $:=$ $\mathrm{Pic}(X)_{\mathbb{R}} = H^{1,1}_{\mathbb{R}}$ (maximal 20)
i.e. $H^{2,0} \oplus H^{0,2}$ is defined over $\mathbb{Q}$. $\Rightarrow$ get transcental lattice T_X has rk 2 and positive.

Prop: let $L \subset V$ be a positive rk 2 sub lattice st. $4 \mid x^2$ $\forall x \in L$, then $\exists$ exc. Kummer X, $\phi: H^2(X,\mathbb{Z}) \to V$ sending t_X isometrically to L.

pf: step 1:
Find exc. 2 tori T st. $H^2(T,\mathbb{Z}) \xrightarrow{\psi} H \oplus H \oplus H = \Lambda^2 \Gamma$ $\subset V$
induces an isometric embedding
$\psi: t_T \xrightarrow{\sim} (L, \frac{1}{2}\langle , \rangle)$. In fact,
$T := $ "$\mathbb{C}^2/\Gamma$" (see back side for details)
— under some embedding $\Gamma \hookrightarrow \mathbb{C}^2$.

$\Gamma = \bigoplus_{i=1}^{4} \mathbb{Z} e_i$
$(u,v) \mapsto \det(u \wedge v)$
$\cup$
L

step 2 : Let X = Kummer of T.

$$\begin{tikzcd} \sigma : H^2(T,\mathbb{Z}) \arrow[r] \arrow[d, "\cdot 4"] & H^2(X,\mathbb{Z}) \arrow[d, "S"] \\ L \subset 3H \arrow[r, leftrightarrow] & V = 3H \oplus (-2E_8) \end{tikzcd}$$

$\sigma(u).\sigma(v) = 2\, u.v$

+ may apply an auto (V) to make diagram comm. and fix. L.

Since $t_T \xleftrightarrow{\sigma} t_X$ (T exc.)

(in fact $\sigma H^{2,0}(T) = H^{2,0}(X)$ since a holo 2-form is σ-inv)

$\sigma(u).\sigma(v) = 2u.v \Rightarrow 4\circ\sigma^{-1}(t_X) = L$. □

Prop (1). the set of all $\mathbb{Q}$-defined 2 planes $P \subset V_{\mathbb{R}}$ with $4 \mid x^2$ $\forall x \in P\cap V$ is dense in $G(2, V_{\mathbb{R}})$ (2). given $x \in L$ with $4 \mid x^2$, $\{y \in L_{\mathbb{R}} \mid \mathbb{R}\langle x,y\rangle \cap L$ has rk 2, $4 \mid$ norm$\}$ dense in $L_{\mathbb{R}}$.

step 1 :

For a lattice $M \ni$ primitive e_0, let $e_0^2 \equiv m\ (n)$

then set of lines $l \subset M_{\mathbb{R}}$ gen. by primitive e with $e^2 \equiv m\ (n)$ is dense in $\mathbb{P}(M_{\mathbb{R}})$.

(pf is just Euclidean algorithm).

step 2 :

let $U \subset G(2, V_{\mathbb{R}})$ any open set. by step 1, may find $P' \in U$ $P' \ni$ primitive e_1, ~~$e_1^2 \equiv 4\ (8)$~~ say.

let $M = e_1^{\perp} \subset V$. Now $\exists$ Aut(V) st. $e_1 \mapsto \in H$.

$\Rightarrow M \supset$ sublattice $\cong 2H \oplus 2(-E_8)$

$\Rightarrow M \ni$ primitive vector of norm $64 = 2^6$ (which one?)

step 1. $\Rightarrow M \ni$ prim. e_2, $e_2^2 \equiv 0\ (64)$ (notice $e_2 \perp e_1$)

st. $\mathbb{R}\langle e_1, e_2\rangle \in U$. Claim: this is the required P.

ie. want $\forall f \in P\cap V$, $f^2 \in 4\mathbb{Z}$:

since $(e_1,e_1)f - (e_1,f)e_1 \in P\cap V$ is $\perp e_1$

$\Rightarrow (e_1,e_1)f = (e_1,f)e_1 + a e_2 \quad (a \in \mathbb{Z})$

$\Rightarrow (e_1,e_1)^2 f^2 = (e_1,f)^2(e_1,e_1) + a^2(e_2,e_2)$. $(*)$

Now $2 \mid f^2$, if $4 \nmid f^2$ then $2^3 \mid (*)$ but $2^4 \nmid (*)$.

but for RHS of $(*)$: $2 \mid (e_1,f) \Rightarrow 2^6 \mid (*)$
$2 \nmid (e_1,f) \Rightarrow 2^3 \nmid (*)$ $\Big\}$ all $\times$.

Theorem: The periods points of marked projective Kummer is dense in Ω. (in fact, special Kummer's are enough) □

Proof of Thm A: All Kähler K3's form a 20-dim'l irreducible families, in particular they are all are diffeomorphic and $\pi_1 = 1$.

pf: Let X be a K3, $\pi: \mathfrak{X} \to S$ be the Kuranishi family.

Fix a trivialization $R^2\pi_* \mathbb{Z}_{\mathfrak{X}} \xrightarrow[\phi]{\sim} V \times S$ gives a period map

$p: S \to \Omega$, variation of Hodge Structures $\Rightarrow$

$$dp_0 : T_0(S) \longrightarrow T_{p(0)}(\Omega) = \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X))$$

$T_0(S) = H^1(X, T_X) \cong H^{1,1}$, and $\mathrm{Hom}(H^{2,0}(X), H^{1,1}(X)) \cong H^{1,1}$ since $K = 0$.

i.e. p is a local isomorphism (Local Torelli thm).

Now all tori are diffeo, hence all Kummer are diffeo.

(e.g. (4) $\mathbb{CP}^3$ is simply connected)

to be refined later!

By ~~density~~, $p(S)$ contain period pt of ~~exc proj. Kummer~~

hence the diffeomorphic statement.

Now our ~~family~~ should be an open subset of $\Gamma \backslash \Omega$, irred. $\square$

Exercise: $\Omega =$ "$O(3,19)/SO(2) \times O(1,19)$" sym. space?

Proof of Thm B: (Torelli Thm): X, X' K3's with an effective Hodge isom. $\phi: H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X',\mathbb{Z})$, then $\phi = f^*$ for an unique isomorphism $f: X' \to X$.

(sketch):

$$\begin{array}{ccc} \mathfrak{X} & & \mathfrak{X}' \\ & \searrow \; \swarrow & \\ & S & \end{array}$$

same unit disk, 2 "different" period map

$p, p': S \to \Omega$, but for a dense set of exc. proj. Kummer period pts (defined purely alg'ly $\in [x^2]$), get isomorphic Kummer, (Apply Torelli for Kummer to $X_{s_i} \to \mathrm{Kummer}_{s_i}$ and to $X'_{s_i} \to \mathrm{Kummer}_{s_i}$ both). Now take limit $s_i \to 0$ ※

For the uniqueness, we claim that: $f: X \to X$ inducing identity on $H^2(X,\mathbb{Z}) \Rightarrow f = id_X$:

pf: $h^0(T) = h^{0,1} = 0 \Rightarrow$ no holo v.f. $\Rightarrow f$ is of finite order (bec. $\mathrm{Aut}(X)$ is cpt)

let $x \in \mathrm{Fix}(f)$ then f^* on T_x

has weight $(n_x, -n_x)$, $n_x \neq 0$ since $f^*\Omega = \Omega$ for Ω holo $(2,0)$ form.

$\Rightarrow x$ is an isolated fixed pt.

Now apply Lefschetz fixed pt thm:

$\# \, |\mathrm{Fix}(f)| = \sum (-1)^k \mathrm{tr}\, f^* : H^k(X;\mathbb{R}) = \chi(X) = 24$.

but since f is holo. and $\det(1-dg)_x \neq 0 \ \forall x \in \mathrm{Fix}(f)$

may also apply ~~holo fixed pt thm~~ (Atiyah-Bott)

$$\sum_{x \in \mathrm{Fix}(f)} \frac{1}{\det(1-dg)_x} = \sum (-1)^k \mathrm{tr} f^* : H^{k,0}(X) = 2.$$

but $\det(1-dg)_x = (1-\lambda)(1-\frac{1}{\lambda}) = 2-(\lambda+\frac{1}{\lambda}) \leq 4$

$0 <$ $\Rightarrow \det(1-dg)_x^{-1} \geq \frac{1}{4}$

(since g is holo.) $\Rightarrow \sum$ over 24 fixed pts $\geq 6 \neq 2$ $\times$. $\square$.

VII. Moduli Space of K3's and Universal families.

consider all Kuranishi families of K3's s.t. $\pi : \mathcal{X} \to U$, $X_0 \to 0$

1). All fiber X_s are Kähler ($s \in U$)

2). π is also the Kuranishi family of X_s ($s \in U$)

3). U is contractible : hence can find a marking $\phi : \pi_* \mathbb{Z}_{\mathcal{X}} \xrightarrow{\sim} V_U$ and hence a period map $p : U \to \Omega$

4). p is an embedding. $\Rightarrow$ no 2 $(X_s, \phi_s), (X_{s'}, \phi_{s'})$ $\cong$.

Define $\mathcal{M} := \coprod \left(\begin{array}{c} \text{marked K3 family} \\ \text{s.t. 1) - 4).} \end{array} \right) / \sim$ as marked K3.

gluing process

$\Rightarrow \mathcal{M}$ smooth (but not Hausdorff, non-separated) manifold of dim 20 with universal marked family $\mathfrak{X} \to \mathcal{M}$, give the universal marked period map

$$p : \mathcal{M} \longrightarrow \Omega.$$

Rmk: For fixed X, $\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(3H \oplus 2E_8) \hookrightarrow O(3,19)$
prev. thm. $\qquad$ Γ $\qquad$ discrete sub gp

for a period pt $s = p(X, \phi)$, $|p^{-1}(s)| = |\mathrm{Aut}(X)|$ which varies irregularly. Although will see p is surj. (Thm C) the actual moduli of K3 $\mathcal{M}' \xrightarrow{\sim} p \backslash \Omega$ is very bad! (why?)