Regular Conn. on Alg. Var. GAGA for D-Modules

$j: X \hookrightarrow V$, $E = V \setminus X$ a div.

$$\overset{\text{alg. mero. conn. } M \in Mod(\mathcal{D}_V) \text{ st. } M \underset{\mathcal{O}_V}{\simeq} M' \in Mod_c(\mathcal{O}_V[E])}{}\qquad \mathcal{O}_V[E] = j_*\mathcal{O}_X$$

$$Conn^{reg}(X) \subset Conn(V;E) \overset{☆}{\xrightarrow{\sim}} Conn(X) \hookrightarrow Mod_h(\mathcal{D}_X)$$

$\Downarrow$ ☆ $= j^{-1}$ with inverse j_*.

$M \in Mod_c(\mathcal{O}_X)$

Def'n: $\overset{\forall}{M}$ $\iff$ i^*M regular $\forall\, i: C \hookrightarrow X$ smooth curve.

Eg. $j: X \hookrightarrow \bar{X}$ a smooth "divisor completion" by Hironaka.

Get $(j_*M)^{an} := \mathcal{O}_{\bar{X}^{an}} \otimes_{\mathcal{O}_{\bar{X}}} j_*M \in Conn(\bar{X}^{an}, E^{an})$

Prop (Easy): $M \in Conn^{reg}(X) \iff (j_*M)^{an}$ regular for some j $\iff \forall j$.

Thm (Deligne): $M \mapsto M^{an}$ induces equiv of cat $*$ and $*'$:

$$\begin{array}{ccc} \text{"}Conn^{reg}(\bar{X};E)\text{"} & \xrightarrow{\ *'\ } & Conn^{reg}(\bar{X}^{an}; E^{an}) \\ \wr\downarrow & & \downarrow \text{. Deligne's RH} \\ Conn^{reg}(X) & \xrightarrow{\ *\ } & Conn(X^{an}) . \end{array}$$

Here $Conn^{reg}(\bar{X},E) \subset Conn(\bar{X},E)$ is defined to be sub-cat. st $M|_X$ reg.

Pf: Enough to prove $*'$ is an equiv. By prop, since reg conn on $\bar{X}^{an}$ is effective, enough to show $Conn(\bar{X},E) \xrightarrow{\sim} Conn^{e}(\bar{X}^{an}, E^{an})$. objects (M, ∇), $M \in \underline{Mod_c(\mathcal{O}_{\bar{X}}(E))}$

$$\cap \qquad\qquad \cap$$

GAGA $\Rightarrow Mod_c(\mathcal{O}_{\bar{X}}(E)) \xrightarrow{\sim} Mod_c^{e}(\mathcal{O}_{\bar{X}^{an}}(E^{an}))$.

$$\nabla \in Hom_{\mathbb{C}}(M, \Omega^1 \otimes_{\mathcal{O}_{\bar{X}}} M) \text{ st } \begin{cases} \nabla(\varphi s) = d\varphi \otimes s + \varphi \nabla s \\ [\nabla_\theta, \nabla_\varphi] = \nabla_{[\theta,\varphi]} . \end{cases}$$

The conditions are $\mathbb{C}$-linear, not $\mathcal{O}$-linear, can't apply GAGA.

Will rewrite them to get $\mathcal{O}$-linear morphism, then DONE!

The 2nd one is easy, since curvature

$$R = \nabla^2 \in Hom_{\mathcal{O}_{\bar{X}}}(M,M) \otimes \Omega^2_{\bar{X}} \text{, and it is just } R=0.$$

For the connection property, let $\Delta: \bar{X} \hookrightarrow \bar{X}\times\bar{X}$ be the diagonal with ideal sheaf J.

It is obvious that

$$N^*_\Delta(\bar{X}) \simeq J/J^2 \text{ and } \Delta^{-1}(J/J^2) \simeq \Omega^1_{\bar{X}} .$$

Exercise: $\nabla \longleftrightarrow \varphi \in \underbrace{Hom_{\mathcal{O}_{\bar{X}}}(\Delta^{-1}(\mathcal{O}_{\bar{X}\times\bar{X}}/J^2) \otimes_{\mathcal{O}_{\bar{X}}} M, \Omega^1_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} M)}_{\cong\, F_1(M,\, \Omega^1_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} M)}$ st. $\varphi|_{\Omega^1_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} M} = id$. #

where $F_p(M,N)$ is the $\mathbb{C}$-v.s. of diff op of order p.

Cor: For X sm alg, $Conn^{reg}(X) \simeq Conn(X^{an}) \simeq Loc(X^{an})$.

If X projective then $Conn(X) \simeq Conn(X^{an})$

Regular Holonomic D-modules, Alg. case

Theorem (Deligne's criterion): $j: X \hookrightarrow \bar{X}$, $E = \bar{X} \setminus X$ NCD, $\Theta_{\bar{X}}\langle E\rangle := \{\theta \in \Theta \mid \theta \mathcal{I}_E \subset \mathcal{I}_E\}$. and $\mathcal{D}_{\bar{X}}\langle E\rangle :=$ alg. gen by $\mathcal{O}_{\bar{X}}$ and $\Theta_{\bar{X}}\langle E\rangle$. $M \in \mathrm{Conn}(X)$. Then   i.e. $x_i \partial_i$ if $x_i|_E = 0$

(i) $M \in \mathrm{Conn}^{reg}(X) \Leftrightarrow$ (ii) $j_* M = \bigcup(\mathcal{O}_{\bar{X}}\text{-coh } \mathcal{D}_{\bar{X}}\langle E\rangle\text{-modules})$

$\Leftrightarrow$ (iii) curve testing $\forall p \in E_{sm}$, $i^* M$ reg at p for $\bar{C} \pitchfork E$ at p.

$$\begin{array}{ccc} C & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ \bar{C} & \hookrightarrow & \bar{X} \end{array}$$

Pf: (i) $\Rightarrow$ (iii) by def^{n}. (iii) $\Rightarrow$ condition (R) $\Rightarrow$ $(j_* M)^{an}$ reg $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): $N = (j_* M)^{an}$ reg along E^{an}. Hence for $\tau: \mathbb{C}/\mathbb{Z} \hookrightarrow \mathbb{C}$ fixed, $\exists$ loc. free $L_T / \mathcal{O}_{\bar{X}}$ st. $N \simeq \mathcal{O}_{\bar{X}}[E^{an}] \otimes_{\mathcal{O}_{\bar{X}^{an}}} L_T^{an} = \bigcup_{k \geq 0} \mathcal{O}_{\bar{X}^{an}}(-kE^{an}) \otimes_{\mathcal{O}_{\bar{X}^{an}}} L_T^{an}$. Now use GAGA.

(ii) $\Rightarrow$ (i): $N := (j_* M)^{an}$ satisfying (R) by restricting to disks B $\Rightarrow$ N reg $\Rightarrow$ M reg. $\#$

Def^{n}: $M \in \mathrm{Mod}_h(\mathcal{D}_X)$ is regular if every comp. factor $L \simeq L(Y, N)$ with $N \in \mathrm{Conn}^{reg}(Y)$.

get ab. sub. cat. $M_{rh}(\mathcal{D}_X)$ and $D^b_{rh}(\mathcal{D}_X) \ni M^\cdot \Leftrightarrow H^i(M^\cdot) \in M_{rh}(\mathcal{D}_X)$ $\forall i \in \mathbb{Z}$. (Δ-sub. cat.)

GOAL: To show that D^b_{rh} is preserved under "six operations".

Lemma: $D^b_{rh}(\mathcal{D}_X) \circlearrowleft \mathbb{D}_X$.

By induction on coh length of $M^\cdot$, may assume $M^\cdot = M \in \mathrm{Mod}_{rh}(\mathcal{D}_X)$

May also assume M simple, ie $M \cong L(Y, N)$, then $\mathbb{D}_X M \simeq L(Y, N^\vee) \subset j_* N^\vee$ is also reg. $\#$

Hence it suffice to show that $\int_f$ and $f^\dagger$ preserve D^b_{rh}. Two special case for $\int_f$:

- Normal crossing extension $j: X \hookrightarrow \bar{X}$. $\int_f = f_*$ is exact. Assume M simple, any comp. factor of $j_* M$ is of the form $L = L(Y, N)$. Claim: N reg. if $j^\dagger L \neq 0$ then $N \simeq M$ is reg. done. Hence assume $j^\dagger L = 0$. Since $j_* M = \bigcup_{k \geq 0} K_k$, with $F_p \mathcal{D}_{\bar{X}} \cdot K_k \subset K_{p+k}$, $K.$ good filt.

  $\Theta_{\bar{X}}\langle E\rangle$ acts by scaling on $\mathrm{gr}^K j_* M$

  $$\Rightarrow \mathrm{ch}(j_* M) = \text{union of } T^*_{E_{I,\alpha}} \bar{X} \quad (\text{using holonomicity}),$$

  where $E = \bigcup_{i=1}^r E_i$, $E_I = \bigcap_{i \in I} E_i = \bigcup E_{I,\alpha}$ irred. decomp.

Let $E_{I,\alpha}$ be a comp of $\mathrm{Supp}\, L$, $i: E^\circ_{I,\alpha} \longrightarrow \bar{X}$. Then Kashiwara's equivalence $\Rightarrow$ $\mathrm{ch}(i^\dagger L) = T^*_{E^\circ_{I,\alpha}} E^\circ_{I,\alpha}$ (zero section)

$\Rightarrow N := i^\dagger L \neq 0$ is an int. conn $\Rightarrow L = L(E^\circ_{I,\alpha}, N)$.

Now we use induction on codim $E^\circ_{I,\alpha}$ to prove regularity of N:

for $E_I = \{x_1 = 0\}$, $\quad i^\dagger L = H^0(i^\dagger L) = \ker(L \xrightarrow{x_1} L)$;

$L = \bigcup_i W_i \quad W_i$ stable under $\theta_i = x_i \partial_i$, $i = 1, \dots, r$

$\Rightarrow$ coherent $\mathcal{O}_{E_I}$ submodules $V_i := W_i \cap \mathrm{Ker}(x_1) \subset i^\dagger L$ stable under $\theta_2, \dots, \theta_r$.

Let $\Delta_{I,\alpha} = E_{I,\alpha} \setminus E^\circ_{I,\alpha} \hookrightarrow E_{I,\alpha}$ (as $X \hookrightarrow \bar{X}$), $E^\circ_{I,\alpha} \xrightarrow{i^\circ} E_{I,\alpha}$. Then

$$i^\circ_* N = \mathcal{O}_{E_{I,\alpha}}[\Delta_{I,\alpha}] \otimes_{\mathcal{O}_{E_{I,\alpha}}} i^\dagger L$$

$$= \bigcup_{i; k \geq 0} \left(\mathcal{O}_{E_{I,\alpha}}(-k\Delta_{I,\alpha}) \otimes_{\mathcal{O}_{E_{I,\alpha}}} V_i\right) \Rightarrow N \text{ is regular } \#$$