• Finiteness Structure

Lemma: $M \in Mod_c(D_X) \Rightarrow M|_U$ is projective$/O_U$ for some open dense $U \subset X$.

pf: let $F_\cdot$ be good for M. $gr^F M$ is coh.$/\pi_* O_{T^*X} \Rightarrow$ free over some open dense U $\Rightarrow$ free over O_U for some smaller U st $T^*X|_U$ trivial. $\Rightarrow F_i M|_U$ hence $M|_U$ proj ∗

Str. Thm. Let $M^\cdot \in D^b_c(D_X)$, TFAE: (i) $M^\cdot \in D^b_h(D_X)$,

(ii) $\exists\ X = X_0 \supset X_1 \supset \cdots \supset X_m \supset X_{m+1} = \emptyset$ by closed sets st. $V_r := X_r \setminus X_{r+1}$ is sm, and $H^k(i_r^\dagger M^\cdot) \in Conn(V_r)$ for $i_r : V_r \hookrightarrow X$.

(iii) $\forall\ i_x : \{x\} \hookrightarrow X$, $H^k(i_x^\dagger M^\cdot)$ is f.d.$/\mathbb{C}$.

pf: (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are easy. Will show (iii) $\Rightarrow$ (ii)* in the stronger form that (ii) holds for any closed $Y \subset X$ st. $Y \supset Supp\, M^\cdot := \bigcup_k supp\, H^k M^\cdot$, and induction on $\dim Y$.

Lemma $\Rightarrow \exists$ open dense $V \subset Y$, with $i : V \hookrightarrow X$, st. $i^\dagger M^\cdot \in D^b_c(D_V)$ and $H^k(i^\dagger M^\cdot)$ is proj$/O_V$

claim: $H^k(i^\dagger M^\cdot) \in Conn(V)$. Indeed, for $j_x : \{x\} \to V$

$$H^k(i^\dagger M^\cdot)_x \otimes_{O_{V,x}} \mathbb{C} \simeq H^{k+d_V}(j_x^\dagger i^\dagger M^\cdot) \simeq H^{k+dV}(i_x^\dagger M^\cdot) \text{ is finite-dim'l by (iii).}$$

projective (why?) $\Rightarrow rk\, H^k(i^\dagger M^\cdot) < \infty \Rightarrow$ coh$/O_V \Rightarrow \in Conn(V)$ ∗

Now let $U \underset{j}{\subset} X$ open st $V = Y \cap U$. Define $N^\cdot$ by dist. $\triangle$:

$$N^\cdot \longrightarrow M^\cdot \longrightarrow \int_j j^\dagger M^\cdot \xrightarrow{+1} \quad \Rightarrow \int_j j^\dagger M^\cdot = \int_i i^\dagger M^\cdot \in D^b_h(D_X)$$

e.g. Kashiwara on k

$i = j \circ k$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "k"] & Y \subset X \\ U \arrow[ur, "j"'] & \end{tikzcd}$$

$\Rightarrow N^\cdot \in D^b_h(D_X)$ with $supp(N^\cdot) \subset Y \setminus V =: Y_1$

Then (ii) follows by descending induction on $\dim Y$. ∗

• Simple Objects and Minimal Extensions

For $M \in Mod_h(D_X)$, had seen $l_{D_X}(M) < \infty$, i.e. $\exists\ M = M_0 \supset \cdots \supset M_{r+1} = 0$ st. M_i/M_{i+1} simple.

To classify all simple holonomic modules, consider $Y \overset{i}{\hookrightarrow} X$ affine, loc. closed, with Y smooth.

Then for $M \in Mod_h(D_Y)$, $i_! M = H^0 i_! M \longrightarrow \int_i M = \int_i^0 M$ in $Mod_h(D_X)$.

Def'n: The image $i_{!*} = L(Y,M) \in Mod_h(D_X)$ is called the minimal extension.

Thm: If M is simple then $i_{!*}M$ is also simple, and it is the unique simple sub D-mod of $\int_i M$, as well as the unique simple quotient D-mod of $i_! M$.

pf: Factorize $i = j \circ k$: $Y \overset{k}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$ st. k is closed, j open.

(a) $E \in Mod^{\bar{Y}}_{qc}(D_X) \Rightarrow H^l i^\dagger E = 0$ for $l \neq 0 \Rightarrow i^\natural = H^0 i^\dagger : Mod^{\bar{Y}}_{qc}(D_X) \longrightarrow Mod_{qc}(D_Y)$ exact ∗

(b) $0 \neq N \hookrightarrow \int_i M$ sub D-mod $\Rightarrow i^\dagger N \simeq M$.

Indeed, $j^\dagger N \to j^\dagger \int_j \int_k M \longrightarrow \int_k M \in Mod^{Y}_h(D_U)$ is simple (K.) $\Rightarrow j^\dagger N \twoheadrightarrow \int_k M \Rightarrow i^\dagger N \twoheadrightarrow k^\dagger \int_k M = M$

by (a), $i^\dagger$ is exact $\Rightarrow i^\dagger N \hookrightarrow i^\dagger \int_i M = M$, hence $i^\dagger N \simeq M$ ∗

(c) $\int_i M$ has a unique simple sub-mod L. Otherwise $L, L' \subset \int_i M$, let $N = L + L' = L \oplus L'$.

(b) $\Rightarrow M = i^\dagger N = i^\dagger L \oplus i^\dagger L' \simeq M \oplus M$ ✗ $i_! M$ has unique simple quot. by duality ∗

$$Hom_{D_X}(i_! M, L) \simeq Hom_{D_Y}(M, i^\dagger L) \simeq Hom_{D_Y}(M, M) \simeq Hom_{D_Y}(M, i^\dagger \int_i M) \simeq Hom_{D_X}(i_! M, \int_i M)$$

$\Rightarrow i_! M \longrightarrow \int_i M$ factors through $i_! M \to L \to \int_i M$. Done ∗

Cor: Any simple $L \in Mod_h(D_X)$ comes from $L = i_{!*} M$ with $M \in Conn(Y)$, $i : Y \hookrightarrow X$ affine sm. locally closed.

$L(Y,M) \simeq L(Y',M') \Leftrightarrow \bar{Y} = \bar{Y}'$, $M|_U \simeq M'|_U$ for U open dense. $\mathbb{D}\, i_{!*} M \simeq i_{!*} \mathbb{D} M$. (Ex.)

Analytic D-Modules

X complex manifold, we mark only the "difference" from the alg. theory.

$\mathrm{gr}\, \mathcal{D}_X$ is a sheaf of comm. alg $/ \mathcal{O}_X$, now only a subalg of $\pi_* \mathcal{O}_{T^*X}$.

Thm: (1) $\mathcal{D}_X$ is a coherent sheaf of rings (based on Oka's thm on $\mathcal{O}_X$),

(2) $\forall x \in X$, $\mathcal{D}_{X,x}$ is Noetherian with left and right global dim $= d_X$.

- Good filtration $F_\cdot$ on $M \in \mathrm{Mod}_c(\mathcal{D}_X)$ only exists locally.

$\Rightarrow$ $Ch(M)$ still defined $= \bigcup Ch(M|_U) \subset T^*X$, and is involutive. $M \in \mathrm{Mod}_h(\mathcal{D}_X) \Leftrightarrow \dim Ch(M) = d_X$

- $D^b_c(\mathcal{D}_X)$, $D^b_h(\mathcal{D}_X)$ are defined st. $H^i(M^\cdot)$ are coh $/ \mathcal{D}_X$, holonomic resp. $(\forall i)$

Operations $\mathcal{H}om$, $\boxtimes$, $\otimes$, $\mathbb{D}$, Lf^*, $\int^+$ defined (via Thm-(2) to get boundedness) on D^b

- $M \in \mathrm{Mod}_c(\mathcal{D}_Y)$, $f: X \to Y$ non-char $/ M$ $\Rightarrow$ $Lf^* M = f^* M \in \mathrm{Mod}_c(\mathcal{D}_X)$, $\mathbb{D}_X f^* M \xrightarrow{\sim} f^* \mathbb{D}_Y M$.

$\int_f$ also defined on D^b (via $\int_f M = Rf_*(DR_{X/Y}(M))$ when $f: X = Y \times Z \to Y$

and then use $R^i f_* K = 0$ if $i \notin [0, 2d_Z]$ to get boundness)

- $M \in \mathrm{Mod}_c(\mathcal{D}_X)$, $f: X \to Y$ proper, if M has a good $F_\cdot$ locally on Y,

$\Rightarrow$ $\int_f M \in D^b_c(\mathcal{D}_Y)$, $\int_f \mathbb{D}_X M \xrightarrow{\sim} \mathbb{D}_Y \int_f M$. (Kashiwara, pf is non-trivial)

Thm (for Mod_h): (1) $f: X \to Y$, $M \in \mathrm{Mod}_h(\mathcal{D}_Y)$ $\Rightarrow$ $Lf^* M \in D^b_h(\mathcal{D}_X)$. (Need b-functions)

(2) $f: X \to Y$ proper, $M \in \mathrm{Mod}_h(\mathcal{D}_Y)$ with good $F_\cdot$ loc. on Y $\Rightarrow$ $\int_f M \in D^b_h(\mathcal{D}_Y)$.

Rmk: In alg setting no conditions are needed in (2). They are needed in analytic setting!

Example: $X = \mathbb{C}^\times \underset{j}{\hookrightarrow} \mathbb{C} = Y$. Then $j_* \mathcal{O}_X = \mathcal{O}_Y[x^{-1}]$ is holonomic. (alg setting)

But $j^{an}_* \mathcal{O}_{X^{an}}$ is much larger, say $\ni e^{1/x}$. It is not even a coherent $\mathcal{D}_{Y^{an}}$-module.

- Kashiwara's equivalence still holds for closed $i: X \hookrightarrow Y$. $\mathrm{Mod}_\#(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}^X_\#(\mathcal{D}_Y)$; $\# = c, h$.

de Rham functor / solution complex, if $M^\cdot = M$ at 0

$$M^\cdot \in D^b(\mathcal{D}_X), \quad DR_X M^\cdot := \omega_X \otimes^L_{\mathcal{D}_X} M^\cdot \simeq [\Omega^0_X \otimes_{\mathcal{O}_X} M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{d_X}_X \otimes_{\mathcal{O}_X} M] : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X),$$

$$\mathrm{Sol}_X M^\cdot := R\mathcal{H}om_{\mathcal{D}_X}(M; \mathcal{O}_X) : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X)^{op}.$$

Prop: $DR_X(M^\cdot) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M^\cdot)[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M^\cdot)[d_X]$.

loc. const. $\mathbb{C}$-mod.

Thm: Let $M \in \mathrm{Conn}(X)$. (i) $H^i DR_X M = 0$ if $i \neq -d_X$, and (ii) $H^{-d_X} DR_X(\cdot): \mathrm{Conn}(X) \xrightarrow{\sim} \mathrm{Loc}(X)$.

pf: (i) by holomorphic Poincaré lemma. (ii) Frob. thm $\Rightarrow$ flat sections has full rk $\divideontimes$

Example: for $f: X \to Y$ hol. map of cpx mfd, $Rf_*(DR_X M^\cdot) \simeq DR_Y(\int_f M^\cdot)$ for $M^\cdot \in D^b(\mathcal{D}_X)$.

Thm (Cauchy-Kowalevski-Kashiwara) If f is non-char $/ M \in \mathrm{Mod}_c(\mathcal{D}_X)$, then

$$f^{-1} \mathrm{Sol}_X M \xrightarrow{\sim} \mathrm{Sol}_Y Lf^* M.$$

pf: May reduce to $Y \hookrightarrow X$ a hyp. surface ($X \subset \mathbb{C}^n$ open) by $z_1 = 0$. As in the alg. case:

$0 \to K \to L = \bigoplus_{i=1}^r \mathcal{D}_X / \mathcal{D}_X P_i \longrightarrow M \to 0$ and Y is non-char $/ P_i$. $\quad Lf^* L = f^* L$

Thm holds for $L = \mathcal{D}_X / \mathcal{D}_X P$: $f^{-1} \mathcal{H}om_{\mathcal{D}_X}(L, \mathcal{O}_X) \cong \{ u \in \mathcal{O}_X|_Y : Pu = 0 \} \xrightarrow{\sim} \mathcal{O}_Y^{\oplus m} \simeq \mathcal{H}om_{\mathcal{D}_Y}(\overset{\shortparallel}{L_Y}, \mathcal{O}_Y)$

via $u \mapsto (u|_Y, \partial_1 u|_Y, \cdots, \partial_1^m u|_Y)$.

$$\begin{array}{ccccccccccc} \Rightarrow & 0 & \to & f^{-1}\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}) & \to & f^{-1}\mathcal{H}om_{\mathcal{D}_X}(L, \mathcal{O}) & \to & f^{-1}\mathcal{H}om_{\mathcal{D}_X}(K, \mathcal{O}) & \to & f^{-1}\mathcal{E}xt^1_{\mathcal{D}_X}(M, \mathcal{O}) & \to \cdots \\ & & & A \downarrow & & S \downarrow & & B \downarrow & & A' \downarrow & \quad S \downarrow \\ & 0 & \to & \mathcal{H}om_{\mathcal{D}_Y}(M_Y, \mathcal{O}) & \to & \mathcal{H}om_{\mathcal{D}_Y}(L_Y, \mathcal{O}) & \to & \mathcal{H}om_{\mathcal{D}_Y}(K_Y, \mathcal{O}) & \to & \mathcal{E}xt^1_{\mathcal{D}_Y}(M_Y, \mathcal{O}) & \to \cdots \end{array}$$

$\Rightarrow$ A inj. $\forall M$, $\Rightarrow$ B inj. $\Rightarrow$ A isom. $\Rightarrow$ repeat get A' isom $\Rightarrow \cdots$ get complete pf $\divideontimes$

Cor. Under the assumption, by duality $\Rightarrow$ $DR_Y(Lf^* M) \simeq f^{-1} DR_X(M)[d_Y - d_X]$ in $D^b(\mathbb{C}_X)$.