Duality.

Def'n: $\mathbb{D} : D^-(\mathcal{D}_X) \longrightarrow D^+(\mathcal{D}_X)^{op}$ by (let $\dim X = d_X = d$)

$$\mathbb{D}M^\cdot = R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee [d] = R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee [d]).$$

Example: $X = \mathbb{C}$, $M = D/DP$. Apply $\mathcal{H}om(\cdot, D)$ to $0 \to P \xrightarrow{\cdot P} D \to M \to 0$ get

$$0 \to \mathcal{H}om_D(M, D) \to D \xrightarrow{P\cdot} D \to \mathcal{E}xt^1_D(M, D) \to 0$$

$\mathcal{H}om_D(M,D) = 0$ since $\operatorname{Ker} P\cdot = 0$; $\mathcal{E}xt^1_D(M,D) = D/PD$ right D-module.

Use side-changing to get left D-module: $\mathcal{E}xt^1_D(M, D) \otimes_{\mathcal{O}} \omega^\vee \simeq D/DP^*$.

In dim $d \geq 1$, will see this characterize "holonomic" modules.

Example: $\mathbb{D}\mathcal{D}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee [d]$.

Lemma: For $U \subset X$ open affine, $\mathcal{E}xt^i_{\mathcal{D}_X}(M, D)(U) = \operatorname{Ext}^i_{\mathcal{D}_X(U)}(M(U), \mathcal{D}_X(U))$. (easy).

Prop: (i) $\mathbb{D} : D^b_c(\mathcal{D}_X) \to D^b_c(\mathcal{D}_X)^{op}$ and (ii) $\mathbb{D}^2 = id$.

pf: (i) follows from lemma & left/right Noetherian, left/right global dim of $\mathcal{D}_X(U) \leq \dim T^*X$, passing to gr.

(ii) $\exists$ canonical morphism $M^\cdot \to \mathbb{D}^2 M^\cdot$. By restricting to affine X, (cf. Appendix D) then it suffices to consider $M^\cdot = \mathcal{D}_X$ since $M^\cdot \simeq N^\cdot \in D^b_c(\mathcal{D}_X)$ with $N^i \simeq \mathcal{D}_X^{n_i}$.

And then the result follows by the above example ✱ (Ex. why?)

Thm (App. D): $M \in \operatorname{Mod}_c(\mathcal{D}_X) \Rightarrow \operatorname{codim}_{T^*X} \operatorname{ch}(H^i(\mathbb{D}M)) \geq d + i$

and $H^i(\mathbb{D}M) = 0$ if $i > 0$ or $i < -\left(d - \operatorname{codim}_{T^*X} \operatorname{ch}(M)\right)$.

Cor: $M \in \operatorname{Mod}_h(\mathcal{D}_X)$, i.e. holonomic $\Leftrightarrow H^i(\mathbb{D}M) = 0 \ \forall i \neq 0$, Also then $\mathbb{D}M = H^0(\mathbb{D}M) \in \operatorname{Mod}_h(\mathcal{D}_X)$

pf: $\Leftarrow$: Set $M^* = H^0(\mathbb{D}M)$, then $\mathbb{D}M^* = \mathbb{D}^2 M \simeq M$ and. $H^0(\mathbb{D}M^*) \simeq H^0(M) \simeq M$

Thm $\Rightarrow \operatorname{codim}_{T^*X} \operatorname{ch}(H^0(\mathbb{D}M^*)) \geq d$, hence $M \simeq \mathbb{D}M^* \in \operatorname{Mod}_h(\mathcal{D}_X)$ ✱

Example/Prop: $M \in \operatorname{Conn}(X) \Rightarrow \mathbb{D}M \simeq \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X) = M^\vee \in \operatorname{Conn}(X)$.

pf: Consider (the Spencer resol of $\mathcal{O}_X$) $\otimes_{\mathcal{O}_X} M$ get loc. free resol of M:

$0 \to D \otimes_{\mathcal{O}} \Lambda^d \Theta \otimes_{\mathcal{O}} M \to \cdots \to D \otimes_{\mathcal{O}} M \to M \to 0$. Hence only have coker of

$$\mathcal{H}om_D(D \otimes_{\mathcal{O}} \Lambda^{d-1}\Theta \otimes_{\mathcal{O}} M, D) \otimes \omega^\vee \to \mathcal{H}om_D(D \otimes_{\mathcal{O}} \Lambda^d \Theta \otimes_{\mathcal{O}} M, D) \otimes \omega^\vee$$

$$\simeq \quad \mathcal{H}om_{\mathcal{O}}(M, \Omega^{d-1} \otimes_{\mathcal{O}} D) \otimes \omega^\vee \to \mathcal{H}om_{\mathcal{O}}(M, \Omega^d \otimes_{\mathcal{O}} D) \otimes \omega^\vee \xrightarrow{\star} \mathcal{H}om_{\mathcal{O}}(M, \omega) \otimes \omega^\vee \to 0$$

Reason for $\star$: M is loc. free, use 2nd Spencer resol. for ω. $\quad \simeq \mathcal{H}om_{\mathcal{O}}(M, \mathcal{O})$ ✱

Rmk: $\operatorname{ch}(M) = \bigcup_{0 \leq i \leq d} \operatorname{ch}(H^i(\mathbb{D}M)) =: \operatorname{ch}(\mathbb{D}M)$, though $\mathbb{D}M$ is a D-mod only if it is holonomic.

Prop: For $M^\cdot \in D^b_c$, $N^\cdot \in D^b$, $R\mathcal{H}om_D(M^\cdot, N^\cdot) \simeq (\omega \otimes^L_{\mathcal{O}} \mathbb{D}M^\cdot) \otimes^L_D N^\cdot [-d] \simeq R\mathcal{H}om_D(\mathcal{O}, \mathbb{D}M^\cdot \otimes^L_{\mathcal{O}} N^\cdot)$

pf: We prove the case $M^\cdot = \mathcal{O}_X$ first: It suffices to prove the case $N^\cdot = \mathcal{D}_X$ in $D^b(\mathcal{O}_X)$.

By Spencer resol of $\mathcal{O}_X$, get $R\mathcal{H}om_D(\mathcal{O}, D) \simeq$ (Spencer resol for) $\omega_X[-d]$ ok.

for $M^\cdot \in D^b_c(\mathcal{D}_X)$, $R\mathcal{H}om_D(M^\cdot, N^\cdot) \simeq R\mathcal{H}om_D(M^\cdot, D) \otimes^L_D N^\cdot \simeq (\omega[-d] \otimes_{\mathcal{O}} \mathbb{D}M^\cdot) \otimes^L_D N^\cdot$

$\simeq \omega[-d] \otimes^L_D (\mathbb{D}M^\cdot \otimes^L_{\mathcal{O}} N^\cdot) \simeq R\mathcal{H}om_D(\mathcal{O}, D) \otimes^L_D (\cdots) \simeq R\mathcal{H}om_D(\mathcal{O}, \mathbb{D}M \otimes^L_{\mathcal{O}} N^\cdot)$ ✱

Cor. Apply $R\Gamma(X, \cdot)$ get $R\operatorname{Hom}_D(M^\cdot, N^\cdot) \simeq R\operatorname{Hom}_D(\mathcal{O}, \mathbb{D}M^\cdot \otimes^L_{\mathcal{O}} N) \simeq \int_p \mathbb{D}M^\cdot \otimes^L_{\mathcal{O}} N^\cdot$ for $p: X \to pt$.

Duality and pullbacks (non-characteristic)

Thm: $f: X \to Y$, $M \in \mathrm{Mod}_c(\mathcal{D}_Y)$. (i) if $Lf^*M \in \mathrm{Mod}_c(\mathcal{D}_X)$ then $\exists\ \mathbb{D}_X(Lf^*M) \xrightarrow{\psi} Lf^*(\mathbb{D}_Y M)$.

(ii) If f is non-char$/M$, then ($Lf^*M = f^*M$ is coh$/\mathcal{D}_X$ and) ψ is isom.

pf: (i) $R\mathcal{H}om_{\mathcal{D}_Y}(M, M) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathbb{D}M \otimes^L_{\mathcal{O}_Y} M) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(Lf^*\mathcal{O}_Y, Lf^*\mathbb{D}M \otimes^L_{\mathcal{O}_X} Lf^*M)$

$\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}Lf^*M, Lf^*\mathbb{D}M)$ get $\mathrm{id}_M \longmapsto \psi$. $\quad f^*\mathcal{O}_Y = \mathcal{O}_X \quad \mathbb{D}(\mathbb{D}Lf^*M)$

(ii) Factorize $f: X \to X \times Y \longrightarrow Y$, may assume f closed imbedding or projection.

- $f: X = T \times Y \to Y$, Y affine. May assume $M = \mathcal{D}_Y$ (why?). Then
  $\mathbb{D}_X Lf^*\mathcal{D}_Y = \mathbb{D}_X(\mathcal{O}_T \boxtimes \mathcal{D}_Y) \simeq \mathcal{O}_T \boxtimes [\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee[d_Y]] \simeq Lf^*\mathbb{D}_Y\mathcal{D}_Y$.
- $f: X \hookrightarrow Y$ closed. May assume codim $= 1$.
  Also it suffices to verify for submodules of the form $M' = \mathcal{D}_Y/\mathcal{D}_Y P$ st. f is non-char$/M'$. i.e. in local coor, may assume $P = \partial_1^m + \sum_{i=0}^{m-1} P_i \partial_1^i$
  $\Rightarrow \mathbb{D}_Y M' \cong \mathcal{D}_Y/\mathcal{D}_Y P^*[-1][d_Y] = \mathcal{D}_Y/\mathcal{D}_Y P^*[d_X]$. Notice that $Lf^*M' \simeq \mathcal{D}_X^{\oplus m}$ (assume Y smaller)
  $\Rightarrow \mathbb{D}_X Lf^*M' \simeq \mathbb{D}_X(\mathcal{D}_X^{\oplus m}) \simeq \mathcal{D}_X^{\oplus m} \otimes_{\mathcal{O}_X} \omega_X^\vee[d_X]$
  similarly $Lf^*\mathbb{D}_Y M' \simeq \mathcal{D}_X^{\oplus m} \otimes_{\mathcal{O}_X} \omega_X^\vee[d_X]$. (Though the isom is up to a sign $(-1)^m$) #

Duality and pushforwards (proper)

Thm: Let $f: X \to Y$ be proper, then $\exists$ canonical $\mathrm{Tr}_f: \int_f \mathcal{O}_X[d_X] \longrightarrow \mathcal{O}_Y[d_Y]$ which induces $\int_f \mathbb{D}_X \simeq \mathbb{D}_Y \int_f : D^b_c(\mathcal{D}_X) \longrightarrow D^b_c(\mathcal{D}_Y)$.

pf: for closed $i: X \hookrightarrow Y$, Tr_i is just given by $\int_i i^\dagger \mathcal{O}_Y = \int_i i^*\mathcal{O}_Y[d_X - d_Y] \longrightarrow \mathcal{O}_Y$. ($i^*\mathcal{O}_Y = \mathcal{O}_X$)

for projection $X = \mathbb{P}^n \times Y \xrightarrow{p} Y$, $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_Y$, reduce to the case $Y = pt$.

$$\int_p \mathcal{O}_{\mathbb{P}^n} = R\Gamma(\mathbb{P}^n, \underset{\Omega^n_{\mathbb{P}^n}}{\underset{\shortparallel}{\mathcal{D}_{pt \leftarrow \mathbb{P}^n}}} \otimes^L_{\mathcal{D}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}) = R\Gamma(\mathbb{P}^n, [\underset{\uparrow\ \text{degree 0 part}}{\mathcal{O}_{\mathbb{P}^n}} \to \Omega^1_{\mathbb{P}^n} \to \cdots \to \Omega^n_{\mathbb{P}^n}])$$

by Spencer resolution I.

$\Rightarrow \int_p \mathcal{O}_{\mathbb{P}^n}[n] \longrightarrow H^n(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}) \simeq \mathbb{C} = \mathcal{O}_{pt}$. (why in H^n?) #

Rmk: Tr_f is indep of the decomp $f = p \circ i$. Also $\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f$. (proof?)

Now for $M^\cdot \in D^b_c(\mathcal{D}_X)$, $\int_f \mathbb{D}_X M^\cdot \stackrel{\Delta}{=} Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes^L_{\mathcal{D}_X} R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X) \otimes^L_{\mathcal{O}_X} \omega_X^\vee[d_X])$

want to construct $\Phi(M^\cdot)$: ① $\to$ ② $\quad \stackrel{①}{\simeq} Rf_*(R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_{X \to Y})) \otimes^L_{\mathcal{O}_Y} \omega_Y^\vee[d_X]$.

$\longrightarrow \mathbb{D}_Y \int_f M^\cdot \stackrel{\Delta}{=} ② \quad R\mathcal{H}om_{\mathcal{D}_Y}(\int_f M^\cdot, \mathcal{D}_Y) \otimes^L_{\mathcal{O}_Y} \omega_Y^\vee[d_Y]$.

since $\int_f \mathcal{D}_{X \to Y}[d_X] \stackrel{\Delta}{=} \int_f f^*\mathcal{D}_Y[d_X] \simeq (\int_f \mathcal{O}_X[d_X]) \otimes^L_{\mathcal{O}_Y} \mathcal{D}_Y$ by the proj. formula.

get $\int_f \mathcal{D}_{X \to Y}[d_X] \longrightarrow \mathcal{D}_Y[d_Y]$ via Tr_f.

$\Phi(M^\cdot)$ is then defined as ① $\to Rf_* R\mathcal{H}om_{f^{-1}(\mathcal{D}_Y)}(\mathcal{D}_{Y \leftarrow X} \otimes^L_{\mathcal{D}_X} M^\cdot, \mathcal{D}_{Y \leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X \to Y}[d_X])$

$\xrightarrow{?} R\mathcal{H}om_{\mathcal{D}_Y}(\int_f M^\cdot, \int_f \mathcal{D}_{X \to Y}[d_X]) \longrightarrow$ ②.

<sketch on "$\Phi(M^\cdot)$ is an isom"> again assume f is an embedding or projection.

As before, locally $\exists$ resol $M^\cdot \simeq N^\cdot \in D^b_c(\mathcal{D}_X)$ st $N^j \simeq \mathcal{D}_X^{n_j}$ $\Rightarrow$ may assume $M^\cdot = \mathcal{D}_X$.

For $f = i$, use Kashiwara equiv. For $f = p$, explicit calculation on $H^*(\mathbb{P}^n, \omega_{\mathbb{P}^n})$. #

Cor. (Adjunction Formula). $f: X \to Y$ proper, $R\mathcal{H}om_{\mathcal{D}_Y}(\int_f M^\cdot, N^\cdot) \simeq Rf_* R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, f^\dagger N^\cdot)$.

where $M^\cdot \in D^b_c(\mathcal{D}_X)$, $N^\cdot \in D^b(\mathcal{D}_Y)$.

Rmk: without $M^\cdot \in D^b_c$ only ok for $f = i$.