GTFT II 5/30, More on BCOV, Quintics.

Quintic Mirror (Greene-Plesser)

V $h^{11}=1$ generic quintic in P^4

V° $h^{21}=1$ 1-d. moduli = res'd of $f(\psi)$: $(x_1^5+\cdots+x_5^5-\psi x_1\cdots x_5)\subset P^4/G$

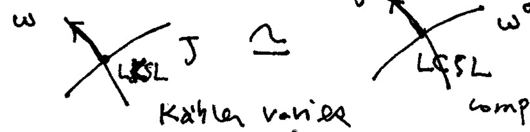
$G=\{(a_1\cdots a_5)\in \mathbb{Z}_5^5,\ \sum a_i\equiv 0\ (5)\}/\mathbb{Z}_5$ $\quad |G|=5^3$ ab gp.

V°_ψ sm except $\psi=\infty$, $\psi=5\mu^i$ $\quad \psi_1^5=\psi_2^5 \Leftrightarrow$ same moduli

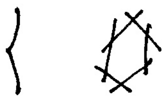
LCSL $\quad$ CONIFOLD

$z=0$ $\quad z=5^{-5}$ $\quad$ inv. coor: or $z:=\psi^{-5}$

MS: A: ω, J $\simeq$ B: J°, ω°

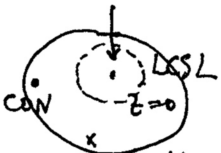


Kähler variee $\quad$ complex variee

Local coor:  q $\xrightarrow{\text{mirror map}}$ z ? 

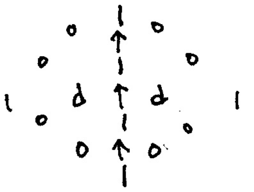
monodromy: $L:=H\cup \curvearrowright \bigoplus_{p=0}^{3} H^{p,p}(V)$ $\quad$ T: q-unipotent

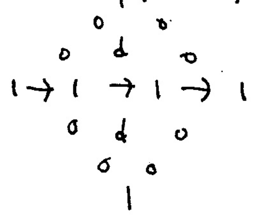
$N=\log T^r$ (or T^r-1)



$L^3\neq 0,\ L^4=0$

acts on $H^3(V^\circ,\mathbb{C})$: KS map





limiting MHS

$\longrightarrow$ pick $z=0$

3pt function:

$\langle H,H,H\rangle = 5+\sum_{d=1}^{\infty} n_d\, d^3 \frac{q^d}{1-q^d}$ $\longleftrightarrow$ Yukawa coupling F_{ijk}, now $= C_{zzz}$

$= K_{ttt}$ now $\quad$? Rmk: for $g\geq 1$ Gopakumar-Vafa inv.

MCF ~~contribution~~ $\quad T\gamma_0=\gamma_0$

" $n_d\in\mathbb{N}$ " ? $\quad T\gamma_1=\gamma_1+\gamma_0$

$q=e^{2\pi i t} \longmapsto e^{2\pi i \frac{\int_{\gamma_1}\Omega_\psi}{\int_{\gamma_0}\Omega_\psi}}$ $\quad$ eg. $\Omega_\psi = \mathrm{Res}\frac{\psi\,\Omega_{P^4}}{f(\psi)}$

PF eq'n: $\quad \delta^4-5^5 z\left(\delta+\frac{1}{5}\right)\cdots\left(\delta+\frac{4}{5}\right)=0$ $\quad e^{2\pi i\left(\frac{x_1}{x_0}\right)} \to z$

$\delta = z\frac{\partial}{\partial z}$

$Y := z^3 C_{zzz} = -\int_X \Omega\wedge\delta^3\Omega$

$\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5} = 2$

$\delta Y = -\int\delta\Omega\wedge\delta^3\Omega - \int\Omega\wedge\delta^4\Omega$

$= -\delta\int\delta\Omega\wedge\delta^2\Omega + \underbrace{\cancel{\int\delta^2\Omega\wedge\delta^2\Omega}}_{=0\ \text{(same odd form)}} - \int\Omega\wedge\frac{2\cdot 5^5 z}{1-5^5z}\delta^3\Omega + \underbrace{\cancel{\cdots}}_{=0}$

$= -\delta\Big(\delta\underbrace{\cancel{\int\Omega\wedge\delta^2\Omega}}_{=0} - \int\Omega\wedge\delta^3\Omega\Big) - \frac{2\,5^5 z}{1-5^5z}Y$

$= -\delta Y - \frac{2\cdot 5^5 z}{1-5^5 z}Y \Rightarrow Y=\frac{c}{1-5^5 z} \Rightarrow C_{zzz}=\frac{c}{z^3(1-5^5z)}$

H unit vector $=\frac{d}{dt} = 2\pi i\, q\frac{d}{dq} \longrightarrow \quad \upsilon = 2\pi i\, q\frac{dz}{dq}\frac{d}{dz} \in H^1(V^\circ, T_{V^\circ})$

$\langle H,H,H\rangle = \left(2\pi i\,\frac{q}{z}\frac{dz}{dq}\right)^3 \delta^3 F = \frac{(2\pi i)^3 c\,(1+770z+\cdots)}{\left(1+5^5\frac{q}{c_1}+\cdots\right)\left(1-240\frac{q}{c_1}+\cdots\right)}$ $\Rightarrow c=\frac{5}{(2\pi i)^3}$

$c_1=1,\ n_1=2875.$

Rmk: MS and GW for <u>quintic</u> (from CLLL String-Math 2015)

$\sum x_i^5 - 5\psi\pi x_i = 0$

$$F_g^{A(X)}(q(t)) = F_g^{B(\check{X})}(t) \qquad t \mapsto q(t) \text{ mirror map}$$

Math: <u>g=0</u> LLY, Givental (1993)

phys: ~~BCOV (g ≥ 1)~~ : explicit recursion $g=1,2$

Katz-Klemm-Vafa $g=3,4$ (1999)

Huang-Klemm-Quackenbush (2007), $\forall g \leq 51$.

by Yamaguchi-Yau & HKQ App III

- <u>holomorphic anomaly eq'ns</u> (BCOV) det. up to $3g-2$ unknowns
  - <u>const map</u> $N_{g,d=0}$ is known (both in A, B) $\Rightarrow 3g-3$ ($g=1$, conifold not needed!)
- <u>Boundary</u> conditions at <u>orbifold pts</u> ($\leftrightarrow$ LG) $\psi=0$. Use global condition $\psi=0$
  - impose $\lceil \frac{3}{5}(g-1) \rceil$ constraints App. I $\qquad \lfloor \frac{2}{5}(g-1) \rfloor$
- <u>Gap condition</u> at <u>conifold pts</u> $\psi=1$ imposes $2g-2$ so $g = 20$ OK.
- Gopakumar-Vafa conj ("$\mathbb{Z}$" + finiteness) enough to fix these unknowns! App. II

Math: now proved by Ionel-Parker Annals (2018), ($\mathbb{Z}$ + local finit)

Doan-Ionel-Walpuski (2021) (preprint 2103.08221) finiteness

Math: explicit determination of $N_{g,d}$

- Li-Zinger (2009): $N_{1,d} = N_{1,d}^{red} + \frac{1}{12} N_{0,d}$ ← failure of hyp plane property, virt cycle not sm
  - Zinger (2009, JAMS): ~~$\mathbb{C}^\times$-localization~~ solves $N_{1,d}^{red}$ and BCOV conj. for $g=1$.
- Gathmann (2002): <u>Algorithm</u> for $N_{1,d}$ via <u>relative GW</u>

Maulik-Pandharipande (2006. Top. view of GW) extended to <u>all $N_{g,d}$</u>!

Q: Better algorithm to fit in the MS structure?

App I. ~~Witten's GLSM: (1993)~~ <u>phases</u> of $N=2$ theories in 2 dimensions.

$$W = \sum_{i=1}^5 x_i^5 : \mathbb{C}^5 \to \mathbb{C} \quad \rightsquigarrow \quad \text{LG } (\mathbb{C}^5, W)$$

diagonal $\mathbb{Z}_5$-action. $\rightsquigarrow$ orbifold LG $([\mathbb{C}^5/\mathbb{Z}_5], W)$

Let $\mathbb{C}^\times \curvearrowright \mathbb{C}^6 = \mathbb{C}^5 \times \mathbb{C} = \{(x_1, \dots, x_5, p)\}$ wt. $= (1, \dots, 1, -5)$

2 GIT quotients of "$\mathbb{C}^6 /\!/ \mathbb{C}^\times$" by removing $\vec{0}$ on both sides!

with function $\widetilde{W} := pW$

$([\mathbb{C}^6/\mathbb{C}^\times], \underline{\widetilde{W}})$ $\quad \mathbb{C}^6 \xleftarrow{(\varphi_i)_{i=1}^5} \Sigma$: Quantization

$\vec{x} \neq 0 \qquad p \neq 0 \qquad$ R.S. $\mathbb{C}^\times \to$ principle $\mathbb{C}^\times$-bundle$/\Sigma$

$$((\mathbb{C}^5 \setminus 0) \times \mathbb{C})/\mathbb{C}^\times = (\underbrace{\mathcal{O}_{\mathbb{P}^4}(-5)}_{=K_{\mathbb{P}^4}}, \underline{w}) \cdots ?? \cdots \mathbb{C}^5 \times \mathbb{C}^\times / \mathbb{C}^\times = ([\mathbb{C}^5/\mathbb{Z}_5], \underline{W})$$

"$\widetilde{W}|_{K_{\mathbb{P}^4}}$" Fayet-Iliopoulos (~~Kähler parameter~~) vs. ~~B-fields~~ ~~Spin F~~ ?

$Q = (W=0) \subset \mathbb{P}^4$ $\qquad r \to +\infty \qquad r \to -\infty$?? [redacted]

$(P=0)$

$=$ critical loci. get GW on Q.

Q: what ~~are the~~ middle phases?

App II. Gopakumar–Vafa conj.

X 3fold, $0 \neq \beta \in H_2(X,\mathbb{Z})$ CY class $\overset{\Delta}{=}$ $c_1(X).\beta = 0$ (Fano if >0)

$$F = \sum_{\beta: CY} \sum_{g=0}^{\infty} N_{g,\beta}\, t^{2g-2} q^{\beta} = \sum_{\beta: CY} \sum_{r=0}^{\infty} n_{r,\beta} \sum_{m=1}^{\infty} \frac{(2\sin(mt/2))^{2r-2}}{m} q^{m\beta}$$

"count" only genus $= r$ embedded curves : BPS invariants

- integrality : $n_{r,\beta} \in \mathbb{Z}$ $\forall r$ $(\beta \neq 0)$
- finiteness : $\exists\, g_\beta \in \mathbb{N}_0$ st $n_{r,\beta} = 0$ $\forall\, r \geq g_\beta$.

~~Example (Faber–Pandharipande)~~ for $X = (\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1)$, $n_{r,d} = 0$ $\forall r > 0$ or $d \neq 1$

no embedded $g = r$ curve

GV $\Leftrightarrow$ MCF, ie. $F(q,t) = \sum_{m=1}^{\infty} \dfrac{q^m}{m\,(2\sin\frac{mt}{2})^2}$.

~~Example (Castelnuovo bound)~~. This is the "effective" g_β.

eg. for $X = \mathbb{P}^3$ (Fano class now), $C \not\subset$ plane $\Rightarrow$ $g \leq \begin{cases} \frac{1}{4}d^2 - d + 1 & 2 \mid d \\ \frac{1}{4}(d^2-1) - d + 1 & 2 \nmid d \end{cases}$.

(Similar bound exists for $X = \mathbb{P}^n$)

App III. Yamaguchi–Yau poly (rel) expr of F_g & HKQ to $3g-2$:

In their notations : $(W = \sum x_i^5 - \underbrace{5\psi^{1/5}}_{\text{our prev } \psi} \prod x_i = 0) =: W$ quintic mirror

PF : $(\delta^4 - \psi^{-1}(\delta - \frac{1}{5}) \cdots (\delta - \frac{4}{5}))\, w = 0$ Solve asymp series at $\psi = \infty$ ie. in $z = \frac{1}{5^5\psi} = 0$

$$w(z,\rho) = \sum_{n=0}^{\infty} \frac{\Gamma(5(n+\rho)+1)}{\Gamma(n+\rho+1)^5} z^{n+\rho} ;\quad D_\rho^k w := \frac{\partial_\rho^k w}{(2\pi i)^k k!}\Big|_{\rho=0}$$

not good ?

$$w_0 = w(z,0) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

$$w_1 = D_\rho w - c\, w_0 = \frac{1}{2\pi i}(w_0 \log z + \sigma_1(z)) ; \quad \sigma_1(z) = 770 z + 810225 z^2 + \cdots$$

$$w_2 = K\, D_\rho^2 w = \frac{K}{2!(2\pi i)^2}\left(w_0(\log z)^2 + 2\sigma_1 \log z + \sigma_2\right) ; \quad \sigma_2 = 1150 z + \frac{4208175}{2} z^2 + \cdots$$

Some top. const.

$$w_3 = K \cdot D_\rho^3 w - c\, w_1 + c\, w_0 = \frac{K}{3!\,(2\pi i)^3}\left(w_0(\log z)^3 + 3\sigma_1(\log z)^2 + 3\sigma_2 \log z + \sigma_3\right)$$

$$\sigma_3(z) = -6900 z - \frac{9895125}{2} z^2 + \cdots$$

$$F_0(q) = -\frac{K}{3!}t^3 - \frac{a}{2}t^2 + ct + \frac{e}{2} + f_{inst.}(q)$$

$q = e^{2\pi i t}$

~~mirror map~~ $2\pi i\, t(\psi) = \dfrac{w_1}{w_0}$

$$\Rightarrow \begin{pmatrix} K = -\log i\,(x^a \bar{F}_a - \bar{x}^a F_a) \\ G^{WP}_{\psi\bar\psi} = \partial_\psi \partial_{\bar\psi} K \end{pmatrix}$$

$= -\log(5^5\psi) + \dfrac{154}{625}\dfrac{1}{\psi} + \cdots$

Def'n : $A_p = \delta_\psi^p G_{\psi\bar\psi}/G_{\psi\bar\psi}$, $B_p = \delta^p e^{-K}/e^{-K}$, $C = \psi^3 C_{\psi\psi\psi}$ (ie. $C_{\psi\psi\psi} = w_0^{-2} F_{0\psi\psi\psi}$)

($\bar\psi \to \infty$ hol part $e^{-K} \sim w_0$, $G_{\psi\bar\psi} \sim \partial_\psi t$) $X = 1/(1-\psi)$.

Prop : $P_g := C^{g-1} F^{(g)}$ $(g \geq 2)$ (has $F^{g}_{A\text{-model}} = \lim\limits_{\bar t \to \infty} w_0^{2g-2}\left(\frac{1-\psi}{5\psi}\right)^{g-1} P_g$)

means non-hol. original GW hol.

(YY) is a deg $3g-3$ inhomog. poly of v_1, v_2, v_3, X.

Thm (HKQ) : Just $3g-3$ poly in X !

(more BCOV)

$A = A_1$, B_1, B_2, B_3

$v_1 - 1 - 2u$ $v_3 - uv_2 \ldots X$ $-\frac{3}{5}uX$

DT (Donaldson-Thomas) Theory (for CY3: X) let $n \in \mathbb{Z}$

(1998) Rmk: MNOP ext to general 3 folds & descendant th.

$I_n(X,\beta)$ = Hilb sch of sub sch $C \subseteq X$, $[C] = \beta \in H_2(X,\mathbb{Z})$, $\chi(\mathcal{O}_C) = n$.

think as ideal sheaf I_C (DT works for general stable sheaves)

$I_{n,\beta} := \int_{[I_n(X,\beta)]^{vir}} 1 \in \mathbb{Z}$! $Z_{DT,\beta}(q) := \sum_n I_{n,\beta} q^n$

- Kuranishi map for moduli of coh sheaves $\quad K: T_{[E]}\mathcal{M} \to Ob(E)$ i.e. section of $T^*Ext^1(E,E) \to Ext^1(E,E)$

$T_{[E]}\mathcal{M} = Ext^1(E,E)$, $Ob(E) = Ext^2(E,E) \simeq Ext^1(E,E)^*$ (X is CY)

$\exists$ $K = df$, $f: Ext^1(E,E) \to \mathbb{C}$ Chern-Simons potential

and $\mathcal{M} \simeq Crit(f)$, near $[E]$. $\Rightarrow$ vir. dim $= 0$

$\exists$ standard perfect ob. th

- Conj (MNOP) (2006): (Maulik, Nekrasov, Okounkov, Pandharipande)

GW/DT correspondence $\quad Z'_{GW,\beta}(u) = Z'_{DT,\beta}(q) : e^{iu} = -q$

$Z'_{GW,\beta}(u) = \sum_g N'_{g,\beta} u^{2g-2}$ poss. disconn. inv $\qquad Z'_{DT,\beta}(q) = \dfrac{Z_{DT,\beta}(q)}{Z_{DT,0}(q)}$ reduced inv.

Amazing already: $N_{g,\beta} \in \mathbb{Z}$!

Trouble: pts can spread out all X!

In fact: $Z_{DT,0}(q) = M(-q)^{\chi(X)}$, $M(q) = \dfrac{1}{\prod_{n\geq 1}(1-q^n)^n}$ MacMahon fcn.

$g \rightsquigarrow g \rightsquigarrow g+1$ eg tw cubic $\to$ cubic

Q: A better compactification st limits do not creat emb. pt?

A: Yes: PT (Pandharipande-Thomas) Theory (2009) Invent.

Def'n: Stable pair (F,s): $I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \in D^b(X)$ for any 3 fold X (primary)

F coh sh. of dim $= 1$. (dim Supp $F = 1$), s i.e. a section

st. (1) F is pure, i.e. any $0 \neq G \to F$ has dim $G = 1$ too

(2) coker(s) is dim 0. (pts still in C!) so no emb. pt

Thm: (Le Potier, P-T) The moduli of stable pairs with $n = \chi(F)$

"$P_n(X,\beta)$" is a proper separated scheme. Moduli is fine! $\ker(s) = I_C$, $[C] = \beta$

i.e. $\beta = $ Supp F

Thm: (PT) $\exists\, [P_n(X,\beta)]^{vir} \in A_{c_\beta}(P_n(X,\beta),\mathbb{Z})$ eg. $\iota: C \hookrightarrow X$, $D \subset C$ Cartier div

here $c_\beta = \int_\beta c_1(X) = -\chi(RHom(I^\bullet, I^\bullet)_0)$ $\quad (\iota_* \mathcal{O}_C(D), s_D)$

with ob th from fixed determinant complexes $\quad D = \emptyset$, get $\mathcal{O}_X \xrightarrow{1} \mathcal{O}_C$

- DT/PT correspondence $\quad Z'_{DT,\beta}(q) = Z_{PT,\beta}(q)$ for CY3: X.

$Z_{PT,\beta}(q) = \sum P_{n,\beta} q^n$, $P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1 \in \mathbb{Z}$.

proved by Bridgeland (2018) PSPUM 97-1, via wall crossing/Hall alg.

We only have invariants for DT, PT (and GV from GW)

but No categorification (structure) yet! phys BPS, phys BCOV, "Math" only $g = 0$.

Examples on $\dot{Z}_{GW,\beta}(u) = Z_{PT,\beta}(q) = Z'_{DT,\beta}(q)$ & GV

$-q = e^{iu}$

• local CY :

$\mathcal{O}(-1)^2 \to \mathbb{P}^1$ $\cancel{\beta = \ell = [\mathbb{P}^1]}$

$P_n(X, \ell)$ = non-zero sections of $\mathcal{O}_p(n-1)$ supp on $\mathbb{P}^1$

$\cong \mathrm{Sym}^{n-1}\mathbb{P}^1 \cong \mathbb{P}^{n-1}$

∃ ob bundle = cotangent bundle

$\Rightarrow \quad P_{n,\ell} = (-1)^{n-1}\chi_{top}(\mathbb{P}^{n-1}) = (-1)^{n-1} n$ for $n \geq 1$, $= 0$ for $n \leq 0$

$Z_{PT,\ell} = q - 2q^2 + 3q^3 + \dots = \frac{q}{(1+q)^2}$

Same as $\dot{Z}_{GW,\ell}$ why? HW (or Rmk)

hint: via GV formula

$\beta = 2\ell$: lowest hol Euler char $= 3$ $(= n)$

• $P_3(X, 2\ell) \cong \mathbb{P}^1 \longleftrightarrow$ choice of subbundle $\mathcal{O}_p(-1) \subset \mathcal{O}_p(-1)^{\oplus 2}$

stable pair $= \mathcal{O}_C$, C = doubling along the subbundle with canonical section

$Z_{PT,2\ell} = \underline{\underline{-2}}\, q^3 + \dots$

• $P_4(X, 2\ell)$ is more interesting, besides $\mathcal{O}_{\mathbb{P}^1}(-2) \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

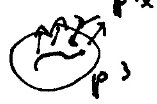
open subset in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(1)^2)) \cong \mathbb{P}^3$

pairs of sections of $\mathcal{O}(1)$ not proportional

when the 2 sections are proportional, find $\mathcal{O}_p(-1)$ subbundle, double the zero set.

$\mathcal{I}_p \leftarrow p \in \mathbb{P}^1$ reduced pt twisted by $\mathcal{O}_p(1)$

$\to$ get $\chi(F) = 4$



$\mathbb{P}^1 \times \mathbb{P}^1$ glue to, as quadric thickening along coor. 1st order deformation $\mathcal{O}(-1)$ direction.

Bernoulli numbers can be defined as

Rmk.

$$\sum_{g \geq 0} B_{2g} t^{2g} = \left(\frac{\sin t/2}{t/2}\right)^{-2}.$$