內在空間的幾何躍遷

Geometric transitions of inner spaces

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大綱

— 幾何學家的世界觀 —

i. 空間

ii. 外維度

iii. 內在空間

iv. 黑洞 vs 蟲洞

v. 量子糾纏

vi. Team TAIWAN

i. 空間

Spaces: flat to curved

Pythagoras ~ 500 BC

Euclid ~ 300 BC

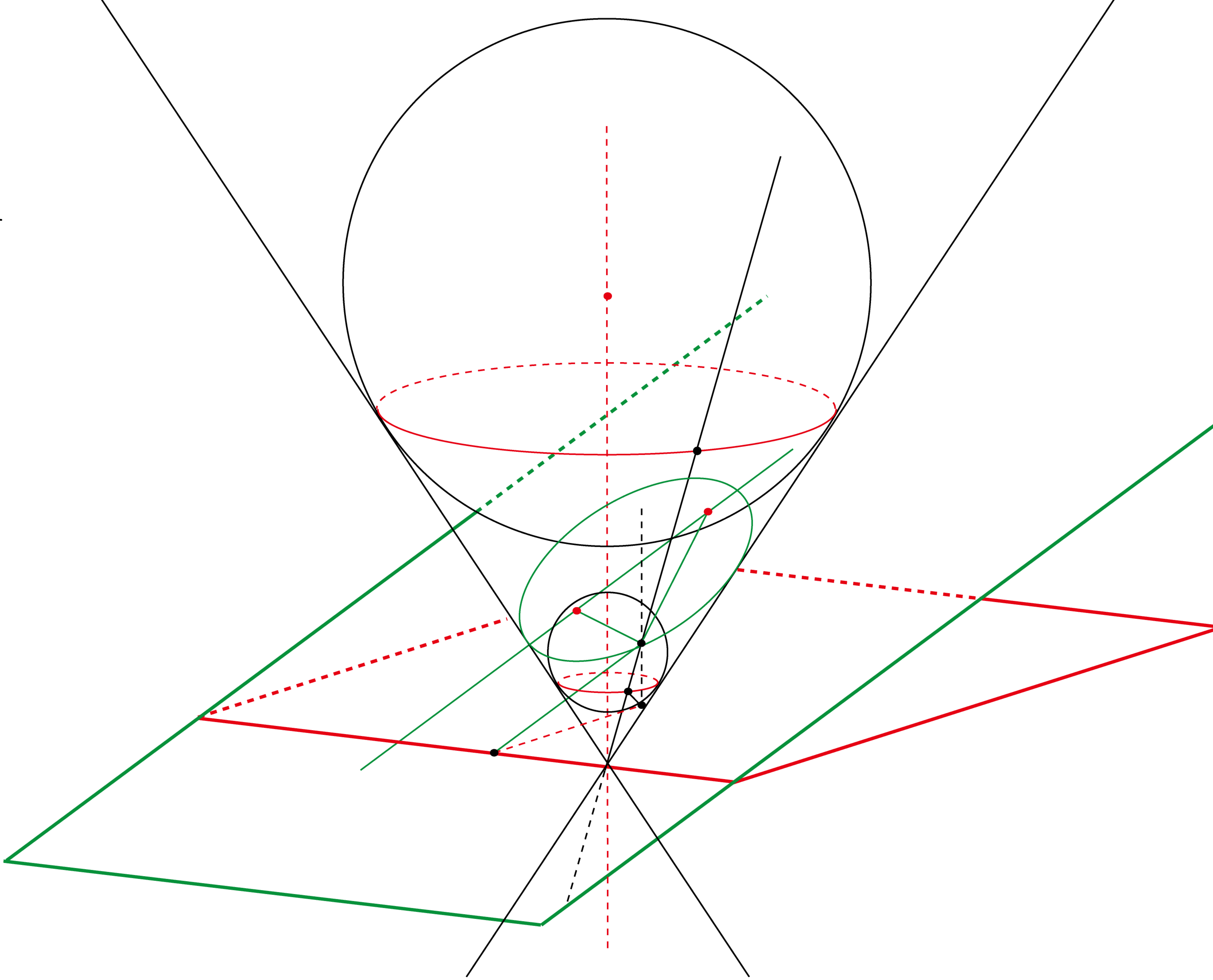
Kepler 1619 — 80 years

Newton 1643-1727

$$\vec{F} = m\vec{a} := m\vec{r}''(t)$$

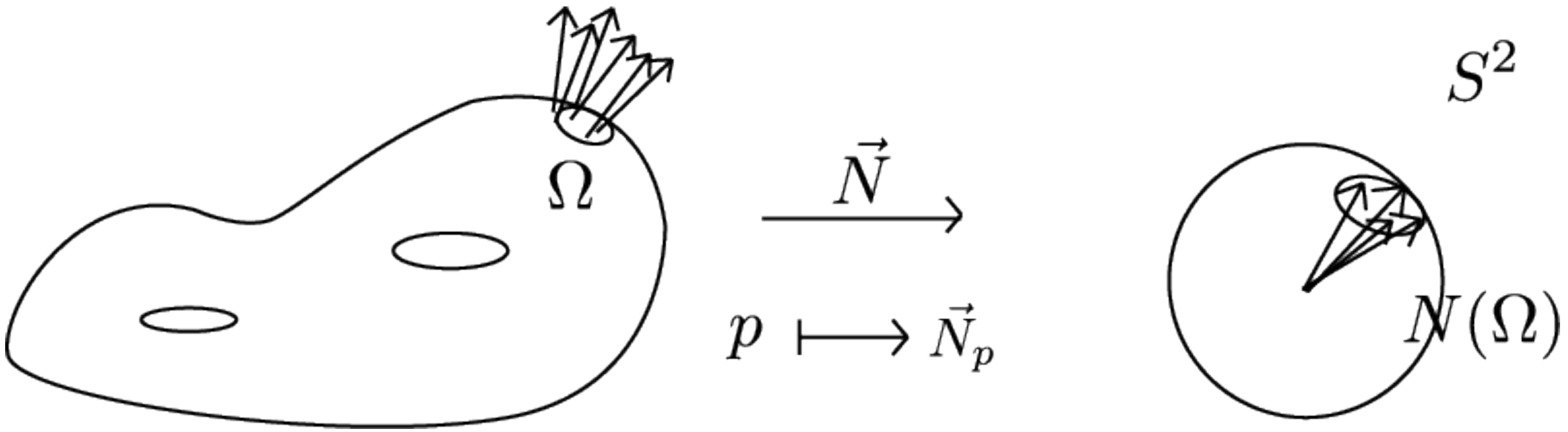
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

Leibniz 1646-1716



曲率的誕生

Gauss 1777-1855

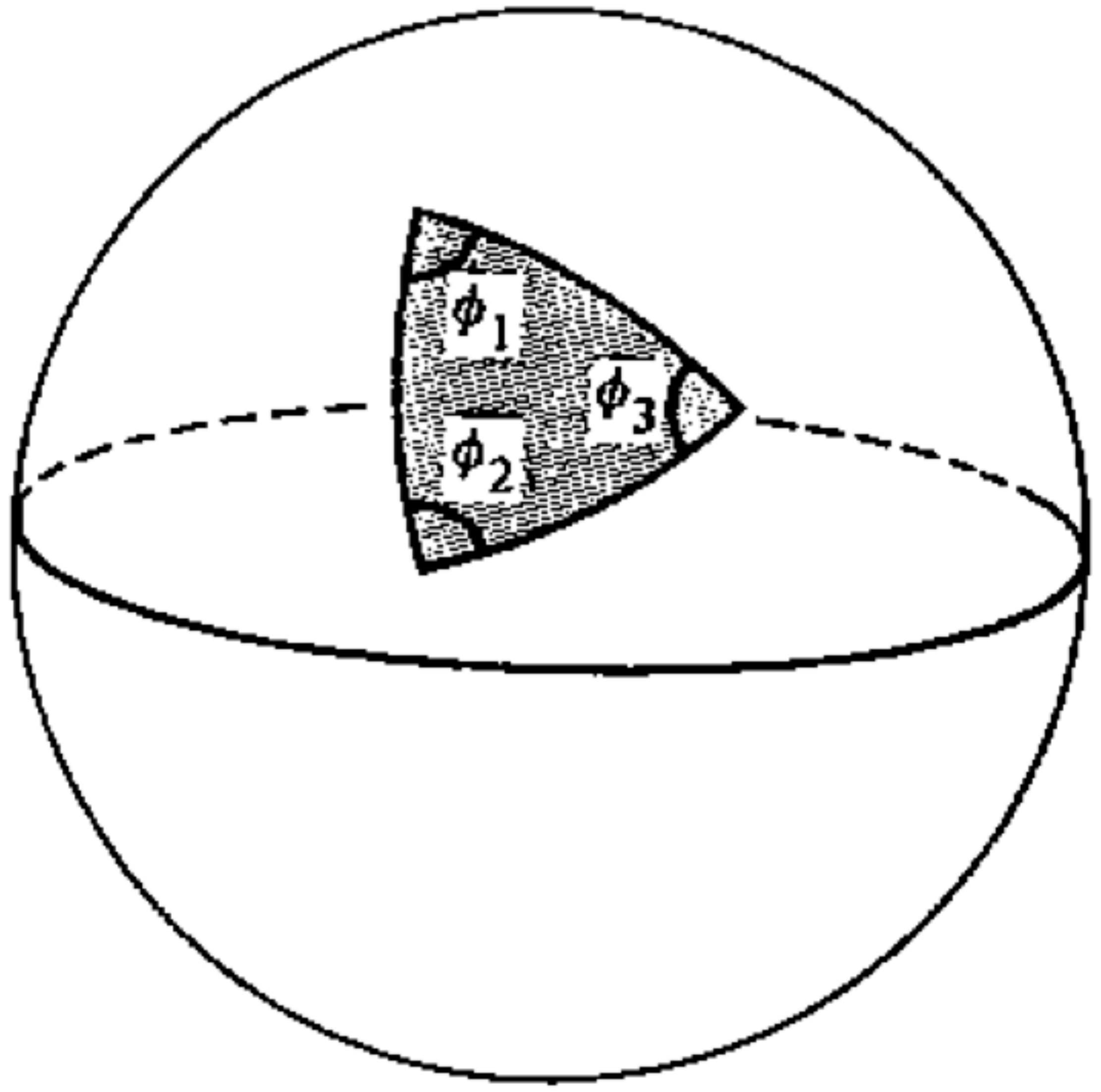


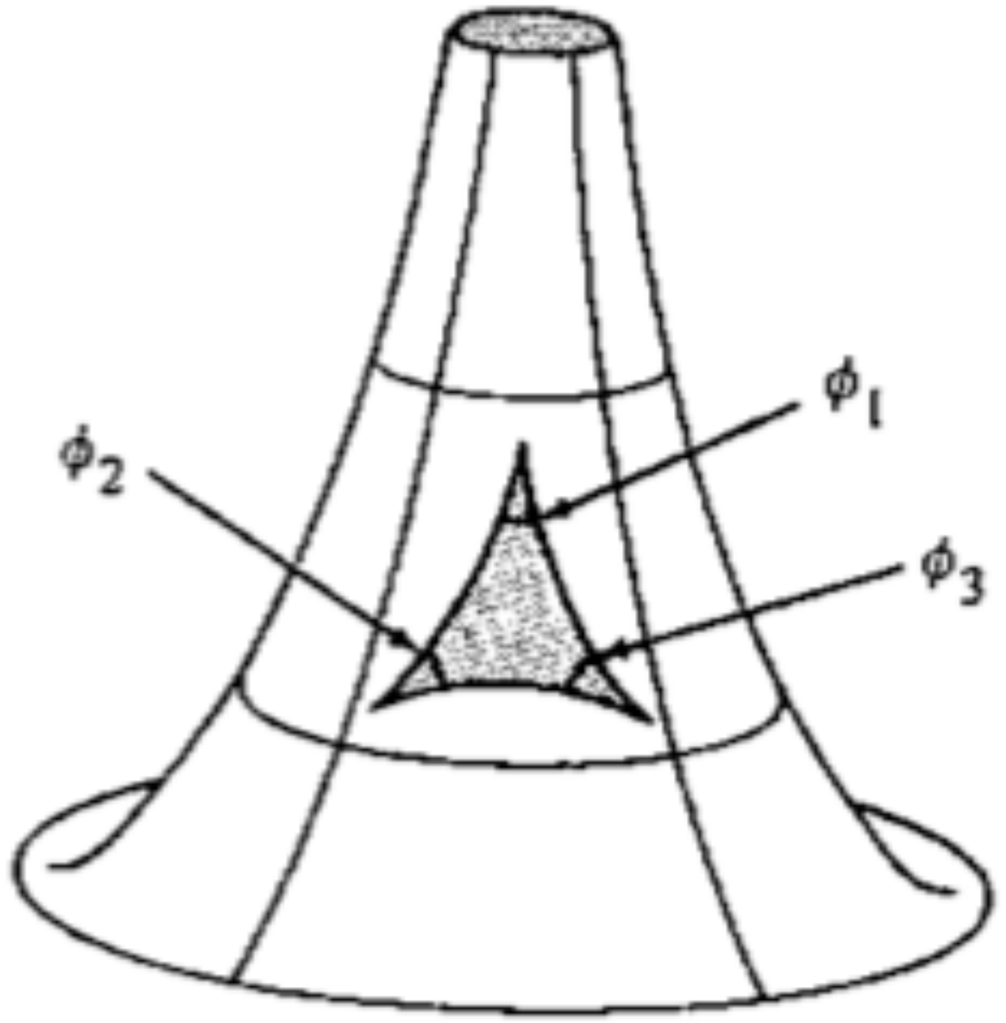
$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2$$

$$K_p = \pm \lim_{\Omega \to p} \frac{|N(\Omega)|}{|\Omega|}$$

$$L(S_r^1) = 2\pi r - \frac{\pi}{3} r^3 K + o(r^3)$$

$$\sum \phi_i^\circ + \int_{\partial S} k_g\,ds + \int_S K\,dA = 2\pi(2 - 2g(S))$$





空間模型 n-dim extended quantity

Riemann 1855 (論幾何學之基礎假說)

$$ds^2 = g_{ij}(x)\, dx^i dx^j$$

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\,\partial_k \qquad\qquad \nabla_{\gamma'(t)}V(t) = 0$$

$$\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\dot{x}^j = 0$$

Christoffel, Levi-Civita: $\quad \Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

Curvature: $\quad R^l{}_{ijk} = \partial_j\Gamma^l_{ik} - \partial_i\Gamma^l_{jk} + \Gamma^l_{jm}\Gamma^m_{ik} - \Gamma^l_{im}\Gamma^m_{jk}$

Ricci: $\quad R_{ij} = R^k{}_{ikj} \qquad$ scalar: $\quad R = g^{ij}R_{ij}$

彎曲的時空模型 (M^n, g) with Minkowski signature (+, -, -, -,⋯) 1908

General relativity

Hilbert 1915: $x^0 = ct$

$$S(g) = \int_M R(g)\sqrt{|\det g_{ij}|}\, dx^0 \cdots dx^{n-1}$$

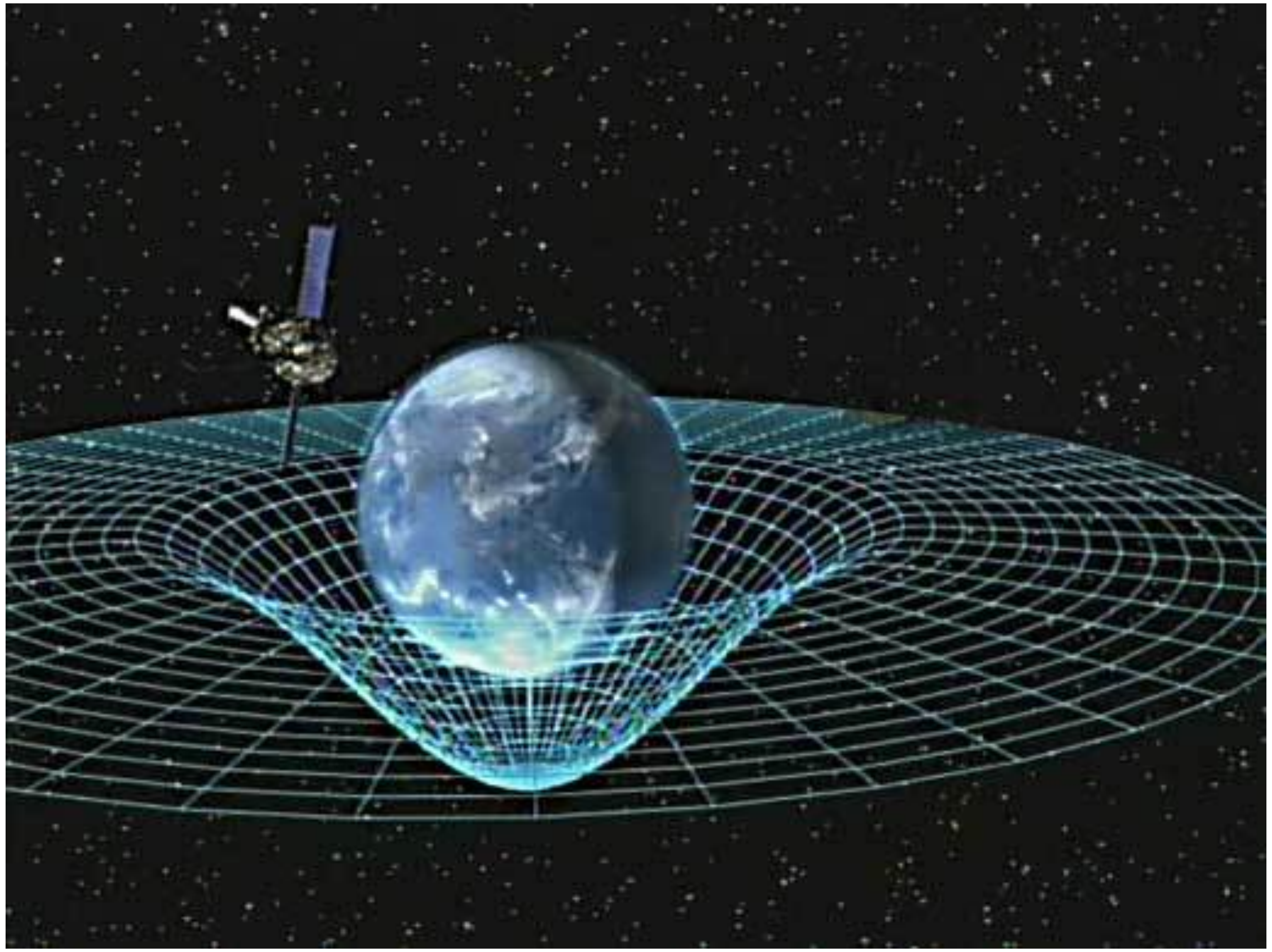
Einstein 1915

$$R_{ij} - \frac{R}{2} g_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

Einstein metrics

$$R_{ij} = \lambda g_{ij}$$

$n \geq 3 \Rightarrow \lambda = C = \frac{R}{n}$. Vacuum $\Rightarrow C = 0$



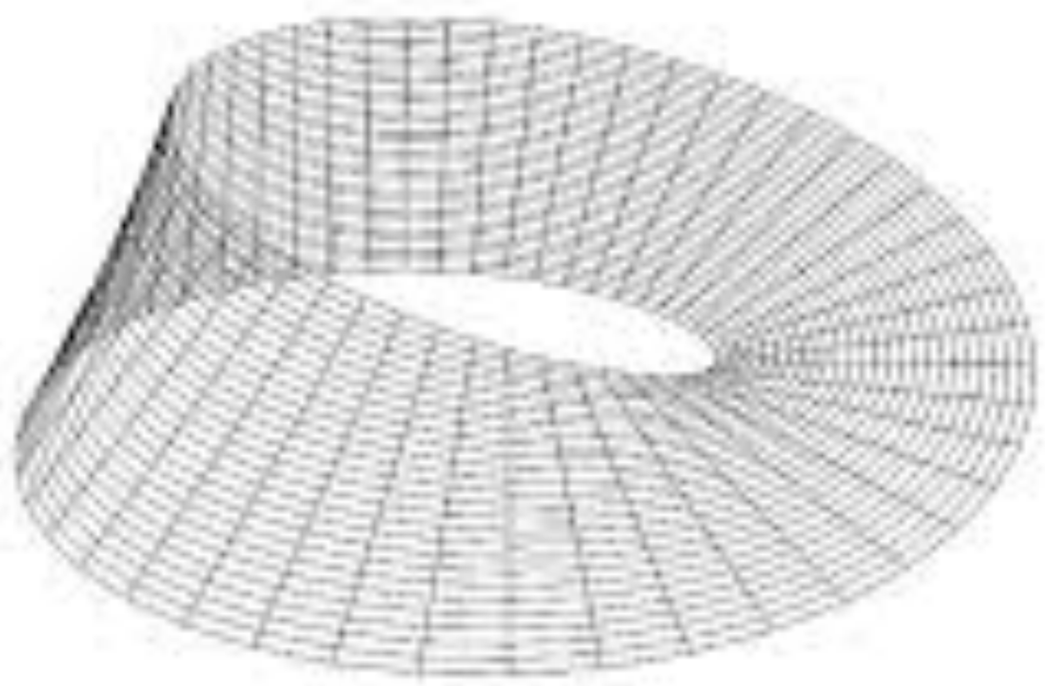
Picture from internet/NASA

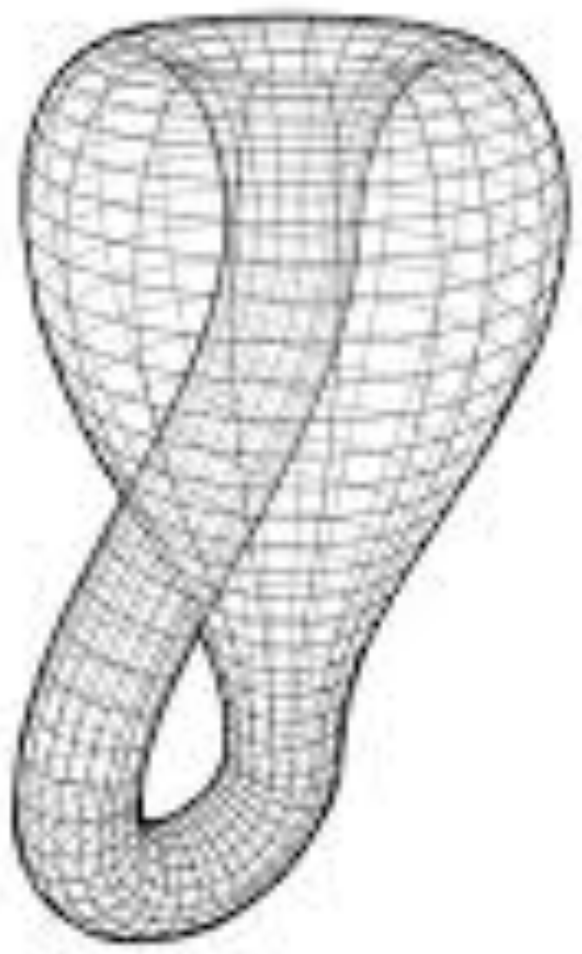
ii. 外維度

Aspect from differential topology

Whitney 1936:

Every C^∞ manifold M^n admits a C^∞ closed embedding into $\mathbb{R}^{2n}$





Klein: $K \not\hookrightarrow \mathbb{R}^3 \Rightarrow K \hookrightarrow \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$

Nash's isometric embeddings

$$f : (M^n, g) \to \mathbb{R}^N, \quad g(X, Y) = df(X) \bullet df(Y)$$

Nash 1954: Any C^1 embedding can be deformed to be isometric, even inside an arbitrarily small ball

Nash 1956: $\exists C^\infty$ isometric embedding for

$$N = \frac{1}{2}n(n+1)(3n+11)$$

$T^2 = \mathbb{R}^2/\mathbb{Z}^2 \hookrightarrow \mathbb{R}^3$

$$T^2 = S^1 \times S^1 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2$$

$$ds^2 = d\theta^2 + d\psi^2$$

Borreli et.al. PNAS 2012

Extra dimensions — D = 1 + 3, 26, 10, 11 ?

弦論與超對稱 String Theory / Super—Symmetry

particles ↦ strings / vibration types

trajectories ↦ surfaces

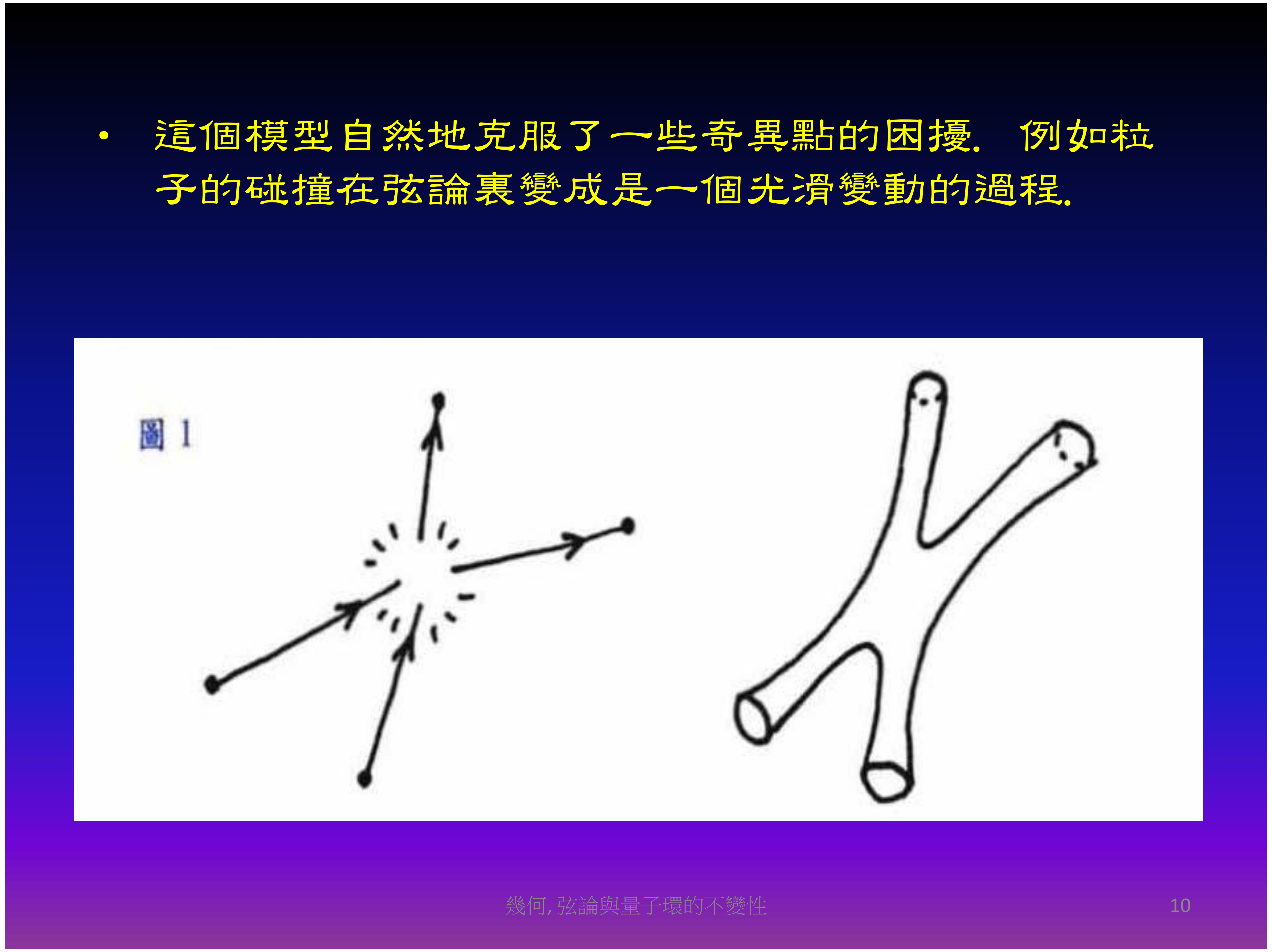
結合重力與量子力學

Bosonic strings: D = 26

Superstrings: D = 10

5 theories: I, IIA, IIB, SO(32), $E_8 \times E_8$

Witten 1995: M theory, D = 11



iii. 內在空間

Inner spaces — Calabi—Yau manifolds

Sigma model:

$$f : \Sigma \to M = \mathbb{R}^{1,3} \times X^6$$

SUSY $\Rightarrow X$ is a complex Kaehler manifold

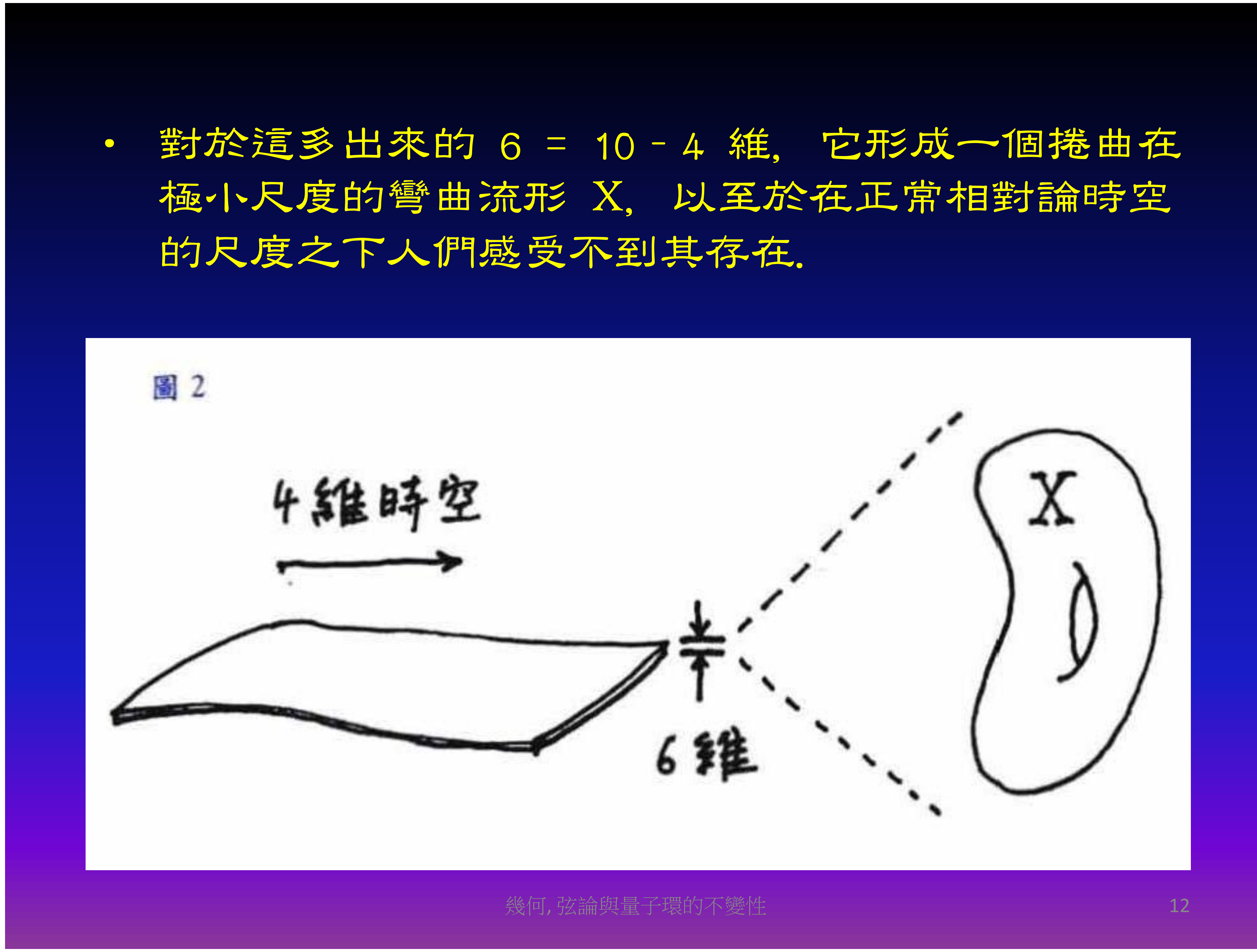
$$R_{i\bar{j}} = 0$$

Yau 1976: Calabi conjecture

Solvable $\Leftrightarrow c_1(X) = 0$ (1st Chern class)

i.e., $\Omega = f(z)\, dz^1 \wedge dz^2 \wedge dz^3 \neq 0$

Algebraic geometry: trivial canonical bundle



内在空間不唯一

Complete intersection Calabi—Yau 3-folds

$\mathbb{C}^3/\Lambda, \quad \Lambda \cong \mathbb{Z}^6$ (not simply connected)

Quintic: $X = (z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0) \subset \mathbb{C}P^4$

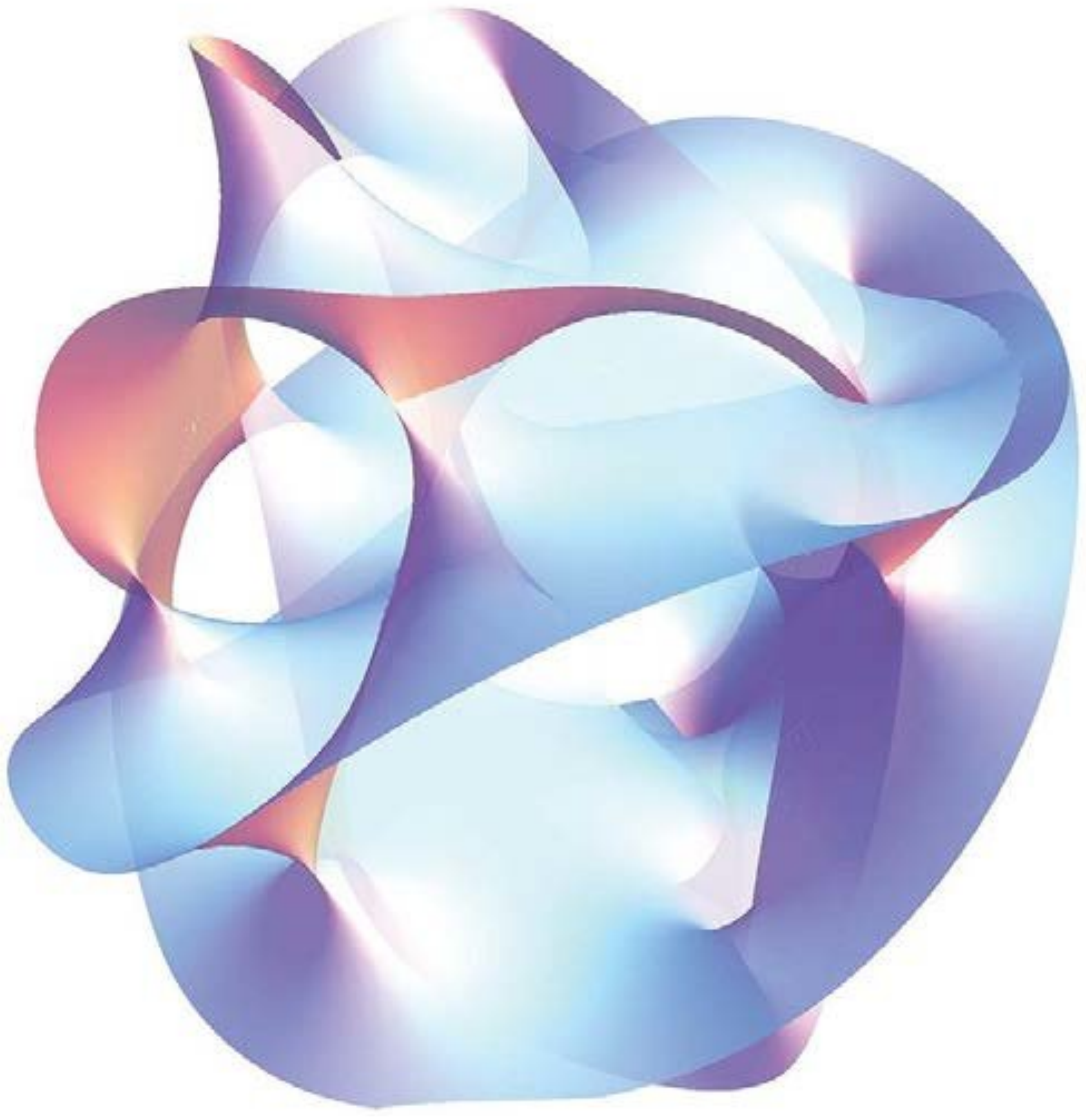
$c_1(\mathbb{C}P^d) = d + 1 \Rightarrow$

$[2,4], \quad [3,3] \subset \mathbb{C}P^5$

$[2,2,3] \subset \mathbb{C}P^6, \quad [2,2,2,2] \subset \mathbb{C}P^7$

$[(1,1),(1,4)] \subset \mathbb{C}P^1 \times \mathbb{C}P^4$

$[(1,1,1),(1,1,1),(1,1,1)] \subset (\mathbb{C}P^2)^3$



A cross section of the quintic by S. Ogmen

CICY web, # = 7890

Green—Hubsch 1988, S.-S. Wang 2018

$\phi : X_{7885} = [(1,1),(1,4)] \subset P^1 \times P^4 \to P^4$

$$l_0(y)\,x_0 + l_1(y)\,x_1 = 0$$

$$q_0(y)\,x_0 + q_1(y)\,x_1 = 0$$

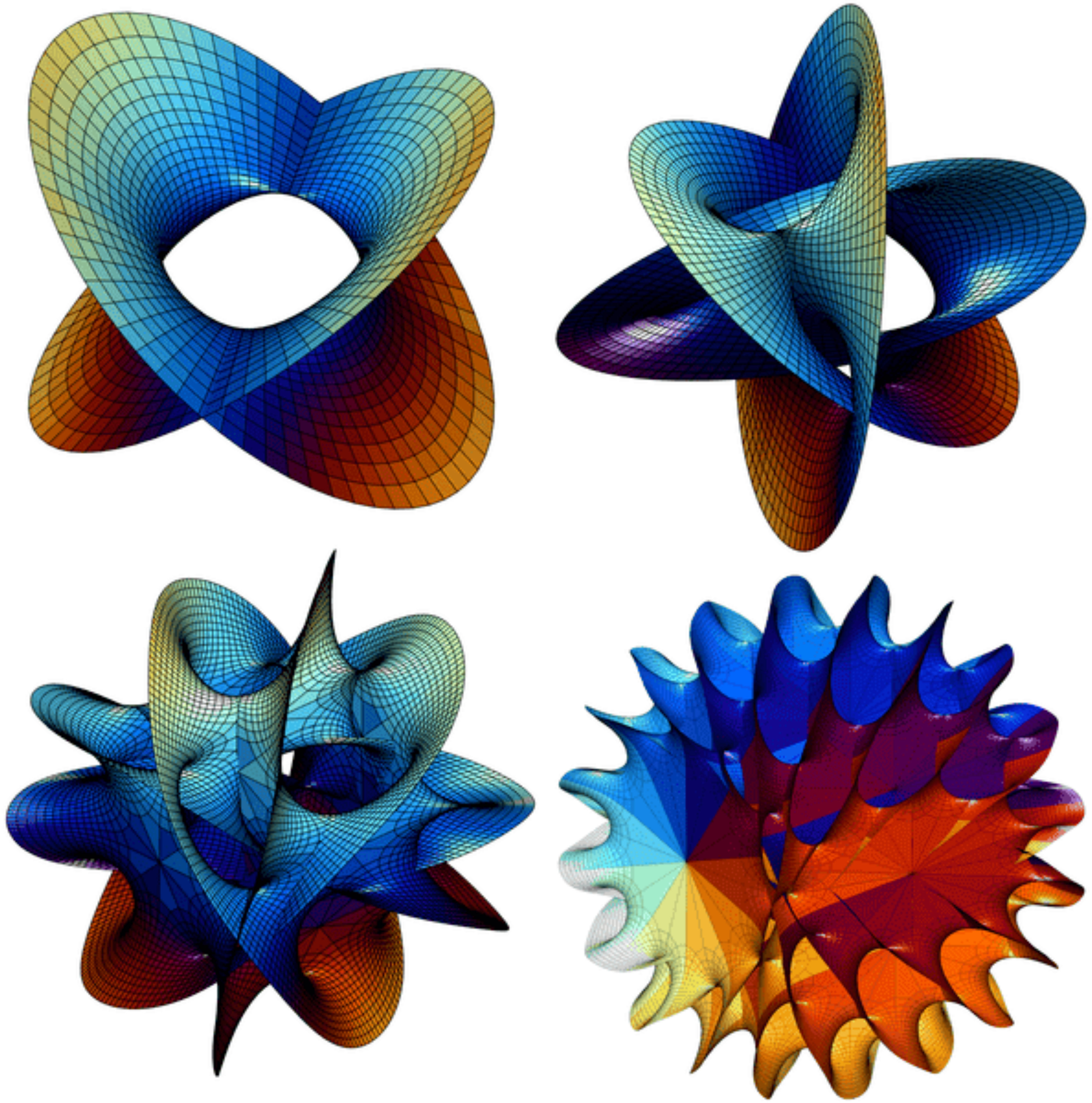
$$Q(y) := l_0(y)q_1(y) - l_1(y)q_0(y)$$

rank = 1: unique solution $[x_0 : x_1] \in P^1$

rank = 0: all P^1 are solutions, $\#P^1 = 16$

Get conifold transition $\phi(P^1) = \mathrm{pt}$ — singular point

$$X_{7885} \to \bar{X} \rightsquigarrow X_{7890} = [5]$$



iv. 黑洞 vs 蟲洞

Black holes versus wormholes — quantum tunneling

Singularity of Einstein equation: Differential geometry — Schwarzchild black hole in $\mathbb{R}^{1,3}$ (Kerr …)

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2\, dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

String theoretic black hole $\mathbb{R}^{1,3} \times \bar{X}$, the singularity could be on $\bar{X}$ instead!

Algebraic geometry — $\bar{X}$ is a singular CY3 with canonical singualrities (Yau 1976, Mori 1984, Wang 1996)

E.g. isolated $\quad (p \in \bar{X}) \sim (f(x,y,z) + w\, g(x,y,z,w) = 0) \subset \mathbb{C}^4$

$$f = x^2 + y^2 + z^{n+1}, \quad x^2 + y^{n-1} + yz^2, \quad x^2 + y^3 + y^{\delta_{0,4}} z^{3,4,5}$$

Conifold: $\quad f = x^2 + y^2 + z^2 + w^2 = 0$

Wormhole — bridge connecting two spacetimes (tunnel solution to Einstein equation)

錐點躍遷與蟲洞 Conifold transitions as wormholes

(Ads/CFT correspondence / Chern—Simons large N duality)

$X = X_{\rm sm} = X_t : z^2 + y^2 + z^2 + w^2 = t$

Real locus: $S^3(\sqrt{t})$,

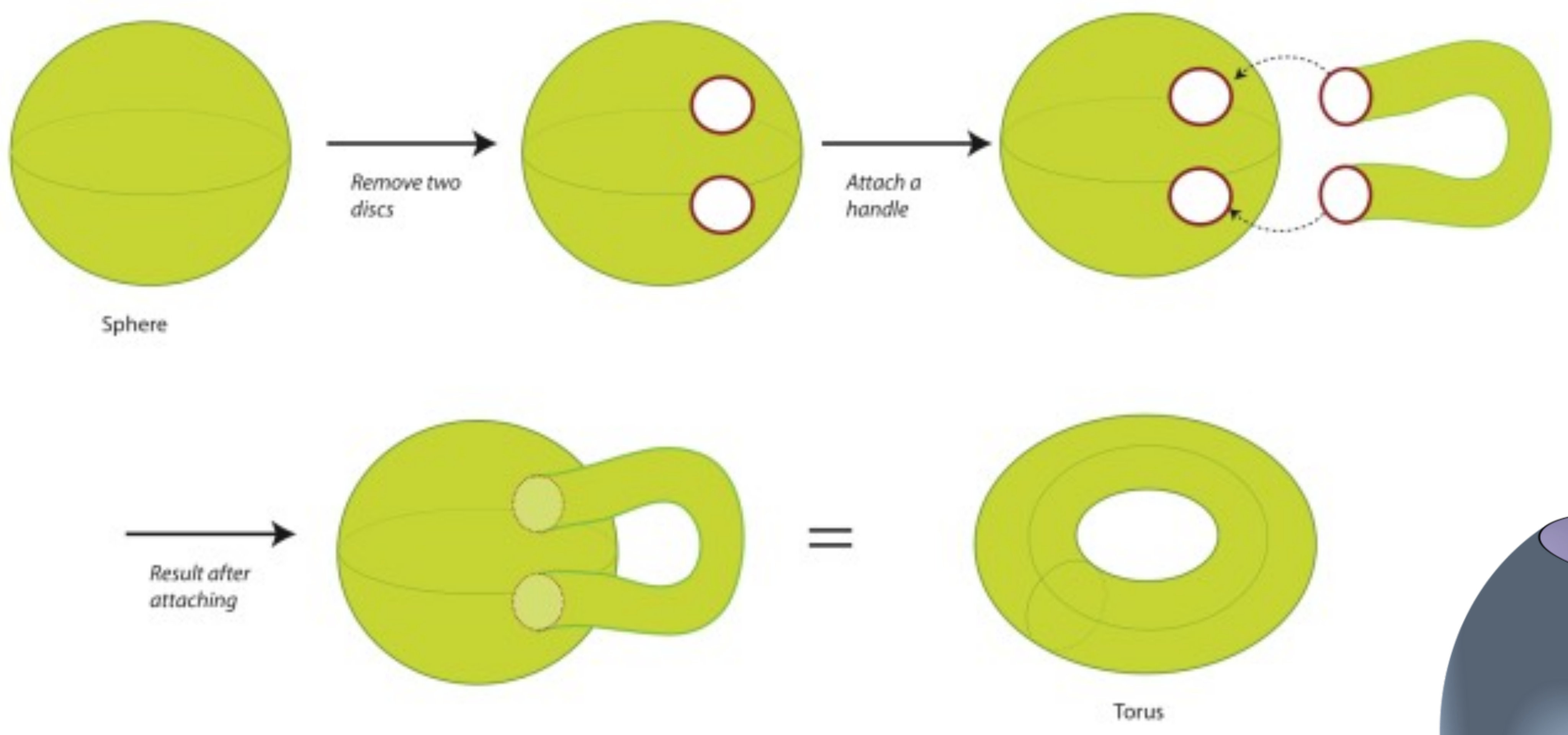
$\text{CY} \;\; \Rightarrow N_S X \cong T^*S^3 \cong S^3 \times \mathbb{R}^3$

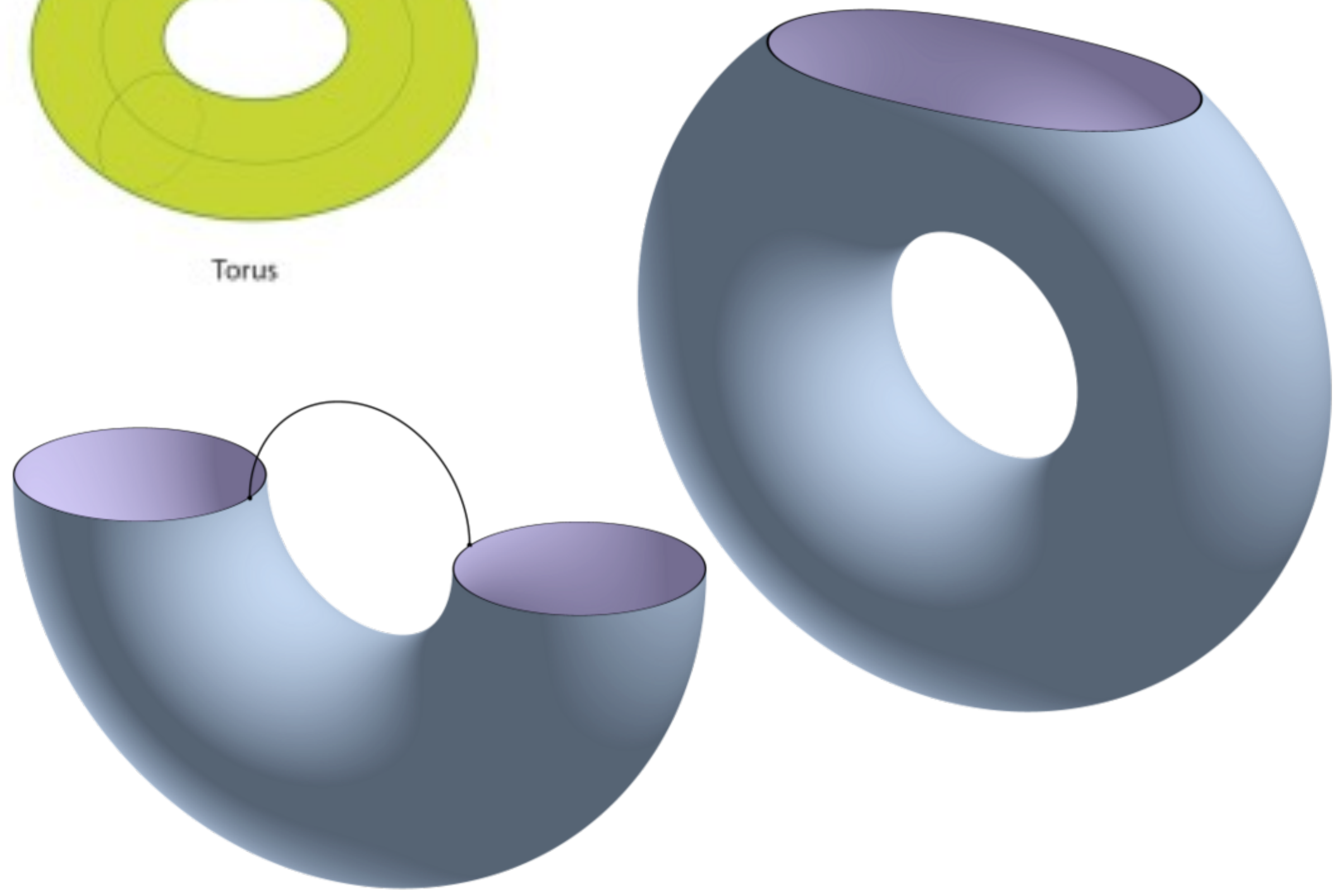
$Y = X_{\rm res} = Y_s \supset P^1 \cong S^2(\sqrt{s})$,

$\text{CY} \;\; \Rightarrow N_C Y \cong \mathcal{O}_{P^1}(-1)^{\oplus 2} \cong S^2 \times \mathbb{R}^4$

Geometric surgery — 空間手術

$\partial(B^4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times B^3)$





Morse Theory

$\partial(B^2 \times S^0) = S^1 \times S^0 = \partial(S^1 \times B^1)$

Weil—Petersson metric on Calabi—Yau moduli

Incompleteness / black hole condensation / entropy

$$g^{WP}_{i\bar{j}} = \partial_i \bar{\partial}_j K, \qquad K = -\log \int_{X_t} \Omega(t) \wedge \overline{\Omega}(t)$$

X_0 is at finite distance $\Longleftrightarrow$ X_0 has canonical singularities (Wang 1996)

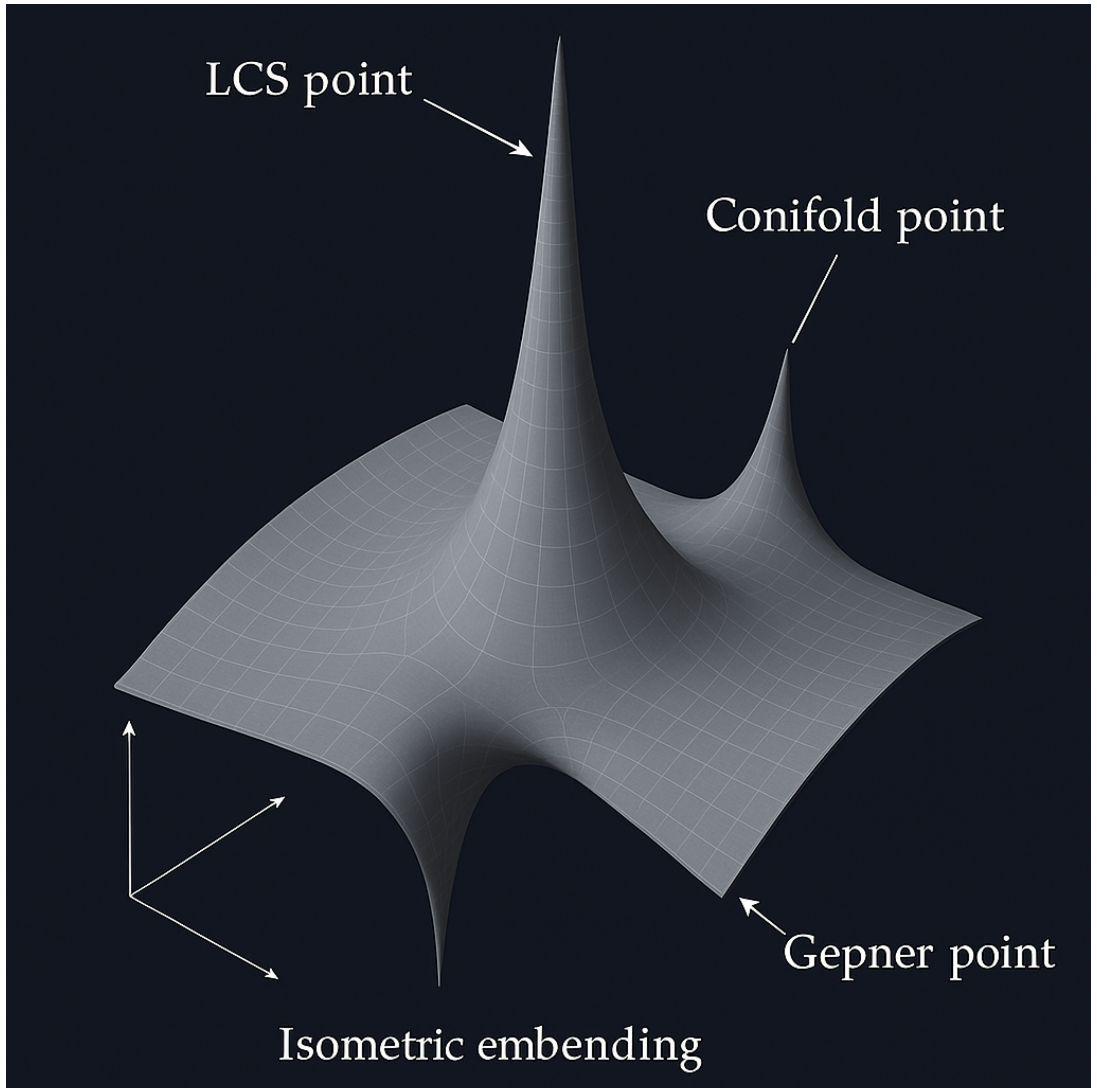
$$R^{WP}_{i\bar{j}k\bar{\ell}} = -\left(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}\right) + e^{2K}\, g^{p\bar{q}}\, C_{ikp}\, \overline{C}_{j\ell q}$$

$C_{ijk} = \int_X \Omega \wedge \partial_i \partial_j \partial_k \Omega$ is the Yukawa coupling (IIB quantum corrections)

near finite distance (one parameter) degenerations

$$g^{WP} \sim c\,|\log|t||^k\,|dt|^2 \to +\infty, \qquad K^{WP} \sim \frac{k}{|t|^2\,|\log|t|^2|^2} \to +\infty$$

$\Rightarrow$ Effective control of entanglement entropy (Wang 2003)



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v. 量子糾纏

Quantum entanglement — toward an unique universe

Type IIA string leads to instanton quantum numbers wrapped on $\beta \in H_2(X)$

Mathematically, it is number of holomorphic curves / maps $f : \Sigma \to X$

For quintic CY $X = [5] \subset P^4$, this was first computed by Candelas et. al. 1990 using mirror symmetry

$$F^0_{ttt}(t) = 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1 - q^d}, \qquad q = e^{2\pi i t}.$$

n_d is the number of degree d rational curves $\Sigma \cong P^1 \hookrightarrow X$

$$n_1 = 2875, \quad n_2 = 609250, \quad n_3 = 317206375, \quad n_4 = 242467530000^*, \quad \cdots$$

The mirror conjecture for all CICY was later proved by Lian—Liu—Yau 1996, Givental 1996

Q — Different physics for different X ?

微觀量子重力猜想

Maldacena—Susskind 2013

ER = Einstein—Rosen bridge

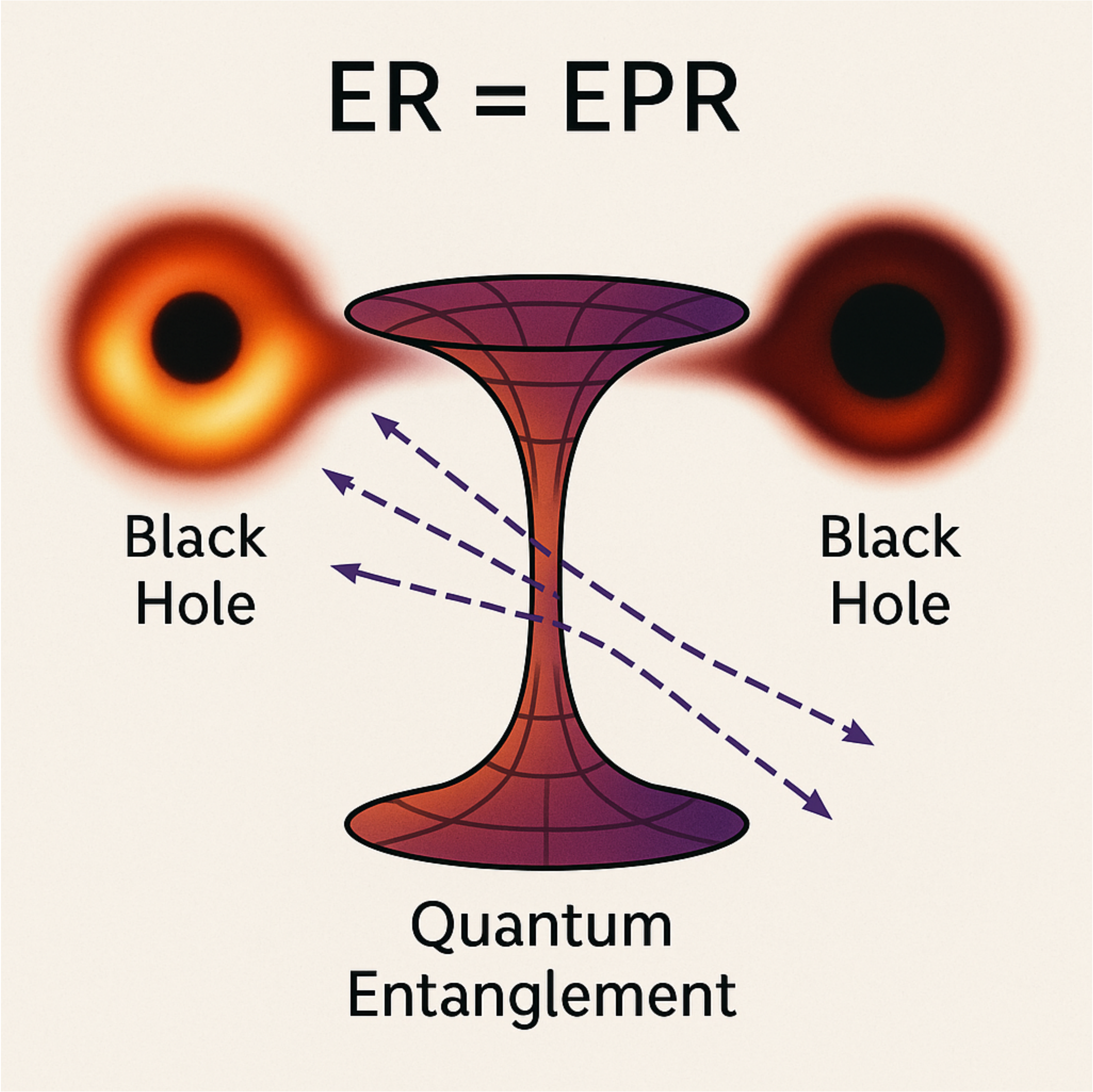
the wormhole

spacetime structure

EPR = Einstein—Podolsky—Rosen

Q-entanglement

information pattern



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超弦理論下的量子糾纏

Let $X \nearrow Y$ be the conifold transition $G(2,4) \rightsquigarrow \overline{P(2,4)} \leftarrow P(2,4)$, $S_i^3 \mapsto S_i^2 \cong \mathbb{C}P^1$.

Instanton splitting: $q^n \mapsto q_1^n q_2^j$, $0 \le j \le n$ (H.-W. Lin, T.-J. Lee 2019):

$$1280 = 640 + 640$$

$$92288 = 10032 + 72224 + 10032$$

$$15655168 = 288384 + 7539200 + 7539200 + 288384$$

$$3883902528 = 10979984 + 753920000 + 2364024560 + 753920000 + 10979984$$

$$1190923282176 = 495269504 + 74132328704 + 520834042880 + 520834042880 + 74132328704 + 495269504$$

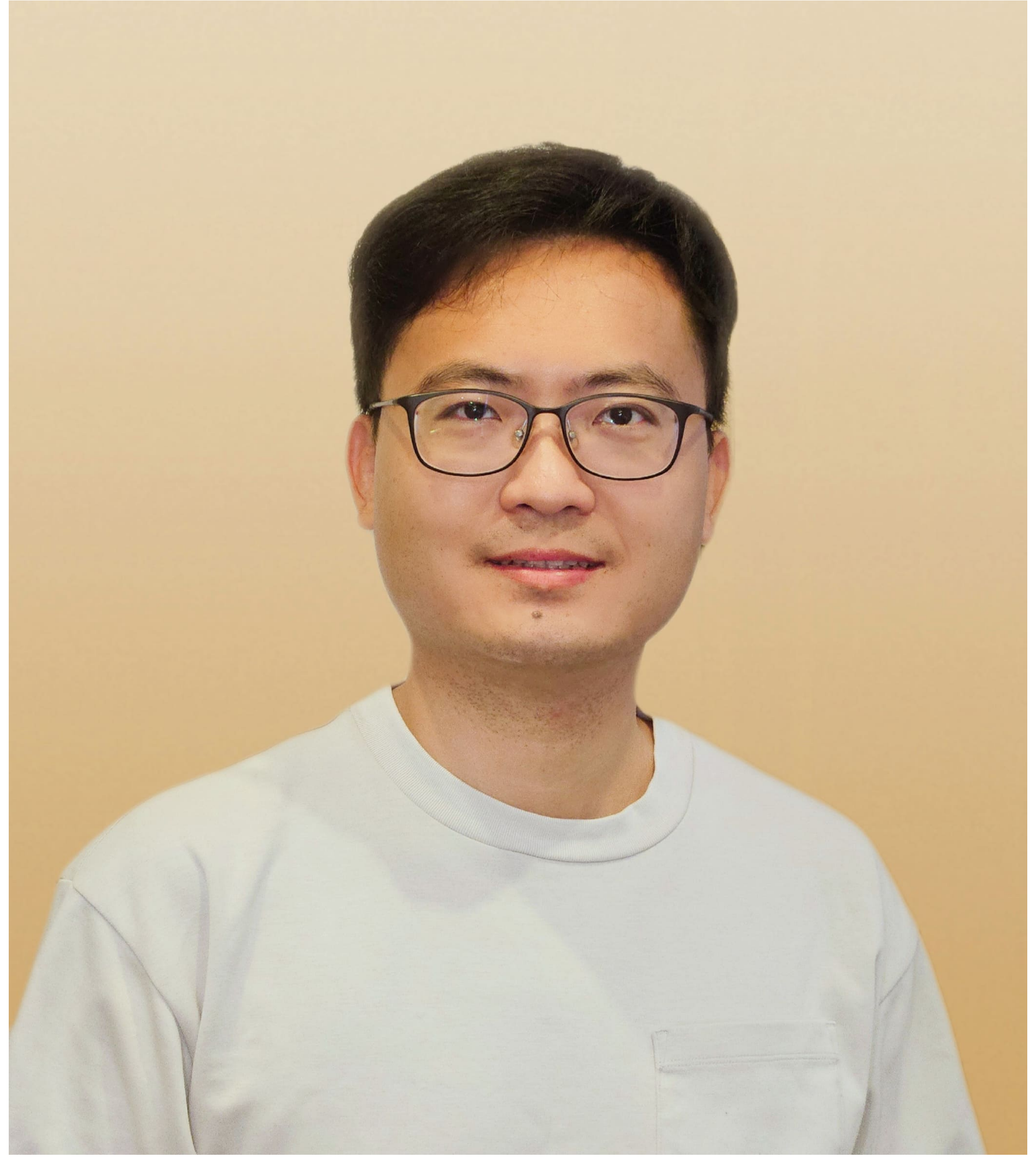
Linking invariants (Wang 2013; Lee—Lin—Wang 2018).

vi. Team TAIWAN

我的研究團隊 at TIMS (台大數學科學中心)

https://ntubeats.ntu.edu.tw/enews/009/01.html

On period map of CY3



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On quantum Riemann—Roch

陳宗震

數學之美　在於真善

不為什麼　好奇而已

看到再有趣的東西

如果你不感到好奇

它都沒有什麼

謝謝