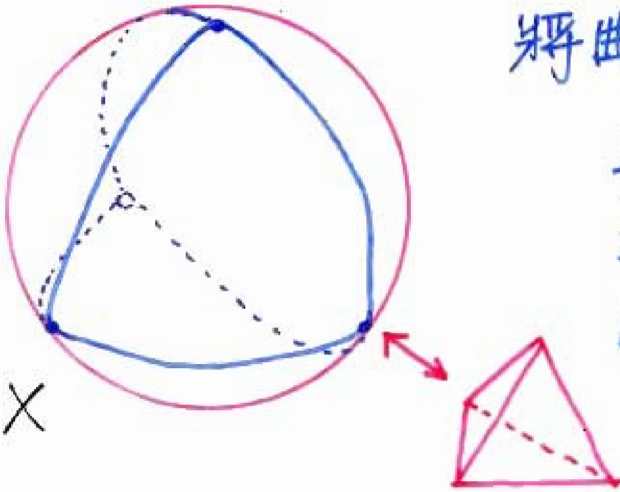
Euclid（歐幾里德，西元前 300 年）

- Elements（幾何原本）

Euler（尤拉，1707–1785）

- 最早的拓撲學概念（topology）



X

將曲面作三角剖分

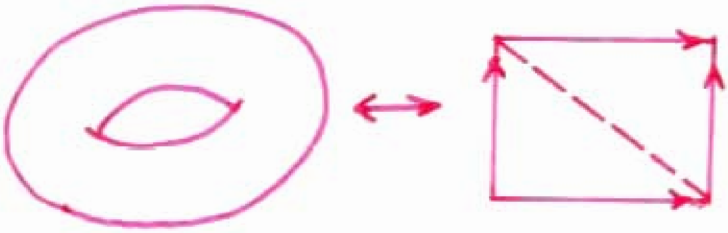
點數 $p = 4$

邊數 $e = 6$

面數 $f = 4$

Euler number $\chi(X) := p - e + f = 2 - 2g$

g = 洞的個數（genus）



$p = 1$

$e = 3$

$f = 2$

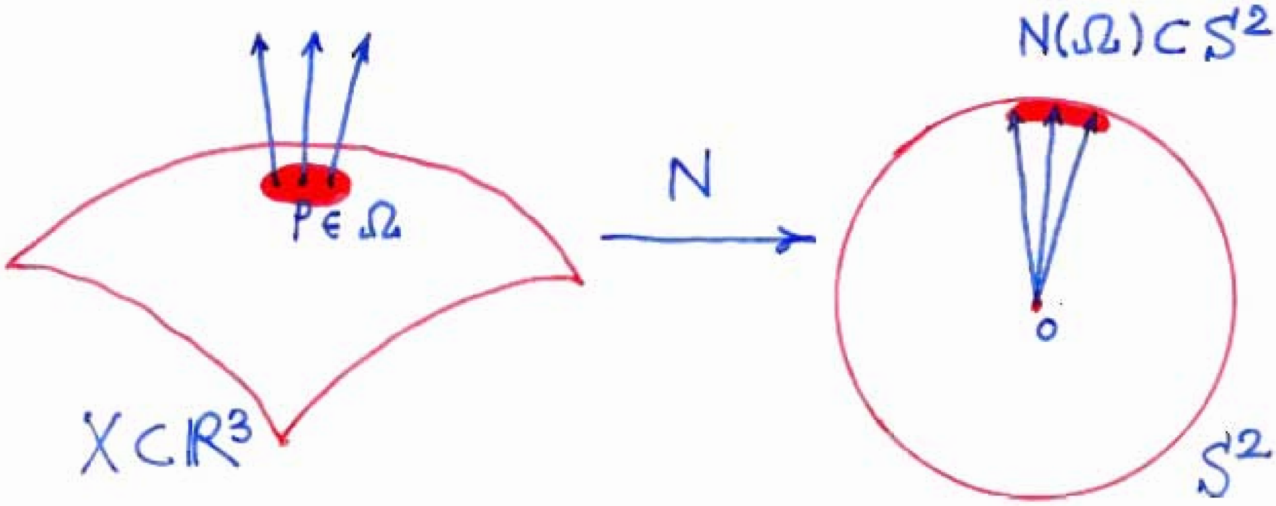
$X : g = 1$

$\chi(X) = 1 - 3 + 2 = 0 = 2 - 2\cdot 1$

- Euler number 是一個拓撲不變量，與三角剖分的方式無關.

Gauss (高斯, 1777–1855)

- 曲率的概念 (curvature)



$$K := \lim_{\Omega \to p} \pm \frac{\text{Area}(N(\Omega))}{\text{Area}(\Omega)}$$

$$= \det(dN_p) \quad (\text{具方向性})$$

- Gauss–Bonnet 定理

$$\boxed{\int_X K\,dA = 2\pi\,\chi(X)}$$

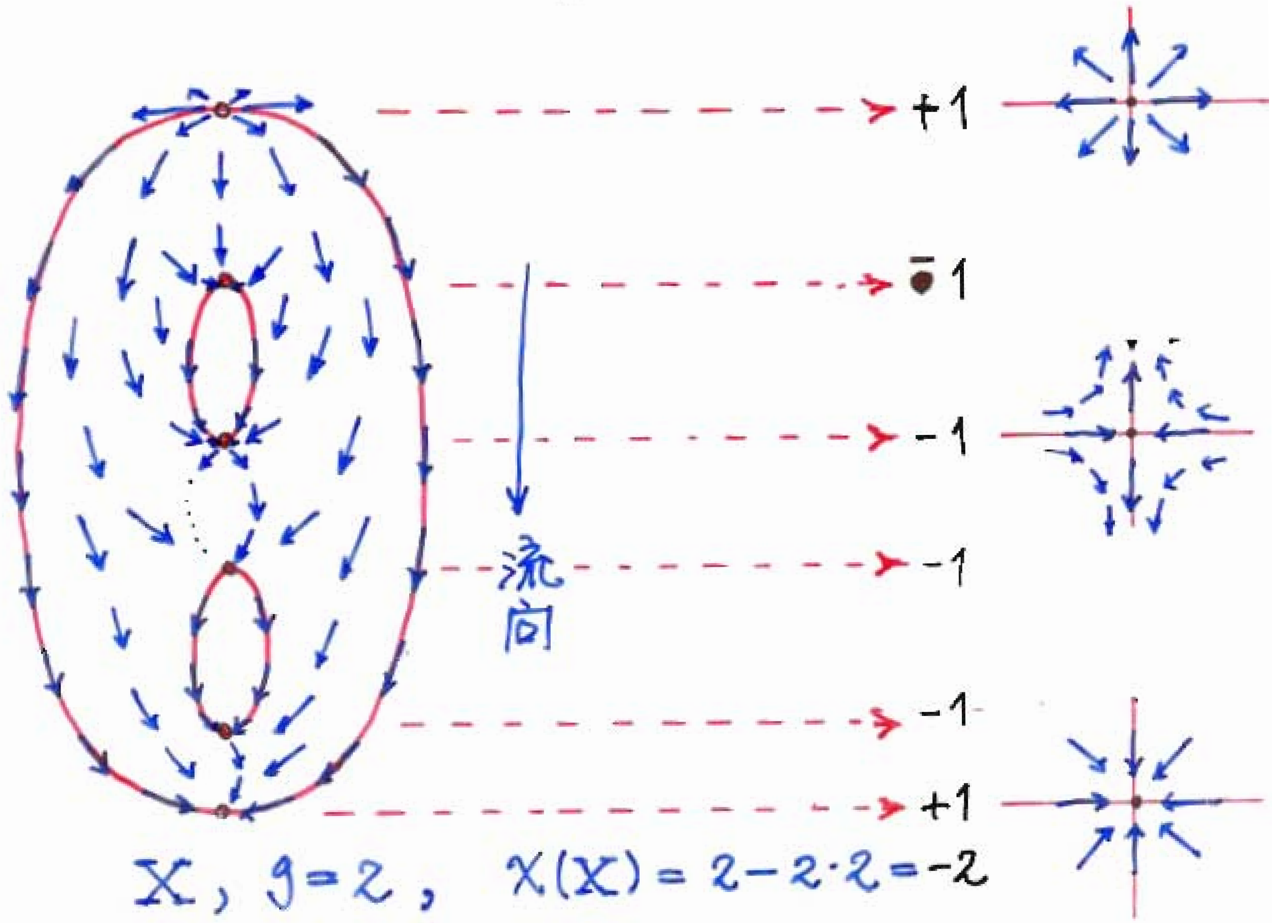
Riemann (黎曼, 1826–1866)

- 高維度空間的曲率

Poincaré

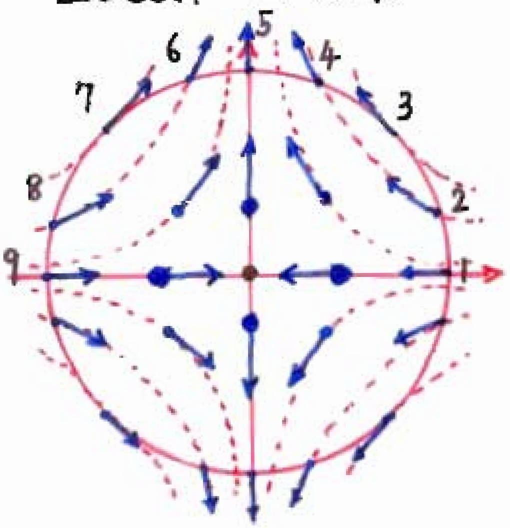
• 動力系統 (Dynamic Systems)

— Vector fields, flows, ODE's —

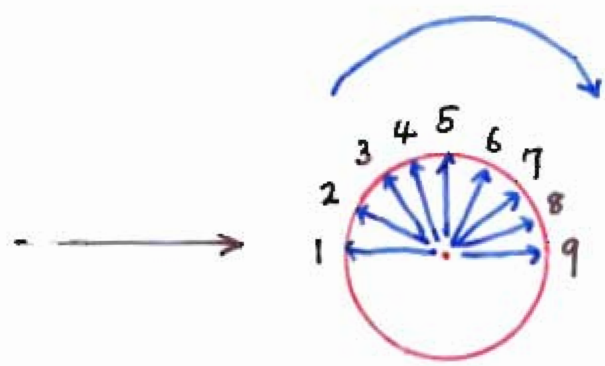


$X,\ g=2,\ \chi(X)=2-2\cdot 2=-2$

Local Index :



繞一圈



繞 k 圈, $k\in\mathbb{Z}$

則 Index $= k$.

- **Hopf–Poincaré Index Theorem** :

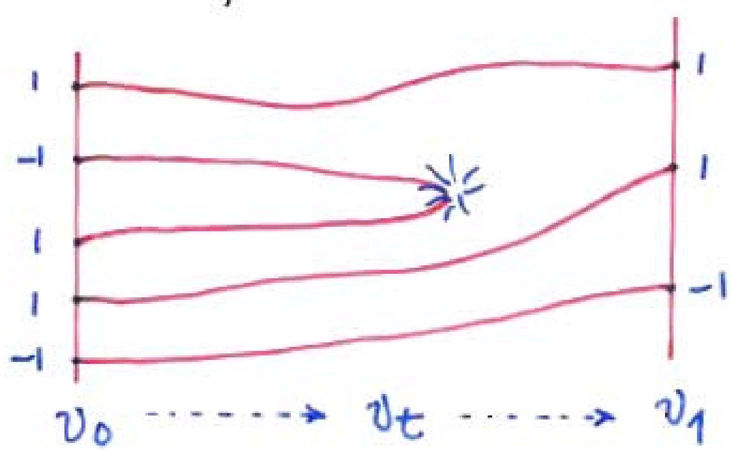
Let X be a compact oriented "space" and let v be a tangent vector field on X with isolated zeros, then

$$\boxed{\mathrm{Index}(v) = \chi(X)}$$

証明:

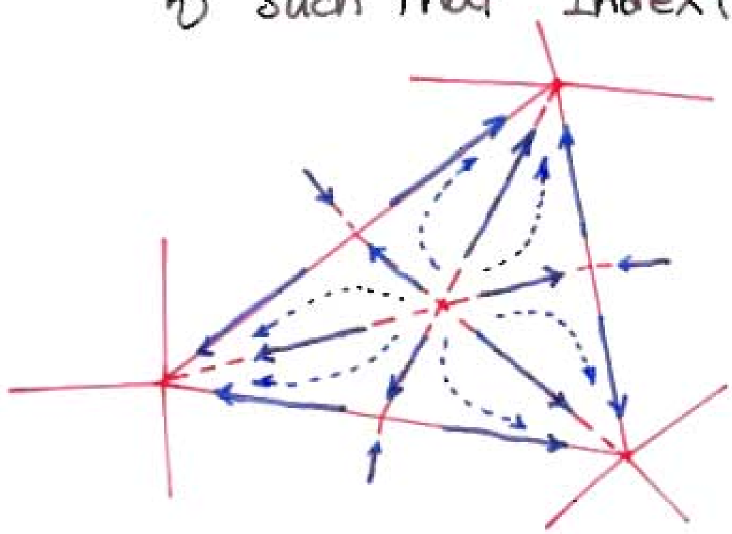
Step 1 : All such vector fields have the same total index :

* Idea : Connect v_0 and v_1 by a one parameter family of vector fields v_t, $0 \leq t \leq 1$.



正負電
相消於某
未知時刻

Step 2 : Explicit construction of a vector field v such that $\mathrm{Index}(v) = \chi(X)$:



点	:	1	p
線	:	-1	$-e$
面	:	1	$+f$
			$\Vert$
			χ

- 陳省身 (Chern, 1944) 將 Gauss–Bonnet 定理推廣至高維度的 Riemannian manifolds
- Gauss–Bonnet–Chern Theorem (special case)
  Let $X \subset \mathbb{R}^{n+1}$ be a compact oriented n-dim'l hypersurfaces, then

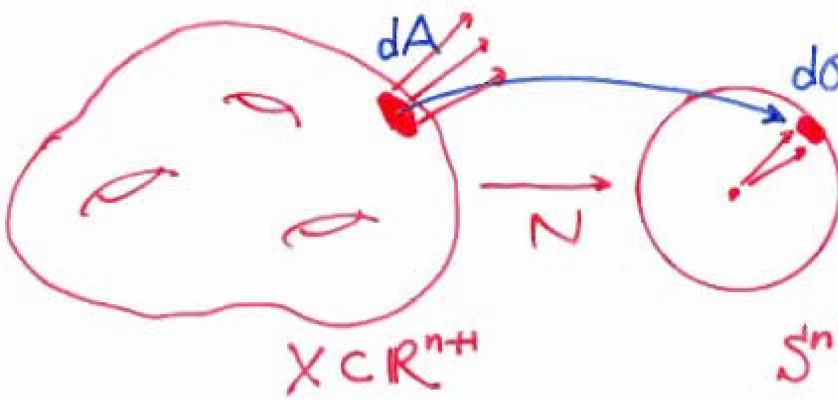
$$\boxed{\int_X K\,dA = \frac{\mathrm{Vol}(S^n)}{2}\cdot \chi(X)}$$

其中 $\chi(X) \overset{\text{def}}{=}$ 點 − 線 + 面 − (3dim) + (4dim) ⋯

證明：

由定义

$$K = \frac{d\sigma}{dA}$$



故 $\displaystyle \int_X K\,dA = \int_X \frac{d\sigma}{dA}\,dA = \int_X d\sigma$

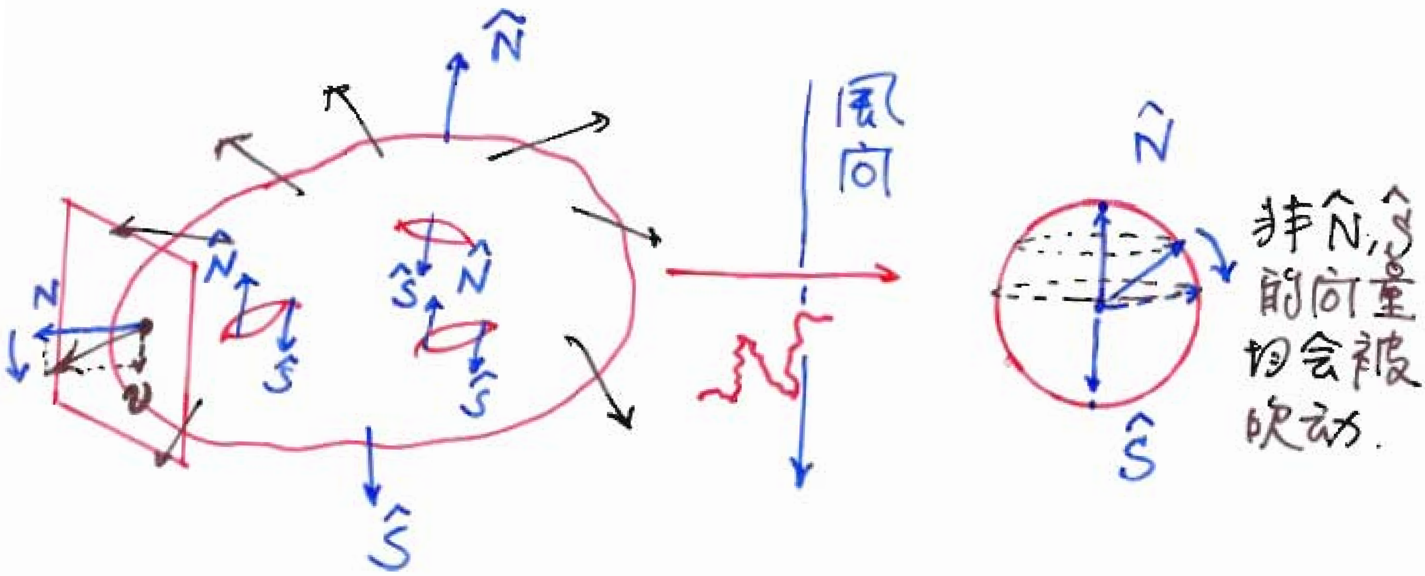
$$= \deg(N)\cdot \int_{S^n} d\sigma = \deg(N)\cdot \mathrm{Vol}(S^n)$$

其中 $\deg(N)$ 表示在 mapping N 作用之下，球面 S^n 被覆蓋的次數！

Question：想証 $\deg(N) = \frac{1}{2}\,\mathrm{Index}(v)$ for some tangent vector field v，但已有的 vector field N 是 normal，非 tangent ??

解決之道：微擾法 (perturbation method)

(間碟法，無風不起浪)



- Let v be the tangent vector field given by the projection of down to the tangent space.
- Then the zeros of v

$$\Updownarrow$$

Points on $\mathcal{X}$ such that $N(p) = \hat{N}$ or $\hat{S}$

- But the local index of v is 1 in both cases

so,

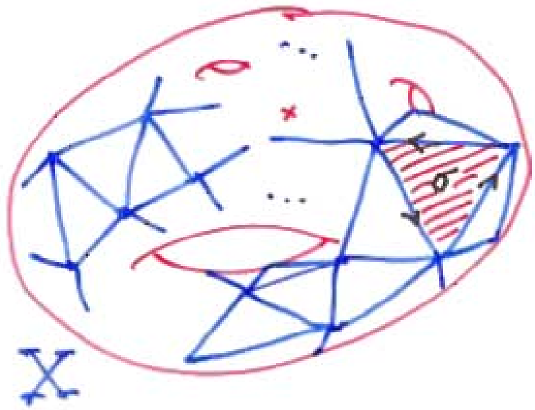
$$\begin{aligned} \deg(N) &= \deg(\;) \\ &= \hat{N}\text{ 被打到的次数} \\ &= \tfrac{1}{2}(\hat{N}\text{ 和 }\hat{S}\text{ 被打}\cdots) \\ &= \tfrac{1}{2}\,\mathrm{Index}(v) = \tfrac{1}{2}\chi(\mathcal{X}) \end{aligned}$$

(By Hopf-Poincare)

$$\Rightarrow \int_{\mathcal{X}} K\,dA = \deg N \cdot \mathrm{Vol}(S^n) = \frac{\mathrm{Vol}(S^n)}{2} \cdot \chi(\mathcal{X}) \quad \text{証完}$$

Poincaré 的另一個推廣 Euler number 的偉大貢獻 —— Homology theory (同調)

- X : A "space" with triangulation (oriented)



$$C_0(X) = \bigoplus_{p_i \text{点}} \mathbb{Z}\, p_i \cong \mathbb{Z}^p$$

$$C_1(X) = \bigoplus_{e_i \text{線}} \mathbb{Z}\, e_i \cong \mathbb{Z}^e$$

$$C_2(X) = \bigoplus_{f_i \text{面}} \mathbb{Z}\, f_i \cong \mathbb{Z}^f \quad \cdots$$

$$\cdots \to C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} \cdots C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

$$\boxed{\text{Key property : } \partial^2 := \partial \circ \partial \equiv 0}$$

故, $\operatorname{Ker} \partial_k$ = "Cycles" =: $Z_k(X)$
$\cup$
$\operatorname{Im} \partial_{k+1}$ = "Boundaries" =: $B_k(X)$

定义 : The k-th homology group of X :

$$H_k(X;\mathbb{Z}) := \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}} \equiv Z_k / B_k$$

- $H_k(X;\mathbb{Z})$ is independent of the triangulations, so are indeed "INVARIANTS" of X.
- ANTI-FUNCTORIAL : If $f : X \to Y$, then

$$f_* : H_k(X;\mathbb{Z}) \longrightarrow H_k(Y;\mathbb{Z})$$

Two Important Properties:

- Euler-Poincare characteristic

  Let $h_i(X) := \dim H_i(X;\mathbb{Z}) \otimes \mathbb{R}$

  (the Betti numbers), then

$$\chi(X) = \sum_i (-1)^i h_i(X) = h_0 - h_1 + h_2 - h_3 \cdots$$

証明: $h_0 - h_1 + h_2 - h_3 \cdots$

$$= (Z_0 - B_0) - (Z_1 - B_1) + (Z_2 - B_2) - (Z_3 - B_3) \cdots$$

$$= C_0 - (C_1 - \cancel{Z_1}) - (\cancel{Z_1} - (C_2 - \cancel{Z_2})) + (\cancel{Z_2} + (C_3 - \cancel{Z_3})) - \cdots$$

$$= C_0 - C_1 + C_2 - C_3 \cdots = \chi(X). \quad \square$$

(Remark: $0 \to B_k \to Z_k \to H_k \to 0$

$0 \to Z_k \to C_k \xrightarrow{\partial} B_{k-1} \to 0$)

- Poincaré Duality Theorem:

$$H_k(X) \otimes H_{n-k}(X) \xrightarrow{\cap} \mathbb{Z}$$

$$(\sigma, \tau) \longmapsto \sigma \cap \tau \in \mathbb{Z}$$

is a perfect pairing, so

$$H_{n-k}(X) \cong H_k(X)^* \quad \text{(dual space)}$$

Lefschetz (1923)

Fixed Point Theorem :

Let $f: X \longrightarrow X$ be a continuous mapping with (transversal) isolated fixed points, then

$$|\mathrm{Fix}(f)| = \sum_i (-1)^i \, \mathrm{Tr}\left(f_* : H_i(X) \longrightarrow H_i(X)\right)$$

- This is a generalization of Hopf-Poincare index theorem :

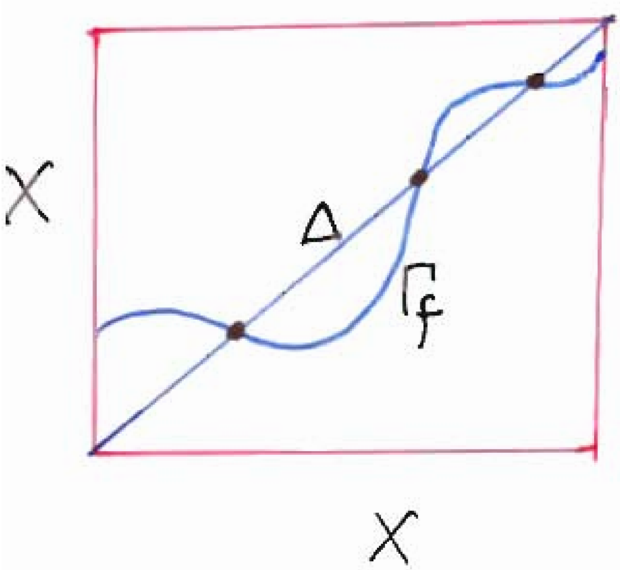
If v is a tangent v.f. on X,
Let Φ be the flow of v, then at time $t=1$,

$f := \Phi|_{t=1} : X \longrightarrow X$

is a map with $\boxed{\mathrm{Fix}(f) \overset{1-1}{\longleftrightarrow} \text{zeros of } v.}$

証明: (Sketch of the idea) :

對角線 $\Delta : X \longrightarrow X \times X$
$x \longmapsto (x, x)$



取 σ_i basis of $H_*(X;\mathbb{R})$,

$$\Delta = \sum_i \sigma_i^* \otimes \sigma_i$$

$$\Gamma_f = \sum_i (f_* \sigma_i) \otimes \sigma_i^*$$

故 $\mathrm{Fix}(f) = \Gamma_f \cap \Delta$

$$= \sum_i (\text{sign}) \underline{\langle f_* \sigma_i , \sigma_i \rangle} .$$

A. Weil (1949): Numbers of Solutions of Equations in Finite fields

X smooth algebraic variety, $\dim X = n$,

$$\boxed{\begin{aligned} f_1(x_1,\cdots,x_m) &= 0 \\ &\vdots \\ f_r(x_1,\cdots,x_m) &= 0 \end{aligned}}$$

with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, a finite field.

Let $X(\mathbb{F}_{p^k})$ = Solutions in $\mathbb{F}_{p^k} \subset \overline{\mathbb{F}_p}$

Frobenius map: $x \longmapsto F(x) := x^p$

$F: X(\overline{\mathbb{F}_p}) \longrightarrow X(\overline{\mathbb{F}_p})$ $\quad F f_i(x) = f_i(F(x))$

Then, $\boxed{X(\mathbb{F}_{p^k}) \equiv \mathrm{Fix}(F^k)}$ (Galois theory)

LFT $\Rightarrow$ 假若存在適當的 Topology 使 F 連續，且 $H_i(X)$ 可定義

$$|X(\mathbb{F}_{p^k})| = \sum_i (-1)^i \, Tr\left(F^k : H_i(X) \to H_i(X)\right)$$

Consider the Zeta function of X:

$$\log Z(X;t) := \sum_{k\in\mathbb{N}} |X(\mathbb{F}_{p^k})| \frac{t^k}{k}, \quad \text{then get}$$

- Rationality of the Zeta function:

$$Z(X;t) = \frac{P_1(t)\cdots\cdots P_{2n-1}(t)}{P_0(t)\, P_2(t) \cdots\cdots P_{2n}(t)}$$

$P_i(t)$ = characteristic polynomial of F on $H_i(X)$.

- Weil conjecture: α a root of $P_i(t)$
  If X is projective $\Rightarrow |\alpha| = p^{-i/2}$.