教育部第一屆高中基礎科學人才培育計畫

桃竹苗區

現代數學簡介 I, II, III, IV

(2001 秋季)

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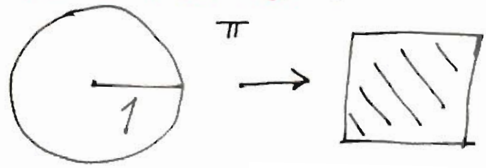
課程綱要

古典几何三大難題（尺規作圖 Ruler & Compass）

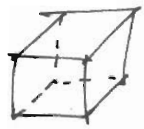
(i) 

a^2

圓化方

可否作 a 使得 $a^2=\pi$

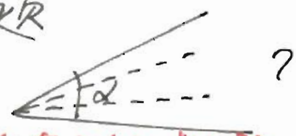
(ii) 倍立方問題



vol=1 可否找 a 使得 $a^3=2$

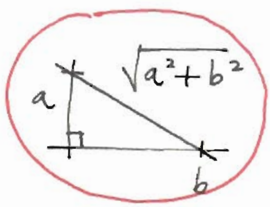
ie $\sqrt[3]{2}$ 可否作圖

(iii) 三分角問題



古希臘數學：數與形之對應

Q: 已知 a, b，作 $\sqrt{ab}$？



尚未登場

$\begin{cases} a, b \mapsto a+b,\ a-b, \\ a \mapsto 2a,\ 3a,\ \ldots\ldots \end{cases}$

$a \mapsto \frac{a}{2}, \frac{a}{3}, \ldots \frac{a}{n}$ $n \in \mathbb{N}$

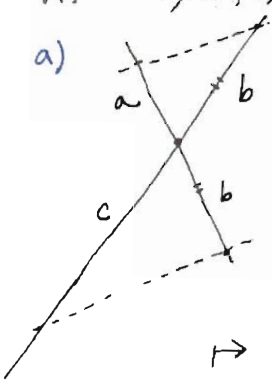
$\mathbb{Q}\langle a, b, c \ldots \rangle$

let $a=1$, $\mathbb{Q}(b, c, d, \ldots)$

All can be constructed

in general, all degree 1 homog. rational expr can be constructed

相似形的應用：

a) 

$\frac{a}{b} = \frac{b}{c}$

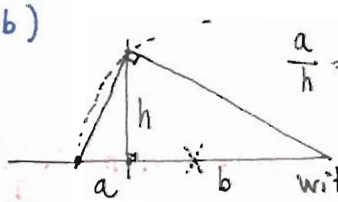
$\Rightarrow b^2 = ac$

ie know a, b may construct $c = \frac{b^2}{a}$

(in particular, if know 1, can get b^2)

$\mapsto \frac{c^2}{a} = \frac{1}{a} \cdot \frac{b^4}{a^2} = \frac{b^4}{a^3} \mapsto \cdots$ if $a=1$, get b^4

b, b^2 get $\frac{(b^2)^2}{b} = b^3$, b^4, $\frac{(b^3)^2}{b} = b^5$, b^6

b) 

$\frac{a}{h} = \frac{h}{b} \Rightarrow ab = h^2$

get $h = \sqrt{ab}$

Set $b=1$, get $1/a$

with 1, as before, can get $(\sqrt{ab})^2 = ab$

Theorem. If $a, b, \cdots, z$ are given "number" by some line portion, then all the constructible numbers are of the form all degree 1 expression in All quadratic extensions (ie $\sqrt{\cdots}$) of

$$\mathbb{Q}(a, b, \cdots, z).$$

pf: it can be constructed by steps I — IV
if can be constructed, then each steps involves at most intersection of lines or circles In Analytic coordinate system equations:

$$\begin{cases} ax + by + c = 0 \\ (x - x_0)^2 + (y - y_0)^2 = r \end{cases}$$

etc.

st a, b, c, x_0, y_0, r constructible #

倍立方 if $\sqrt[3]{2} = \frac{a}{b}$, $a, b \in \mathbb{N}$, $(a, b) = 1$

then $2b^3 = a^3$, $\Rightarrow 2|a$ so $a = 2\bar{a}$

$\Rightarrow 2b^3 = (2\bar{a})^3 = 8\bar{a}^3 \Rightarrow b^3 = 4\bar{a}^3$

$\Rightarrow 2|b \Rightarrow 2|(a, b)$, ※

Fermat's ~~A~~ descent method

三分角 $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\uparrow$ given a, $4x^3 - 3x + a = 0$

say, $a = 1$.

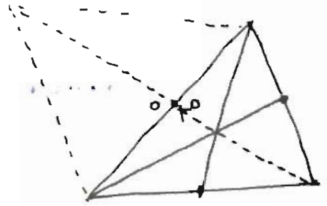
如何解 $x^3 - \frac{3}{4}x + \frac{a}{4} = 0$

in general, $x^3 + px + q = 0$? Ans: 不能尺規作圖

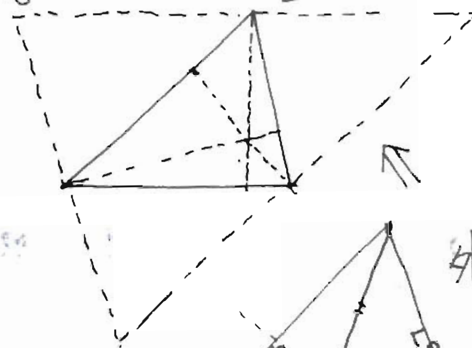
圓化方: π can't be got from square roots, or more generally, it is transcental (Harder)

三角形的代數與幾何

重心

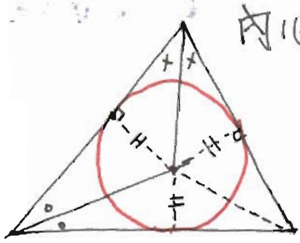


垂心

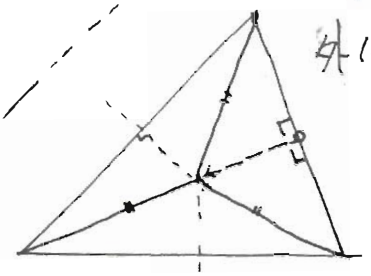


All these "centers" can be constructed by ruler & compass

內心

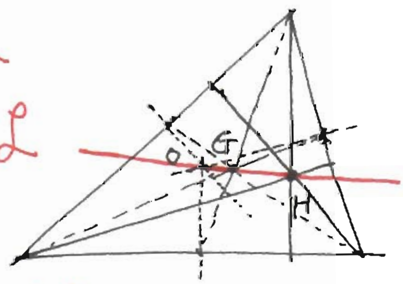


外心



Ex 尤拉線

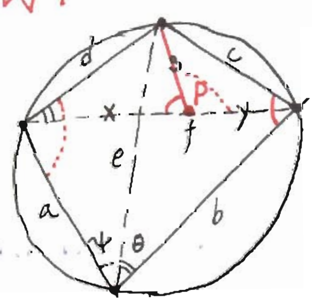
$\mathcal{L}$



提示 使用向量

$\overrightarrow{OH} = 3\overrightarrow{OG}$

Ex 西瓦(師氏)



$$\begin{cases} \dfrac{c}{y} = \dfrac{e}{a} \\ \dfrac{b}{x} = \dfrac{e}{d} \end{cases} \qquad \begin{aligned} ye &= ca \\ xe &= bd \end{aligned}$$

$$\Rightarrow ef = ac + bd$$

弦函數(源自天文觀測)

eg $c = s(\theta)$ $\quad f = s(\theta+\psi)$

$d = s(\psi)$

take $e =$ 直徑 $= 1$, get

$$s(\theta+\psi) = f = (ac+bd) = (a\, s(\theta) + b\, s(\psi))$$

$$= \left(\sqrt{1-s^2(\psi)}\; s(\theta) + \sqrt{1-s^2(\theta)}\; s(\psi)\right)$$

$$= s(\theta)\, c(\psi) + c(\theta)\, s(\psi)$$

在此定義下

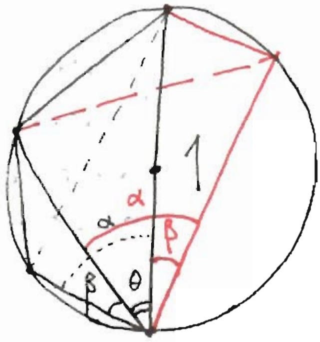
$s = \sin$ 最自然 (正弦)

$c = \cos$ 較不自然 (餘弦)

正弦和角公式

Ex. 最好的方法是華氏定理 (與座標系)

Q：餘弦和角公式？ let $\theta = \alpha - \beta$ so $\theta + \beta = \alpha$



$$c(\alpha-\beta) \overset{?}{=} c(\alpha)c(\beta) - s(\alpha)s(\beta)$$

can't get through! (Challenge)

- $c(\theta)s(\beta) + c(\beta)s(\theta) = s(\theta+\beta)$

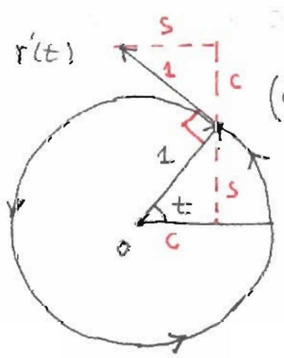
使用 c 與 s 的對偶關係

$$\Rightarrow \quad c(\alpha+\beta) = s\left(\frac{\pi}{2}-\alpha-\beta\right)$$
$$= s\left(\frac{\pi}{2}-\alpha\right)c(-\beta) + c\left(\frac{\pi}{2}-\alpha\right)s(-\beta)$$
$$= c(\alpha)c(\beta) - s(\alpha)s(\beta)$$

$$\Rightarrow \quad c(2\alpha) = c^2(\alpha) - s^2(\alpha) = c^2(\alpha) - (1 - s^2(\alpha))$$
$$= 2c^2(\alpha) - 1$$

$$\Rightarrow \quad c(3\alpha) = c(2\alpha+\alpha) = c(2\alpha)c(\alpha) - s(2\alpha)s(\alpha)$$
$$= (2c^2(\alpha)-1)c(\alpha) - 2s(\alpha)c(\alpha)s(\alpha)$$
$$= 2c^3(\alpha) - c(\alpha) - 2(1-c^2(\alpha))c(\alpha)$$
$$= 4c^3(\alpha) - 3c(\alpha) \;\#$$

三角函數的微積分 (Newton, Lebnitz)



$(\cos(t), \sin(t)) = \vec{r}(t)$ $\quad \vec{r}'(t) \perp \vec{r}(t)$

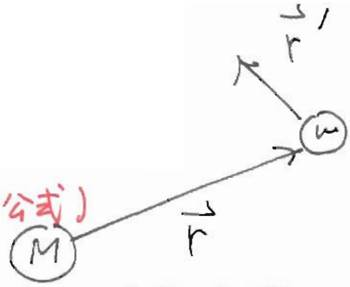
等速圓周運動 $\quad |\vec{r}'(t)| = 1$

$$\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$
$$= (\cos'(t), \sin'(t))$$
$$= (-\sin(t) \;\; \cos(t))$$

Ex. Lebnitz rule $\quad (fg)' = f'g + fg'$

Also for vector valued functions, eg $\vec{f}\cdot\vec{g}$, $\vec{f}\times$

" $F = \frac{GMm}{r^2}$ "



$$\begin{cases} \vec F = -\dfrac{GMm}{r^2}\hat r & \text{牛頓重力(引力公式)} \\ \vec F = m\vec a = m\vec r\,'' & \text{牛頓第二運動定律} \end{cases}$$

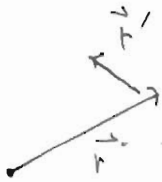
$\hat r := \frac{\vec r}{|\vec r|}$ 單位向量

ie $\quad \vec r\,'' + \frac{GM}{r^3}\vec r = 0$

Kepler's Law:
克卜勒行星運動定律

$$(\vec r\,'\times\vec r)' = \underbrace{\vec r\,''\times\vec r}_{0} + \underbrace{\vec r\,'\times\vec r\,'}_{0}$$

中心力場: for any central force field $\quad r'' = \frac{\vec F}{m} \parallel \vec r$

① 

Let $\vec n := \vec r\,'\times\vec r$

$\Rightarrow$ $\vec n$ is constant w t

ie any $\vec r(t) \perp \vec n$, so $\vec r(t) \in$ plane $\vec n^{\perp}$

say (x,y) plane

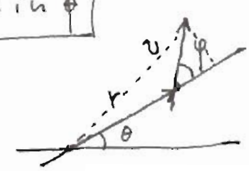
M = origin

② $\vec r = r(\cos\theta, \sin\theta)$ $\quad$ ∴ $r(t), \theta(t)$

$\vec r\,' = r'(\cos\theta, \sin\theta) + r(-\sin\theta, \cos\theta)\,\theta'$

$$\begin{vmatrix} r\cos\theta & r\sin\theta \\ r'\cos\theta - r\theta'\sin\theta & r'\sin\theta + r\theta'\cos\theta \end{vmatrix}$$

$\boxed{r\,v\sin\phi} = r\cdot r\theta'$



$= r^2\theta'$ this is a constant

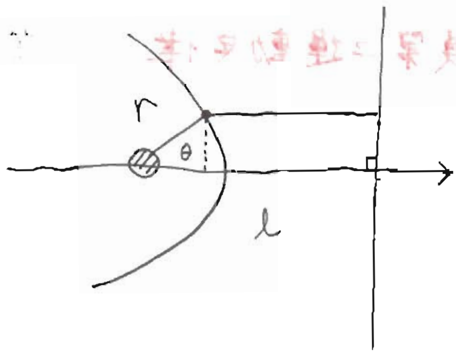
Both ①, ② are true for central force fields

$\vec r = r\vec e_1$; $\vec r\,' = r'\vec e_1 + r\vec e_1\,' = r'\vec e_1 + r\theta'\vec e_2$

$\vec r\,'' = r''\vec e_1 + r'\theta'\vec e_2 + r'\theta'\vec e_2 + r\theta''\vec e_2 + (-r\theta'^2\vec e_1)$

$= (r'' - r\theta'^2)\vec e_1 + (2r'\theta' + r\theta'')\vec e_2$

Review of ellipse "etc" (圓錐曲線，二次曲線)



$$\frac{r}{l - r\cos\theta} = e$$

$$r = le - re\cos\theta$$

$$r(1 + e\cos\theta) = le$$

$$r = \frac{le}{1 + e\cos\theta}$$

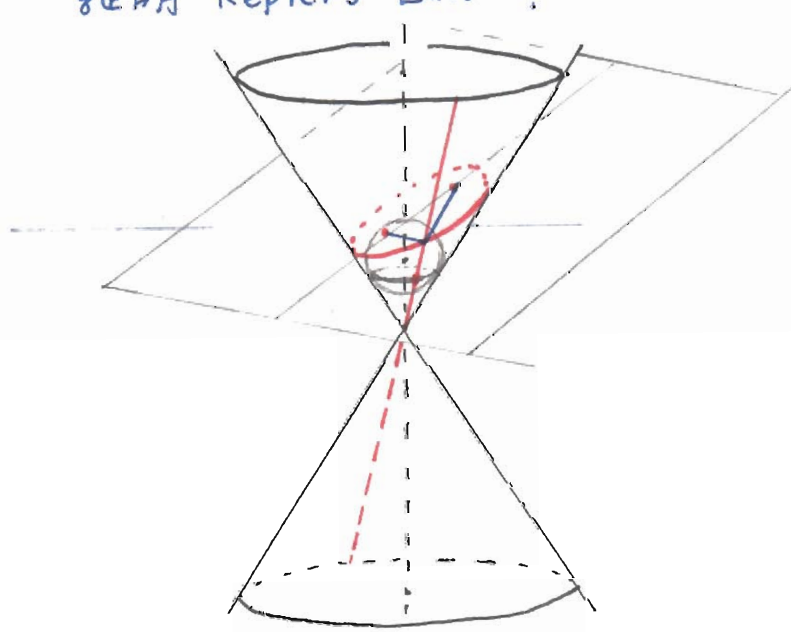
$$r(\theta) = \frac{1}{\frac{p}{\lambda^2} + A\cos\theta} = \frac{\lambda^2/p}{1 + A\,\lambda^2/p\,\cos\theta}$$

牛頓發明微積分
證明了 Kepler 定律
Principia（稍後討論）

so 離心率：$e = A\lambda^2/p$

準線：$l = 1/A$.

Q 挑戰 如何使用古典幾何與力的平衡（不使用微積分）證明 Kepler's Law ?



$$\vec{r}'' + \frac{GM}{r^2}\hat{r} = 0 \qquad (\text{let } \rho = GM)$$

③ $\Leftrightarrow \begin{cases} r'' - r\theta'^2 + \dfrac{\rho}{r^2} = 0 \\ 2r'\theta' + r\theta'' = 0 \quad (\text{this} \Leftrightarrow r^2\theta' = \text{constant } \lambda) \end{cases}$

$$(r^2\theta')' = 2rr'\theta' + r^2\theta'' = r(2r'\theta' + r\theta'')$$

Goal: Eliminate t to get eq'n of r with θ as variable (1st integral)

Conservation v.s Symmetry: let $u = \frac{1}{r}$, so $\theta' = \lambda u^2$

$$\frac{dr}{d\theta} = \frac{dr}{dt}\Big/\frac{d\theta}{dt} = r'/\theta' = \frac{1}{\lambda}r^2 r'$$

or, $r' = \theta' \dfrac{dr}{d\theta}$

$$r'' = \theta''\frac{dr}{d\theta} + (\theta')^2\frac{d^2r}{d\theta^2}$$

* Eq'n: $\theta''\dfrac{dr}{d\theta} + (\theta')^2\left(\dfrac{d^2r}{d\theta^2} - r\right) + \dfrac{\rho}{r^2} = 0$

" $\dfrac{du}{d\theta}, \dfrac{d^2u}{d\theta^2}$ " ; $\dfrac{dr}{d\theta} = \dfrac{-1}{u^2}\dfrac{du}{d\theta}$, $\dfrac{d^2r}{d\theta^2} = \dfrac{2}{u^3}\left(\dfrac{du}{d\theta}\right)^2 - \dfrac{1}{u^2}\dfrac{d^2u}{d\theta^2}$

$$\theta'' = -2r'\theta' u = -2\theta'\frac{dr}{d\theta}\theta' u = -2(\theta')^2 u \frac{-1}{u^2}\frac{du}{d\theta}$$

plug-in *: $\cancel{-2\theta'^2\dfrac{1}{u^3}\left(\dfrac{du}{d\theta}\right)^2} + \cancel{2\theta'^2\dfrac{1}{u^3}\left(\dfrac{du}{d\theta}\right)^2} - \dfrac{\theta'^2}{u^2}\dfrac{d^2u}{d\theta^2} - \dfrac{\theta'^2}{u} + \rho u^2 = 0$

Magic

$$-\lambda^2u^2\frac{d^2u}{d\theta^2} - \lambda^2u^3 + \rho u^3 = 0$$

ie $\dfrac{d^2u}{d\theta^2} + u = \dfrac{\rho}{\lambda^2}$ ← constant

$$\frac{d^2\left(u - \frac{\rho}{\lambda^2}\right)}{d\theta^2} + \left(u - \frac{\rho}{\lambda^2}\right) = 0$$

solution $u(\theta) = \dfrac{\rho}{\lambda^2} + A\cos(\theta - \theta_0)$

ie $r(\theta) = \dfrac{1}{\frac{\rho}{\lambda^2} + A\cos(\theta)}$

How to solve ODE ?

sep of variable

1st order Eq'n $y' + p(x) y = g(x)$

2nd order Eq'n constant coefficient

$$y'' + 2y' - 3y = g(x)$$

$$\| $$

$$(D-1)(D+3) y = g(x)$$

Think as $(D-1) g = g(x)$, once get g then solve $(D+3) y = g$

Consequence : order n will get exactly n constants (i.e. solution space is n-dim'l)

$$y' - \lambda y = g(x) \qquad *$$

$$e^{-\lambda x} y' - \lambda e^{-\lambda x} y = e^{-\lambda x} g(x)$$

$$(e^{-\lambda x} y)' \quad \Rightarrow \quad y = e^{\lambda x}\left[\int e^{-\lambda x} g(x) + c\right]$$

trivial case $g(x) = 0$

notion of "vector space"

$$P(D)\, e^{\lambda x} = p(\lambda)\, e^{\lambda x}$$

characteristic polynomial

all candidates

for multiple roots ?

e.g. in $*$ $g(x) = c_1 e^{\lambda x}$

then get

$$y = e^{\lambda x}(c_1 x + c_2)$$

new base $e^{\lambda x}$, $x e^{\lambda x}$, etc

結語 Unification of Math