幾何, 弦論與量子環的不變性
Geometry, Strings and Quantum Invariance

(In celebration of the opening of CMTP)

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1. Prologue

- In the early 20th century, i.e. about 100 years ago, mathematics and theoretic physics took a very close relationship with each other.

- In large scale, Riemannian geometry forms the natural language in Einstein's theory of general relativity. At the same time, relativity provides natural subjects to study for frontier researches in differential geometry.

- In small scale, quantum mechanics had involved various subjects in mathematics including

- Analysis (variational methods, operator theory in Hilbert spaces, PDE),
- Geometry (Hamiltonian mechanics, symplectic geometry),
- Algebra (group represetation theory) and
- Statistics/probability.

- Its development is completely parallel to those corresponding mathematical subjects.

- That was a golden age, the achievement had formed a solid foundation for modern sciences and technologies.

- However, soon (before the 2nd world war) mathematics and physics had faced intrinsic challenges from their own fields.

- There were consistency problems in the mathematical foundation. Mathematicians were forced to investigate their axiom system and logic. No branches in mathematics could stay away from this serious challenge which might affect the thousand years basis of Mathematics.

- Except few instances like the invention of computers (Von Neumann), applied mathematics had been out of the core Math. The Bourbaki school in Paris was typical in this period. Some mathematicians were proud of being working on researches totally unrelated to the real world.

- In mid 70's, through Gauge Theory and vector bundle theory, mathematics and physics seemed to find their common playground again. Yet, mathematicians like Dieudonne still insisted that Number Theory will save a place for math that will never be polluted by sciences.

- Researches in recent years showed that even the relation between number theory and physics is getting closer.

- On the other side, in physics, it became a dream to search for an ultimate unified theory to explain all basic forces from both the large scale and small scale (the so called Grand Unified Theory).

- Many attempts had required non-existing mathematical models. The Feymann path integral is perhaps the most basic (popular?) one.

- The integral has to be taken in an infinite dimenisonal path space. So far such a theory still could not be constructed rigorously. But it has been widely used in physics for many decades, and its crucial role is non-challengeable.

- String Theory is a theory attempting to unified all basic forces. After several revolutions, it became mature in early 80's. Through many people's efforts, including Witten's, some people believe that - The Theory of Everything - has been found.

- However, string theory uses huge amount of deep mathematics. Also the related quantities can't be observed in current experiments. Thus most physicists do not believe it. They simply regard it as a tough mathematical game (or magic).

- Even worse, string theory attempts using far more mathematical argument without rigorous foundation. Thus most mathematicians did not believe in it as well.

- In 1980's, a few dramatic events completely changed our, or some of our, viewpoints on string theory.

- First in 1983 Donaldson used the very basic topological property of solution space of Yang-Mills equations to achieve breakthroughs on 4 dim manifolds. E.g. there are infinitely many non-standard structures on R^4.

- This is the first example that modern physical ideas feedback advanced mathematical research (through rigorous mathematical formulation).

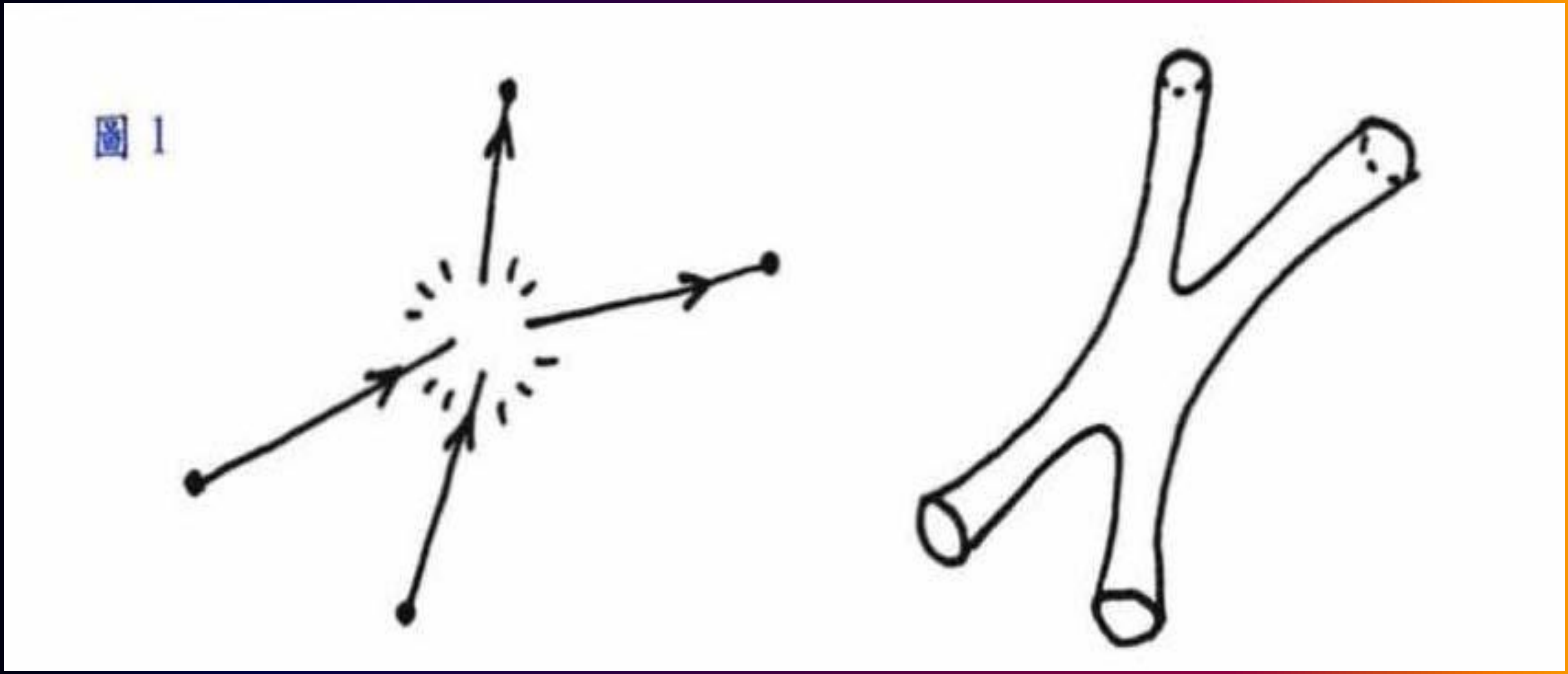
- At that time most people still regarded this as an isolated event. (Why Yang-Mills?)

- A few years later, further impact came from the mirror symmetry phenomenon, predicted by string theory, on Calabi-Yau manifolds and their moduli spaces.
- This indeed opened a new era since the most frontier ideas in Math and Phys had merged again.
- The aim of this talk is to report on this development from a geometric viewpoint as well as on my own related researches in recent years.

2. Geometric View of Strings

- Since mid 20th century, many new particles were found. String theory says that particles are indeed strings (an interval or a circle).
- Different vibration patterns determine their particle behavior under the classical observation.
- The trajectory of motion in string theory becomes a 2 dimensional surface (world sheet).

- This model overcomes the puzzle arising from singularities. E.g. collision is indeed a smooth process.

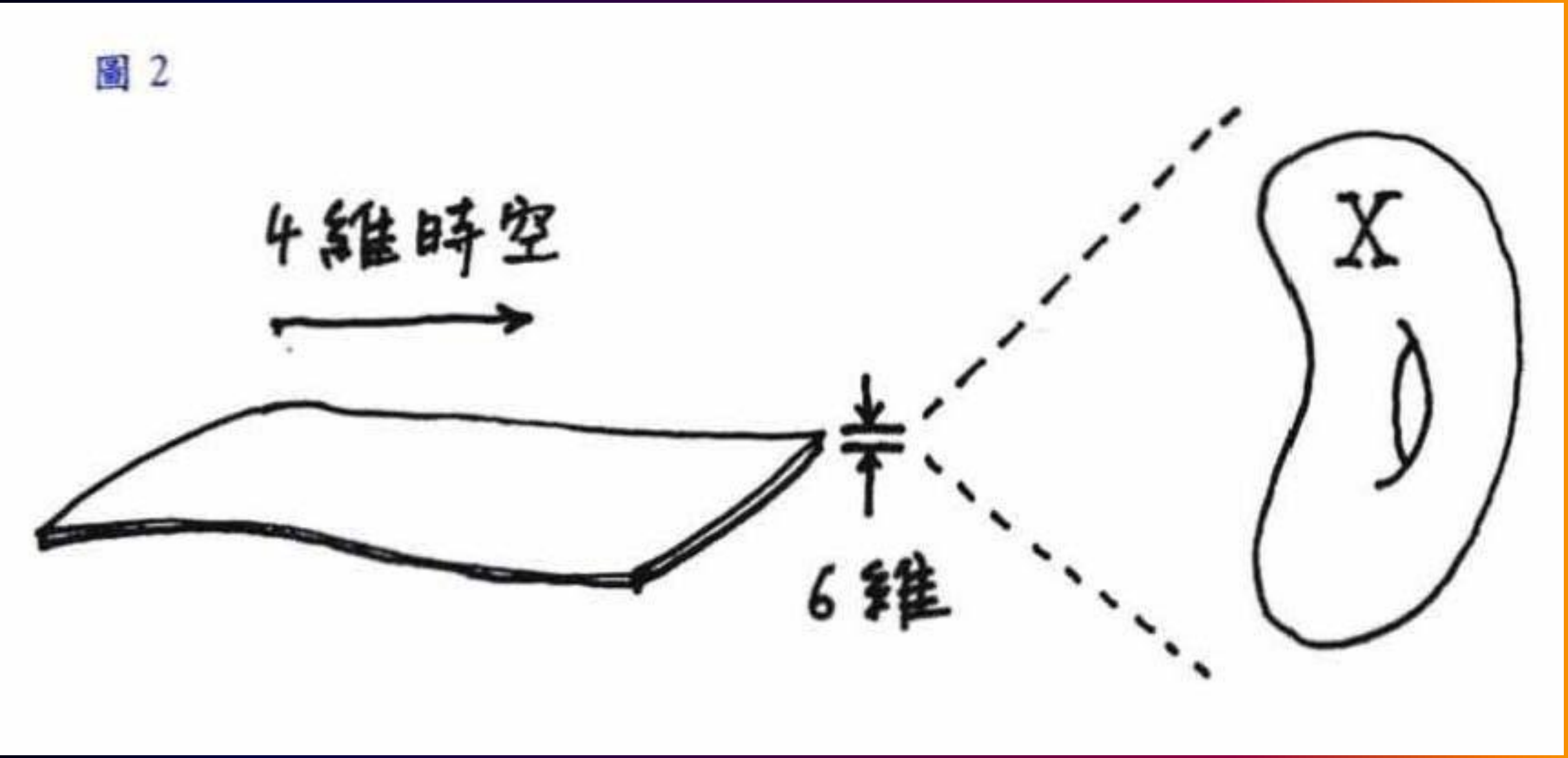


$D = 26, 10$, or 4?

- Yet, according to ST the space-time dimension is 26, instead of 4 . This predicts the existence of extra dimensions as well as one or several ways to approximate the classical limit (compactifications).

- Already in the 80's, there were 5 known ST's and corresponding compactifications. It is the Heterotic ST to be addressed here. It contains a special 16D gauge group $E_8 \times E_8$, so the space-time can be constructed on a $10 = 26 - 16$ dim model.

- For the extra 6 = 10 − 4 dim, it rolls into a tiny scale curved manifold X, so small to be observed in the normal scale of daily life.



Quantum effects

- But since the scale of X is small, the quantum effects ought to be apparent. Through physical theory, the effects turns out can be described mathematically in exact geometric terms.

- Although we can’t test ST in current experiments, it is possible to test it through mathematical proofs!

Calabi-Yau manifolds

- In general, the compactification from 10D to 4D relies on a 6D manifold X, which has a metric g satisfying the (Riemmanian) Einstein's vacuum equation

$$\mathrm{Ricci}(g) = 0.$$

- All these Ricci flat solutions g form the moduli space of string theory, denoted by M(g) or simply by M.

- Heterotic theory is an N = 2 super-symmetric conformaly invariant quantum field theory.
- It implies that X has a complex structure J compactible with the Ricci flat metric g, i.e. holonomy = SU(3).
- The general constructions of such X are dated back to Shing-Tung Yau (丘成桐) in his 1976 paper on proving the Calabi conjecture. So (X, J, g) is now called a Calabi-Yau manifold.

Which one?

- There are many Calabi-Yau's and string theory does not indicate the one which leads to our world.
- As no Calabi-Yau is superior than the others, a brave guess is that any Calabi-Yau should do the job - Namely all the corresponding QFT's are all equivalent!
- This leads to puzzels immediately since different Calabi-Yau's could have different topology!!

Local structure of moduli

- We need some more geometric concepts to proceed. For general (X, J), the metric g uniquely corresponds to a symplectic structure w: A non-degenerate closed 2 form

$$dw = 0,$$

w is also called a Kaehler metric.

- Yau's theorem implies that every [w] (cohomology class of w) uniquely corresponds to a Ricci flat g.

- So there is a local 1-1 correspondence between Ricci flat g and (J, [w]), i.e. M(g) = M(J) x M(w) locally. M(J) is the complex moduli of X and M(w) is the Kaehler (class) moduli.

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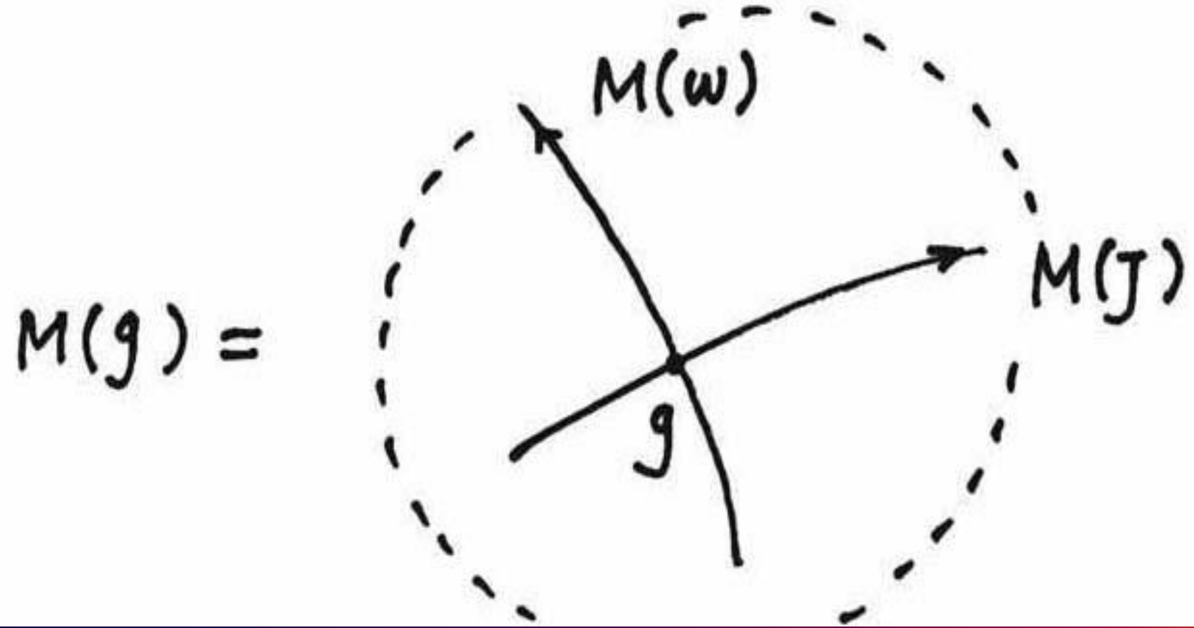


Hilbert spaces

- The tangent space of M(J) at X is $H^1(X, T)$, and the tangent space of M(w) at X is $H^1(X, T^*)$, here T is the holomorphic tangent bundle of X.

- These vector spaces and higher Dolbeault cohomology groups form the Hilbert space of states H = H(X) of the QFT. Elements in them are called fields.

- According the complex manifold theory, every field corresponds to a tensor field on X.

3. Mirror Symmetry

- In the heterotic theory, denote by (Q, Q') the left/right generators of the $N = 2$ susy algebra $u(1) \times u(1)$, then $H^1(T)$ and $H^1(T^*)$ are the $(1, 1)$ and $(-1, 1)$ eigenspaces.

- The choice of Q and Q' is only up to sign, hence we may as well choose (−Q, Q') as generators.

- Key point: When we use Calabi-Yau X to construct H(X), the sign-change switches the types of the two eigenspaces. But $\dim H^1(X, T) \neq \dim H^1(X, T^*)$!?

Exchanging moduli

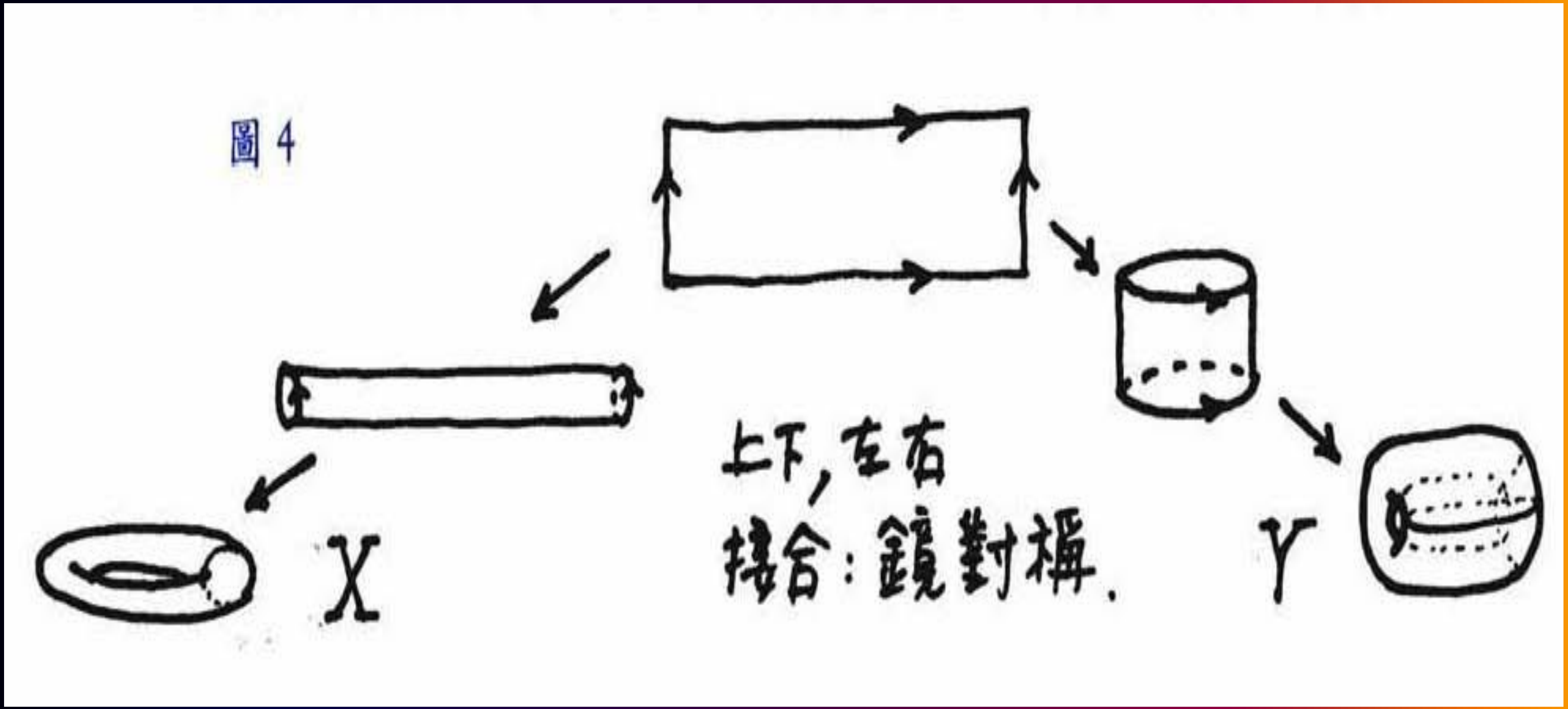
- There must be another Calabi-Yau Y such that

$$H^1(Y, T) = H^1(X, T^*),$$
$$H^1(Y, T^*) = H^1(X, T).$$

- Since X and Y should lead to the same QFT, we expect that X and Y exchange their M(J) and M(w).
- This is called Mirror Symmetry, and Y is the mirror manifold of X.

Mirror symmetry

a naïve illustration



Correlations

- The Hilbert space H involves only the number of fields. The full quantum theory describes also the correlation functions between fields a, b, c, … denoted by

$$\langle a, b, c, \ldots \rangle = \int \text{(some path integral)}$$

- Here the variables of the functions are the string moduli parameters in M(g).
- We consider twisted theories when we set restrictions on the variables.

From SCFT to AG

- According to the principle of path integral, correlation can be calculated as weighted integral among all possible paths (world sheets) of strings. The weight of each path is determined by an action functional. As said, this infinite dimensional integral is still lacking a rigorous mathematical definition.

- For Conformal FT, using localizations on weights, string theorists may reduce the integral (in twisted theories) to an integral over finite dimensional moduli. This transforms the problems into algebraic geometry.

A = Gromov-Witten theory

- For heterotic theory, there are A-model (J fixed, [w] varies) and B-model ([w] fixed, J varies) as twisted theories. (Also known as σ models.)

- In A-model, correlation becomes intersection theory in the moduli of holomorphic maps from Riemann surfaces to X, now called Gromov-Witten invariants.

- This is classically known as enumerative geometry, with many unsolved problems since hundred years ago.

Quantum ring

- Before 1990, algebraic geometers had not found systematic ways to handle these problems, even on P^2.

- In the classical limit, correlations are reduced to topological intersections in X, in the ring $H^*(X)$.

- In A-model, through the WDVV equations, Vafa found that all genus $g = 0$ correlations gives rise to a new big quantum cohomology ring structure on $H^*(X)$, denoted by $QH^*(X)$. The existence of ring structure already solves the enumerative problem in the P^2 case.

B = Kodaira-Spencer theory

- In B-model, the correlations are called the Yukawa coupling. At least for the genus g = 0 case they can be computed by Kodaira-Spencer theory in algebraic geometry.

- Specifically, the correlation functions satisfy the Picard-Fuchs equations arising from the theory of variations of Hodge structures. Thus they can be solved by methods in classical differential equations.

A(X) = B(Y)

- If X and Y are mirror manifolds to each other, the striking consequence of mirror symmetry is that it switches the A-models and B-models of X and Y.

- Hence the quantum ring on X, which is difficult to compute, can be transformed into the much easily handled Picard-Fuchs equations on Y!

Candelas formula

- Given a Calabi-Yau X, mirror symmetry predicts the existence of Y, but says nothing on the constructions.

- The first explicit construction is due to Greene and Plesser in 1990: Orbifold construction on quintic CY hypersurfaces in 4 dim projective space.

- In 1991, by mirror symmetry, Candelas, de la Ossa, Green and Parkes used this Y to derive the exact formulae for all genus 0 enumerative invariants on X.

Mirror conjecture

- Mathematically, the Candelas formula can only be regarded as a conjecture, but the power to make such a prediction was beyond the ability of Math at that time.

- Mathematicians started to watch the message sent from nature. To string theorists, no doubt they are confident with the predictions. Still, they expect for rigorous mathematical justifications.

- Through many efforts, this mirror conjecture was finally solved by Givental and Lian-Liu-Yau in 1996. This is the first key mathematical justification for string theory.

Duality

- In the 90's, Batyrev extended the orbitfold constructions to toric geometry. He used reflexive polytope to achieve some combinatorial duality.
- The real breakthrough is the Strominger-Yau-Zaslow conjecture in 1996. They proposed geometric constructions of mirror manifolds using T-duality.
- Duality is the most important direction of ST since 1990's. The 5 known ST's are all equivalent to a single theory under duality: Witten's M theory (physically)!

SYZ conjecture

- SYZ predicts that complex 3D Calabi-Yau X admits a fibration structure over S^3, with general fiber being special Lagrangian 3D tori T^3.
- The mirror Y is simply the dual fibration of $X \to S^3$.
- This is now one of the hottest research directions in the study of Calabi-Yau manifolds.

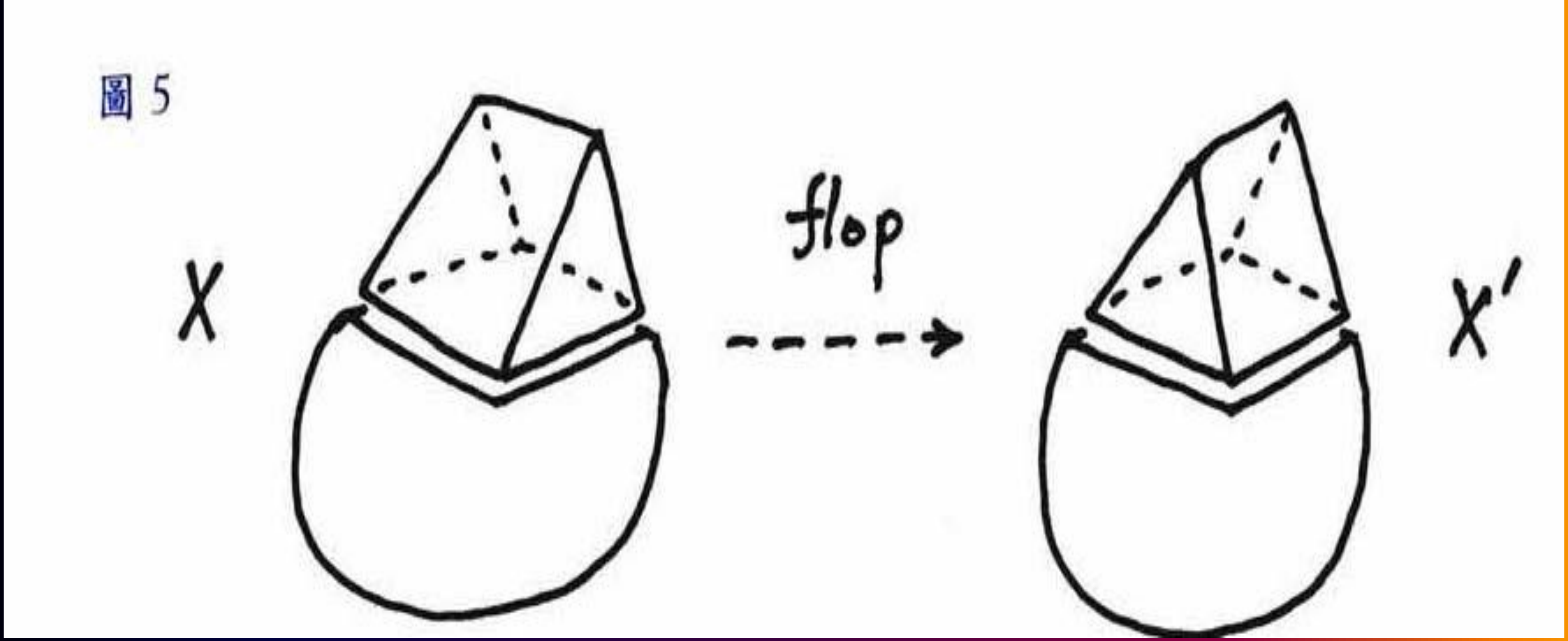
4. K equivalence and flops

- In Heterotic theory, there is a more basic observation which connects to algebraic geometry: If the compactified Calabi-Yau X and X' differs only within a small range around sub-manifolds, we want to know how the QFT's correspond.

- Type I: Birational case. In algebro-geometric language, X and X' are a pair of 3D birational minimal models.

- Kollar and Mori in 1990 proved that X can be connected to X' though flops (space surgery). 3D flops preserve complex moduli M(J) and cohomology groups. So X and X' have the same B-model and same number of fields in A-model.

Type I: Flops (not necessarily for Calabi-Yau)



Invariance of QH

- But the classical intersection product is not preserved under flops. Thus it is expected that X and X' have isomorphic A-model and quantum rings.

- Yet, the isomorphism can not be achieved in the naïve way. Witten in 1992 noticed the necessity to use analytic continuations in the A-model moduli M(w), since the M(w)'s for X and X' have no common part in H^2.

- In 3D, this was solved by Li-Ruan in 2000. Thus I am interested in the higher dim case and Type II case: Extremal transitions. (Here we discuss only HD Type I.)

Minimal model program

- The central topic in HD algebraic geometry is Mori's minimal model program (MMP) invented in 1982. Mori finished the 3D MMP in 1988. However, the existence in HD and non-uniqueness are big issues of the MMP.

- Recently Hacon and McKernan combined Siu (蕭蔭堂) and Shokurov's works and obtained important existence results.

- Also Kawamata proved that birational minimal models can be connected by flops. Unfortunately HD flops are rather complicate and hard to understand.

Ordinary flops

- In 1998 I raised the notion of K-equivalence in order to study the non-uniqueness. K-equivalence is a crucial generalization of flops. It allows us to extend the study on Calabi-Yau or minimal models to the general cases.

- There is no hope to classify HD flops, I thus try to modify classical MMP by adding symplectic deformations into it, with the hope to decompose K-equiv into ordinary flops.

- This holds in 3D, which is crucial in Li-Ruan's theorem. But the only known proof uses MMP and classification of 3D singularities. It can not be generalized to HD.

My earlier works

- By extending Kollar's 3D results, I had shown around 2000 that the Betti and Hodge numbers are invariant under K-equiv. I also found all the curvature integrals that are invariant under K-equivalence.

- Curvature integrals are exactly Chern numbers, e.g. Euler number = Gauss-Bonnet integral. The result is that the K-invariant integrals are precisely the complex elliptic genera. They can be interpreted as the equivariant index of Dirac operators on loop spaces.

Topological evidence

- This has important applications on manifolds with Ricci curvature close to each other. It is conjectured that they can be connected to each other by ordinary flops up to symplectic deformations.

- Modulo complex cobordism, this conjecture follows from my elliptic genus result. Thus it gives a partial topological solution to the non-uniqueness problem in all dimensions and in a more general context.

- Q: Replace cobordism by symplectic deformations.

5. Quantum invariance in HD

- K-equivalence preserves cohomology groups, but not the product structure. For ordinary flops, the topological defect had been determined by H.-W. Lin (林惠雯) and I in 2004. Meanwhile we extended the Lian-Liu-Yau 1996 theory in mirror conjecture to the local case and proved the invariance of (local) small quantum ring.

- Indeed, the topological defect is completely remedied by the quantum corrections attached to the Mori extremal rays. Our major calculation is on the 3-point (field) correlation function (called the generalized multiple cover formula). This extends Witten's 1992 3D result to all dimensions.

Analytic continuations

- Noticed that, the n-point functions on X and on the flopped manifold X' are analytic functions defined on their Kaehler cones respectively, and the invariance is under analytic continuations. This is a brand new phenomenon in HD algebraic geometry.

- Two remaining steps are needed to attack the full QH.
- (1) Non-extremal curve classes.
- (2) Big quantum ring, i.e. n-point functions with $n > 3$.

- In the fall of 2005, with NCTS visitor Y.-P. Lee (李元斌, Utah), we found a new proof of the multiple cover formula via the Divisor Relation on stable map moduli (2003, Lee and Pandharipande, Princeton).

- The proof consists of reductions of 3-point functions into 1-point functions with gravitational descendents. This new method also extends our result to (local) big quantum ring.

- In early 2005, we had started using the degeneration formula of Gromov-Witten invariants (due to Ruan and Li) to reduce the problem to local models (projective bundles).

2001 – 2004 – 2006

- In early 2006, we solved the local model case completely. Our proof uses a 5 level induction on functional equations and analytic continuations.

- Later in April, we overcame the technical details to reduce the general case to local models, hence completely solved the invariance of QH under simple flops in all dimensions.

- This is one of the best results in Gromov-Witten theory in recent years. It also put a landmark in K-equivalence theory. [1] LLW, accepted by Annals of Mathematics, 2007.

Recent works at NCU-CMTP

- With Baohua Fu (U. Nantes, visited CMTP in April 2007):
  [2] Motivic and quantum invariance under stratified Mukai flops. J. Differential Geometry 80, 2008.

- With Y.-P. Lee (U. Utah, visited CMTP and NCU, Dept. of Math. in November 2007 and May 2008) and H.-W. Lin:
  [3] Invariance of Gromov-Witten theory under simple flops, submitted. Preprint 2008, arXiv: 0804.3816.

6. Epilogue

- In this talk I use the recent interactions between algebraic geometry and heterotic string theory to highlight that some parts of Math and Phys are going together now.

- In fact, in the applied area there is always close relationship and collaboration between Math and Phys and other sciences. Sometimes Math even plays the key role.

- For historical reasons, for a long period in the 20th century, the core Math had been separated from other sciences (purposely) in order to face her own intrinsic problems.

- Now the nature offers again vast subjects and great ideas for mathematical studies. Meanwhile, the abstract objects invented by purely intrinsic study of Math also found their roles in nature, sometimes quite unexpectedly.

- Rigorous mathematical proofs may even play the role of experiments to test the law of nature.

- The atmosphere is similar to the one in early 20^{th} century. It is our belief that Mathematics and Sciences will deeply interact with each other in the 21^{st} century. More surprises are waiting for us!

*** The end ***

Thank you!