A Beautiful Mind
The Mathematical Life of John Nash

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一個偉大傳奇數學家的故事

- 從許多觀點看, Nash 都極可能是二十世紀最具原創性的數學家. 從 1950 畫時代的對局論 Ph.D 論文以及首創實代數幾何的研究, 到 1953 解決微分幾何的世紀懸案, 到 1957 石破天驚地解決了非線性 PDE 解的連續性問題. Nash 創造了許多深刻亦又自然的新思維. 然而, 正值其人生璀璨之 30 歲, Nash 陷入失心與幻想. 1960 至 1990 Nash 鬼魅似的出現在 Princeton 校園, 又於 1990 傳奇地甦醒, 並旋及穫頒 Nobel 經濟獎.

Part I: The way I knew Nash

- 1986: The visit of Chern to Taiwan, from Gauss-Bonnet Theorem to isometric embedding problems.
- 1988: The course in Differential Geometry, the unbelievable theorem of Nash.
- Geometry = Intuition?

- 1989: The curve shortening problem. Hamilton's Ricci flow. The Nash-Moser Implicit Function Theorem.
- 1989: C.-S. Lin's work on isometric embedding of surfaces into $\boldsymbol{R}^3$.

- A recall of 1985: Newton's iteration for solving equations. Smale's Dynamical Systems.

- 1992: Fields medals? A dead man? 1940-1980: an era of Topology and Algebraic Geometry. A general ignorance of Analysis.
- 1994: New York Times: Nash won the Nobel for economics!
- 1996: The Duke Math J. Vol. 81: A Celebration of John F. Nash. Nash’s 1968 paper “Arc Structures of Singularities”.
- 1998: Nasar: A Beautiful Mind.

Part II: The Real Life of Nash

- **Birth: 1928, June 13. Bluefield, West Virginia.**
- **4-th grade: `B-' in Mathematics.**
- **12 years old: T. Bell: Man of Mathematics.**
- **Carnegie Inst of Tech: 1945, Course by Synge: Relativity and Tensor Calculus. Bott, Weistein.**
- **1947: Putnam Math Competition. Young Gauss.**
- **1948: Harvard vs Princeton**

- **From Hilbert, Weyl to Von Neumann.**
- **Lefschetz, Annals of Math, school of genuis.**
- **The way to learn is to do research on it!**
- **Disagreement with Artin.**
- **Game Theory, bargaining and axioms.**
- **Game Theory, Nash equilibrium.**
- **Von Neumann’s against.**
- **Childish behavior and jealousy, Shaply.**
- **RAND.**

The Start of Mathematics

- **A beautiful theorem.**
- **MIT.**
- **Ambrose's challenge.**
- **C^1 isometric embedding. Artin's against.**
- **Eleanor. A nurse for Nash's surgery.**
- **1954: homosexual at RAND, depression.**
- **Alicia.**
- **The embedding theorem.**

- **Sloan fellowship.**
- **1956: NYU. Marriage and death.**
- **Nirenberg’s problem.**
- **De Giorge, Rota’s comment in 1994.**
- **1958 Fields Medalist: Roth and Thom.**
- **30 years old. Riemann Hypothesis (1859, 33).**
- **1962 Fields Medalist: Hormander and Milnor.**
- “無情的自我超越” 以療傷.

- **Hospitalization.**
- **1960: Absolute zero.**
- **1963: Nash Blowing-Up (Hironaka).**
- **Divorce.**
- **1965 – 1967: Back to Boston**
- **Short recovery: Two papers in 40’s.**
- **A man all along in a strange world.**
- **A ghost in Princeton.**
- **Alicia’s endless love, 1970 -.**
- **Slow recovery 1990. Nobel 1994.**

Part III: Nash's Research Works

- **1. Game Theory**
- **2. Real Algebraic Geometry**
- **3. Embedding Problems**
- **4. PDE: Fluid Flows**
- **5. Singularities: Arc Spaces**

1. Game Theory

Cooperative vs Non-Cooperative Games

- **Non-Cooperative Games: The Nash Equilibrium**
- Pre-play Games, Unbounded Rationality
- **Theorem (Nash 1951):** *Each individual simply finds strategies for his maximal profit, the society then reach its (non-unique) equilibrium*

- History: Von Neumann and Morgenstern: 1944, **The Theory of Games and Economic Behavior**
- Cooperative Game Theory Min-Max Theorem (1928)
- Implication: Importance of Government, laws, contracts and regulations

Bargaining Problems
(Nash 1950)

- **Bargaining Problem I** Axiomatic Approach (Nash, 1950)
- **Bargaining Problem II** Transfer Cooperative Games into Non-Cooperative Games

2. Real Algebraic Geometry

- A **real algebraic set** is a set in Euclidean space consists of all *real solutions* of a finite number of polynomial equations in finite variables.
- **Theorem** (Nash 1952, Tognoli 1973) *Every compact manifold can be approximated by (hence diffeomorphic to) real algebraic manifolds*.
- Hironaka (1982): Nash Blowing-Up.

3. Isometric Embeddings

- A **Riemannian metric** g on a manifold M is a family of smoothly varied inner products, with one on each tangent space $T_p M$ of M. (Riemann 1850)
- An **isometric imbedding** of M into an Euclidean space $\boldsymbol{R}^n$ is to regard M as a sub-manifold such that g coincides with the restriction of the standard inner product $<u,v> = \Sigma\, u^i v^i$.
- **Question:** Does isometric imbedding exist?

Density Theorems and *h*-Principle

- **Whitney's Theorem** (1936, 1943): *Every smooth manifold can be smoothly imbedded into* $\boldsymbol{R}^{2n}$*, and into* $\boldsymbol{R}^{2n+1}$ *freely (imbeddings are dense in the space of smooth maps).*
- **Nash's** C^1 **Isometric Imbedding Theorem** (Nash 1954, Kuiper 1955): *Any topological* C^k *imbedding of* (M, g) *in* $\boldsymbol{R}^N$ *with* $N > n$ *and* $k > 0$ *can be* C^0 *approximated by (deformed into) a* C^1 *isometric imbedding in the same* $\boldsymbol{R}^N$*.*

Nash's Implicit Function Theorem and the Full Isometric Imbedding Theorem

- **Nash** (1956): *Every Riemannian manifold can be isometrically imbedded into any arbitrarily small region (volume) of* $\boldsymbol{R}^N$. *For manifolds of dimension* n, $N = (n + 2)(n + 3)/2$ *is sufficient.*
- **Step 1:** Implicit function theorem for *tame maps* in Frechet spaces.
- **Step 2:** Existence of an initial *non-degenerate* approximate imbedding.

4. PDE: Fluid Flows and Heat

- **Nash 1958:** Continuity of solutions of (linear) parabolic and elliptic equations (with measurable coefficients).
- Here, although I did succeed in solving the problem, I ran into some bad luck, since, without my being sufficiently informed, it happened that I was working in parallel with de Giorgi of Pisa, Italy. And de Giorgi was first actually to achieve the ascent of the submit at least for the particularly interesting case of "elliptic equations".

Parabolic equation in divergence form

$$\frac{\partial T}{\partial t} = \sum_{i,j} \frac{\partial}{\partial x_i} \left(C_{ij}(x_1, \cdots, x_n) \frac{\partial T}{\partial x_j} \right).$$

Assume $c_2 \geq c_1 > 0$ **being uniform bounds of eigenvalues of** C**. Then for** $|T| < B$ **and** $t_2 \geq t_1 > t_0$**, there are bounds** A **and** α **depend only on** c_1**,** c_2 **and** n **such that**

$$\begin{aligned} |T(x_1, t_1) \; - \; & T(x_2, t_2)| \\ & \leq BA \left[\left(\frac{x_1 - x_2}{\sqrt{t_1 - t_0}} \right)^{\alpha} + \left(\sqrt{\frac{t_2 - t_0}{t_1 - t_0}} \right)^{\frac{\alpha}{1-\alpha}} \right]. \end{aligned}$$

5. Nash's Arc Spaces

- **Hironaka 1964:** Resolution of Singularities of algebraic varieties over a field of characteristic zero.
- **Nash 1968:** Singularities can be studied through arcs (or jets), which encodes all the infinitesimal structures. This is particularly useful in studying bi-rational maps.

1. For an algebraic variety X, let $\mathcal{L}(X) := X(k[\![t]\!]) = \mathrm{Mor}_k(\mathrm{Spec}\, k[\![t]\!], X)$ be the Nash space of (formal) arcs in X (power series solutions of equations defining X).

2. Let X, Y be smooth of dimension n and $\phi : Y \to X$ be a birational morphism. ϕ induces an algebraic map $\tilde{\phi} : \mathcal{L}(Y) \to \mathcal{L}(X)$. Let $J(\phi)$ be the (usual) Jacobian computed from $\phi^*\Omega^n_X / \Omega^n_Y$.

3. Problem: let $S_r \subset \mathcal{L}(Y)$ be the subset such that $\mathrm{ord}_t J(\phi) = r$. What is the structure of $S_r \to \tilde{\phi}(S_r) \subset \mathcal{L}(X)$?

Answer: For each r, $\phi_r : S_r \to \tilde{\phi}(S_r) \subset \mathcal{L}(X)$ is a piece-wise trivial affine A_k^r fibration.

Two algebraic manifolds X and X' are called K-equivalent if there exists smooth Y with birational morphisms $(\phi, \phi') : Y \to X \times X'$ such that $\phi^* \Omega_X^n = \phi'^* \Omega_{X'}^n$. Example: Birational Calabi-Yau manifolds.

Application (Wang 1998): $X =_K X'$ implies that $[X] = [X']$ in the Grothendieck ring of varieties $K_0(\mathsf{Var}_k)$. In particular, $h^{p,q}(X) = h^{p,q}(X')$.

Work in progress: Nash's arc space indeed provide a space to find "natural cycle" in $\mathcal{L}(X) \times \mathcal{L}(X')$ which will produce a cycle $W \subset X \times X'$ extending the graph closure $\bar{\Gamma} \subset X \times X'$, such that W induces a canonical isomorphism $H^{p,q}(X) \cong H^{p,q}(X')$.

Expected application: birational maps between Calabi-Yau manifolds X and X' deform in families. Hence X and X' have canonically isomorphic moduli spaces.

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