幾何学的昨日與今日

王金龍

Euclid · 直線

Newton · 二次曲線（圓錐曲線）

Gauss · 一般曲線（曲面）

2010.11.17 於 鳳山高中

古典幾何學

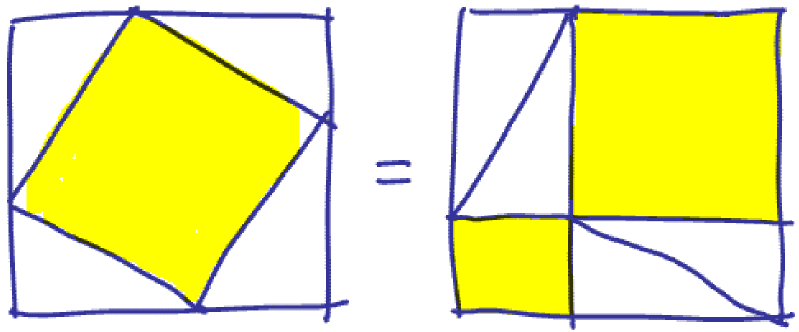
Pythagoras 570 – 495 B.C.

Euclid's Elements 300 B.C.

- 公理化數學 / Logic
- 尺規作圖（三大難題）

畢氏定理（几何的起源）

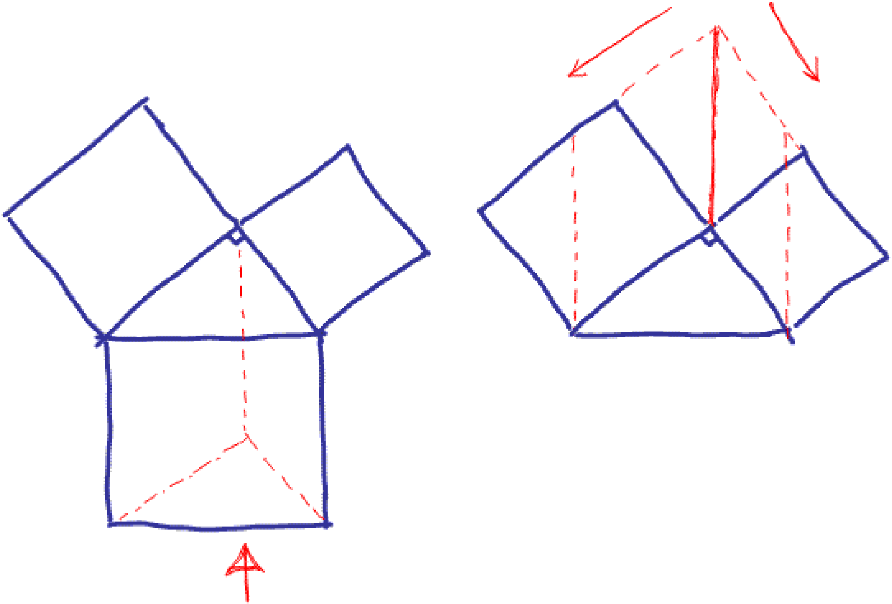
①



② $(a+b)^2 = a^2 + 2ab + b^2$

$$\begin{array}{c}(a+b)^2 \\ \| \\ c^2 + 4 \times \dfrac{ab}{2}\end{array} \quad \Rightarrow \quad a^2 + b^2 = c^2$$

Dynamical Proof 動畫式証明



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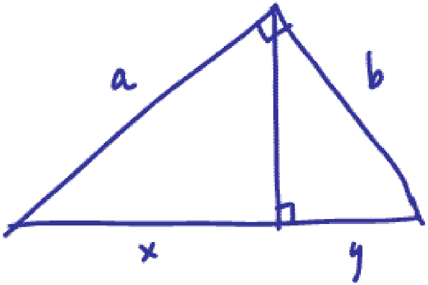


$$\frac{y}{b} = \frac{b}{c} \Rightarrow b^2 = yc$$

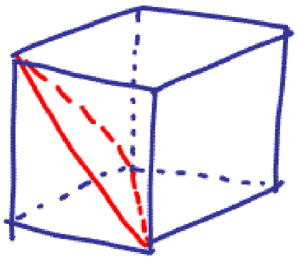
$$\frac{x}{a} = \frac{a}{c} \Rightarrow a^2 = xc$$

$$a^2 + b^2 = (x+y)c$$

$$= c^2$$

方法的沿伸

①'



$Vol = \frac{1}{6}$

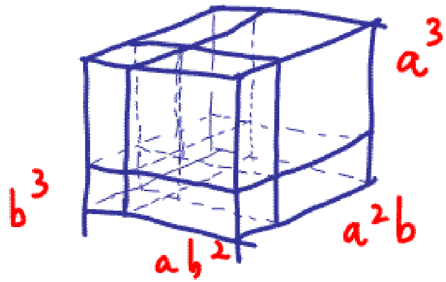
锥体积 $= \frac{1}{3}$ 底面积 $\times$ 高

Key formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

How can one "find this" ?

2' $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$



$$2^3 = (1+1)^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1$$

$$3^3 = (2+1)^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1$$

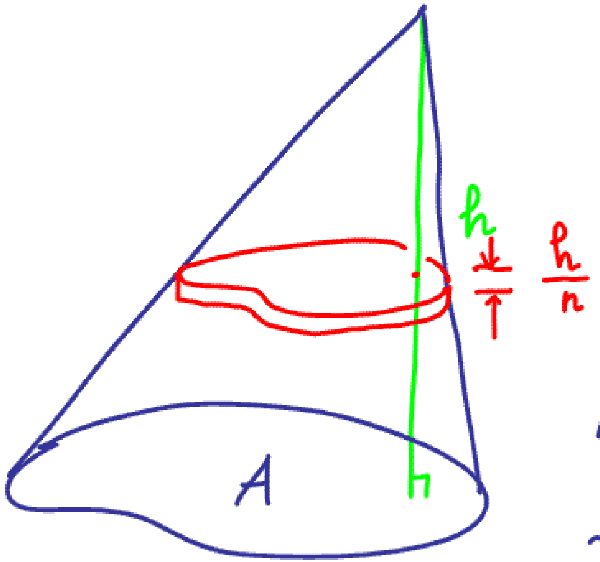
$$4^3 = (3+1)^3 = 3^3 + 3\cdot 3^2 + 3\cdot 3 + 1$$

$$\vdots$$

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$\Rightarrow \quad (n+1)^3 = 1 + 3\left(1^2 + 2^2 + \cdots + n^2\right) + 3\frac{n(n+1)}{2} + n$$

$$\Rightarrow \quad 1^2 + \cdots + n^2 = \frac{1}{3}\left((n+1)^3 - 1 - \frac{3}{2}n(n+1) - n\right)$$



阿基米得一祖冲之

第 k 層的体積

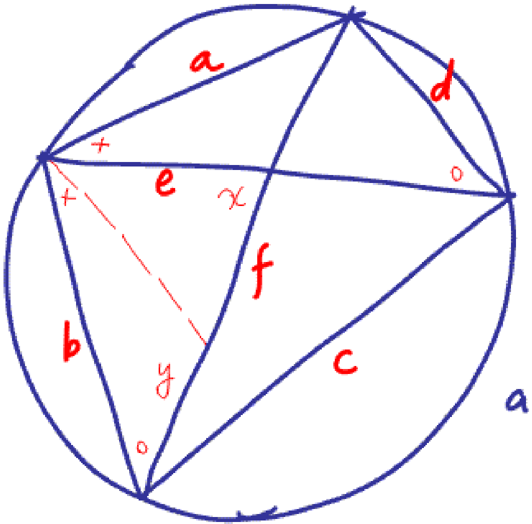
$A\cdot\left(\frac{k}{n}\right)^2\cdot\frac{h}{n}$

加起來 =

$Ah\,\frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)$

$\xrightarrow[n\to\infty]{}\ \frac{1}{3}Ah$

③' Ptolemy (AD 90 ~ 168) $ac + bd = ef$



$$\frac{y}{b} = \frac{d}{e}$$

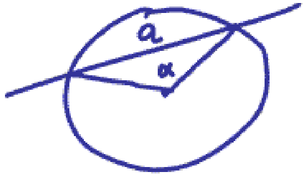
$$\frac{x}{a} = \frac{c}{e}$$

$$ac + bd = (x+y)e$$

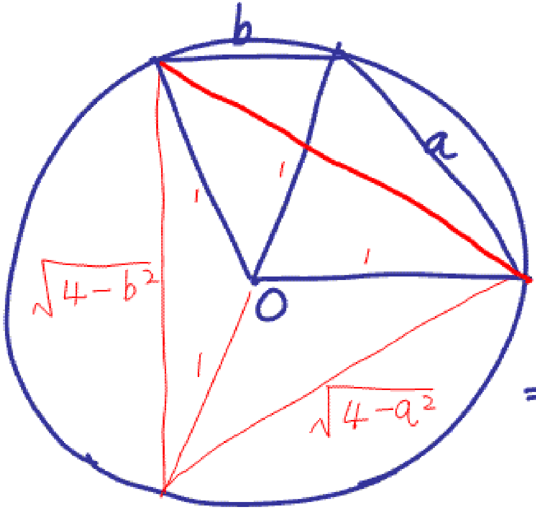
$$= fe$$

古代的正弦和角公式

（天文觀測）



$$a = \text{sine}(\alpha)$$
$$= 2\sin\left(\frac{\alpha}{2}\right)$$



$$2\,\text{sine}(\alpha+\beta)$$
$$= a\sqrt{4-b^2} + b\sqrt{4-a^2}$$

幾何 (imagination)

代數 (量化)

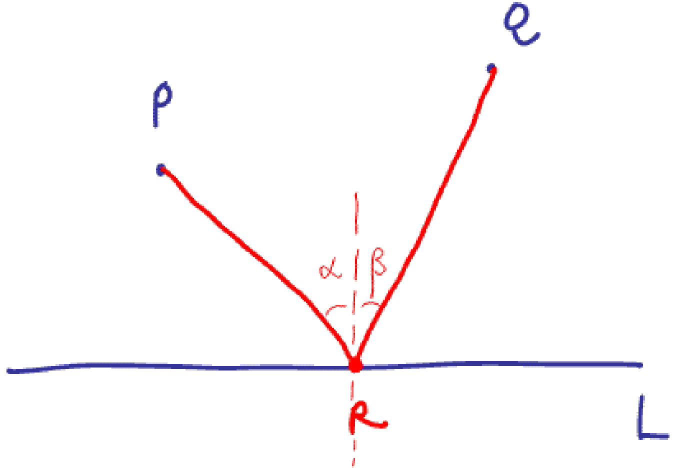
Descartes 17 世紀

座標化 (笛卡兒)

獨立變數的概念

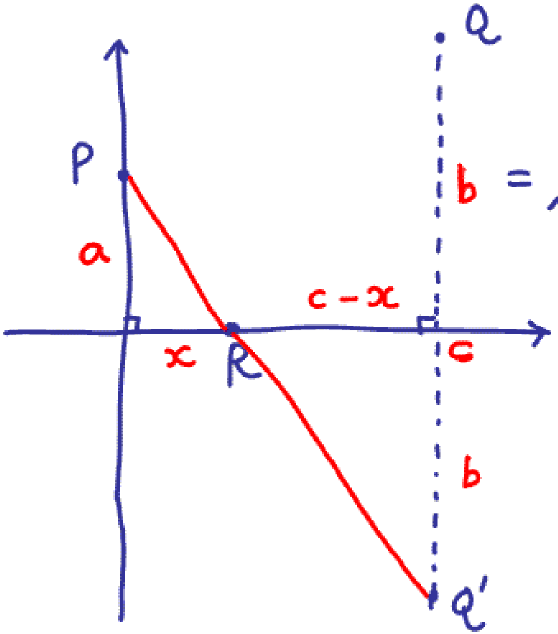
解析幾何 (analysis)

Fermat, Newton, Gauss



Find $R \in L$
such that

$\overline{PR} + \overline{QR}$ 最小



$$\ell(x) = \ell_1(x) + \ell_2(x)$$

$$= \sqrt{a^2+x^2} + \sqrt{b^2+(c-x)^2}$$

R：仲介者

極值發生時必有

$$\ell'(x) := \lim_{h\to 0} \frac{\ell(x+h)-\ell(x)}{h}$$

$$= 0$$

$l_1'(x)$: $\frac{1}{h}\left(\sqrt{(x+h)^2+a^2}-\sqrt{x^2+a^2}\right)$

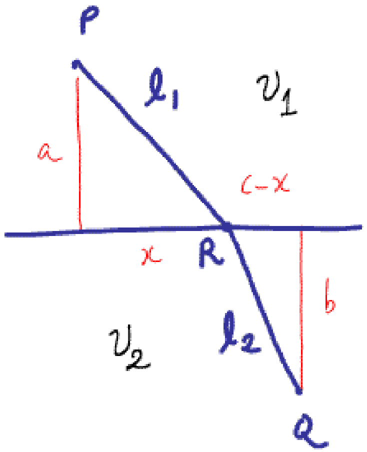
$$= \frac{1}{h}\,\frac{(x+h)^2+\cancel{a^2}-x^2-\cancel{a^2}}{\sqrt{(x+h)^2+a^2}+\sqrt{x^2+a^2}}$$

$$= \frac{1}{\cancel{h}}\,\frac{2x\cancel{h}+h^{\cancel{2}}}{\sqrt{(x+h)^2+a^2}+\sqrt{x^2+a^2}} \xrightarrow[h\to 0]{} \frac{x}{\sqrt{x^2+a^2}}$$

同理： $l'(x) = \frac{x}{\sqrt{x^2+a^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$

故 $l'(x)=0 \iff \sin\alpha = \sin\beta \iff \alpha=\beta$

折射定律



Fermat's Least Action Principle

最小作用原理

$$T(x) = \frac{l_1(x)}{v_1} + \frac{l_2(x)}{v_2}$$

$$T'(x) = \frac{l_1'(x)}{v_1} + \frac{l_2'(x)}{v_2}$$

$$T'(x) = 0 \iff \frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}$$

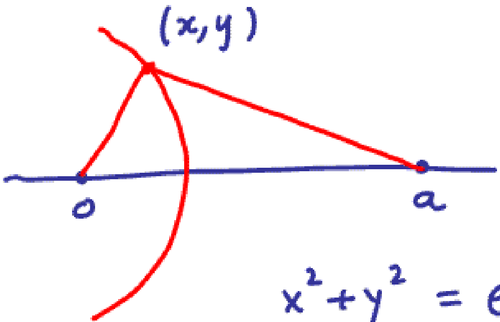
How about $\overline{PR}/\overline{QR}$?

$$\left[\frac{l_1(x)}{l_2(x)}\right]' : \quad \frac{1}{h}\left[\frac{\sqrt{(x+h)^2+a^2}}{\sqrt{(x+h-c)^2+b^2}} - \frac{\sqrt{x^2+a^2}}{\sqrt{(x-c)^2+b^2}}\right]$$

$$= \frac{1}{h}\left(\frac{\sqrt{(x+h)^2+a^2}\,\sqrt{(x-c)^2+b^2} - \sqrt{x^2+a^2}\,\sqrt{(x+h-c)^2+b^2}}{\sqrt{(x+h-c)^2+b^2}\,\sqrt{(x-c)^2+b^2}}\right)$$

$$= ???$$

Apollonius (BC 260 ~ 190) 圓



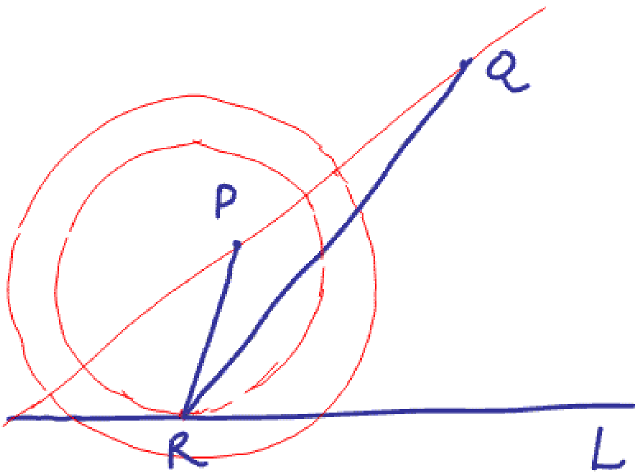
求 $\dfrac{\sqrt{x^2+y^2}}{\sqrt{(a-x)^2+y^2}}$ = 定值 e

之軌跡

$$x^2+y^2 = e^2(x^2-2ax+a^2+y^2)$$

$$(1-e^2)x^2+(1-e^2)y^2+2ae^2x = e^2a^2$$

$\longrightarrow$ circle



l_1/l_2 之極值
發生於當
L 為 Apollonius
circle 之切線時

Homework : $R = ?$

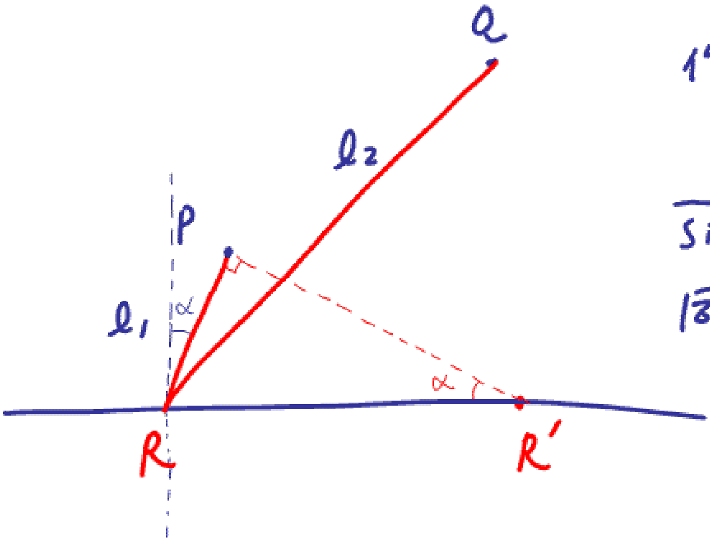
解析法 $\left(\frac{l_1}{l_2}\right)' = \frac{l_1' l_2 - l_1 l_2'}{l_2^2} = 0$

$$\Longleftrightarrow \quad \frac{l_1'}{l_1} = \frac{l_2'}{l_2}$$

i.e. $\frac{\sin\alpha}{l_1} = \frac{\sin\beta}{l_2}$

Remark: Lebnitz 1646-1716 (莱布尼茲)

$$(fg)' = f'g + fg'$$



作 $\overline{PR'} \perp \overline{PR}$

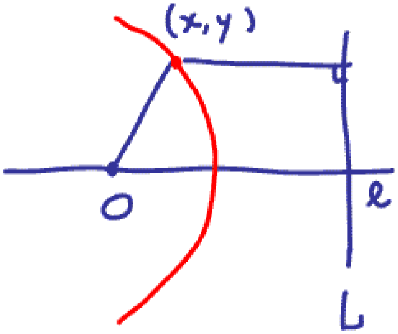
$$\frac{l_1}{\sin\alpha} = \overline{RR'}$$

同理，作 $\overline{QR''} \perp \overline{QR}$

$$\frac{l_2}{\sin\beta} = \overline{RR''}$$

$$\Rightarrow R' = R''$$

二次曲線 (beyond circles)



準線

$$\frac{\sqrt{x^2+y^2}}{l-x} = \text{定值 } e$$

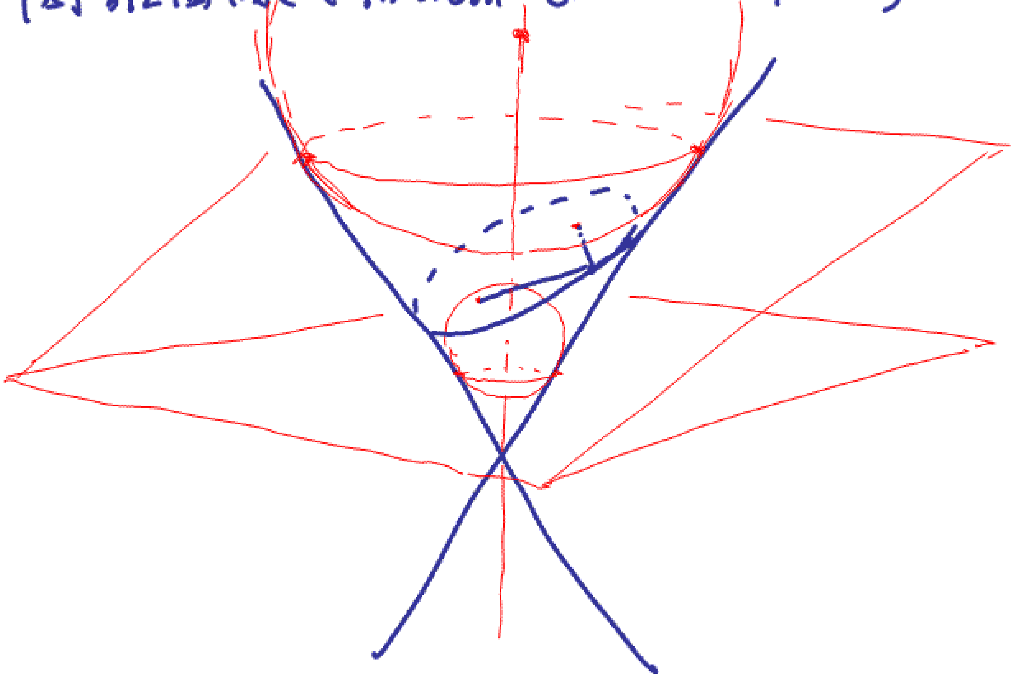
$$x^2+y^2 = e^2(x^2-2lx+l^2)$$

$$(1-e^2)x^2+y^2+2le^2x = e^2l^2$$

Polar coor. $\frac{r}{l-r\cos\theta} = e \Rightarrow r = \frac{le}{1+e\cos\theta}$

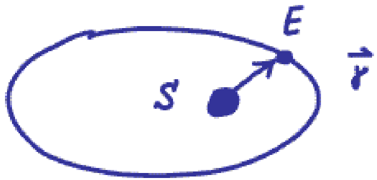
(r,θ)

圓錐曲線 (Ancient Greek's viewpoint)



Kepler's Law 1619

80年天文觀測



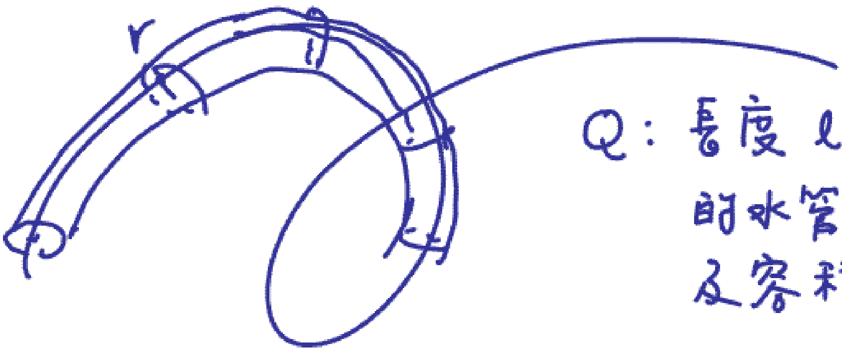
Newton 牛頓 1643-1727

Physics : $\vec{F} = m\vec{a}$; $\vec{F} = -\frac{GMm}{r^2}\hat{r}$

Mathematics : CALCULUS

Principia Mathematica 自然哲学的数学原理

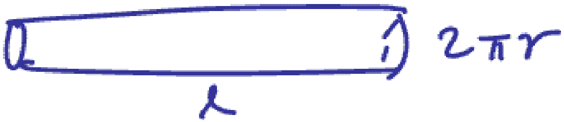
更一般的曲線理論

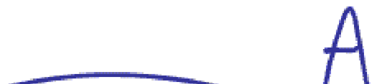


Q: 長度 ℓ，半徑 r 的水管表面積及容積 = ?

$$\text{Area} = 2\pi r \ell$$

$\pm$ (....) ?

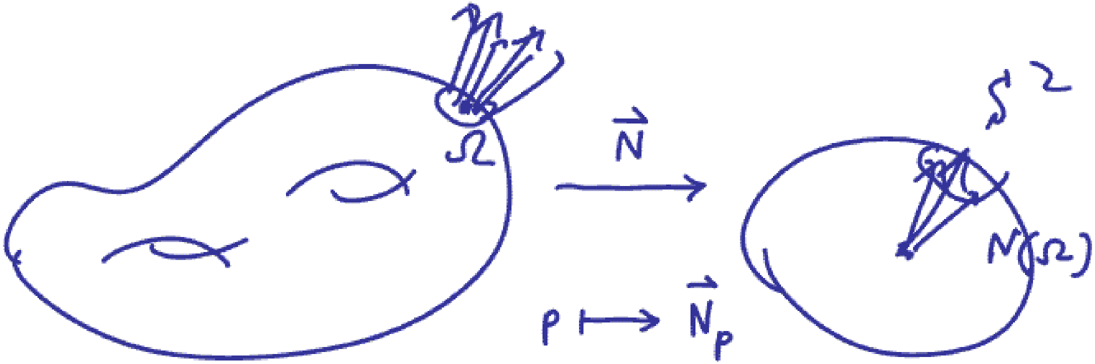




曲率的概念 (Curvature)

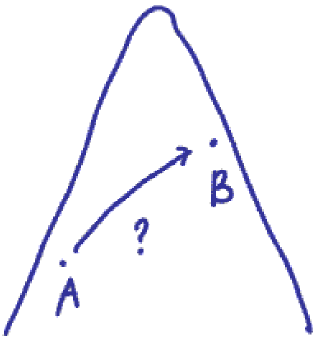
Gauss 高斯 1777—1855

1818 地形測量 ⟶ 微分幾何誕生

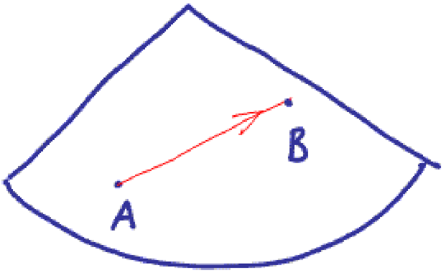


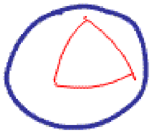
曲率 $K = \pm \lim_{\Omega \to p} \frac{|N(\Omega)|}{|\Omega|}$

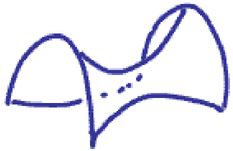
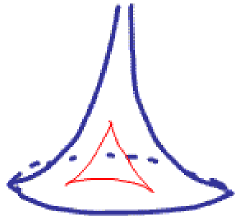
這是一個不變量 (invariant)



測地線



$K > 0$  半徑 R 的球， $K = \frac{1}{R^2}$

$K < 0$  or 

Gauss 的偉大發現

三角形之內角和 $\neq 180^\circ$ $\left(= \pi + \int_\Omega K\,dA\right)$

半徑 r 的圓周長 $L = 2\pi r - \frac{\pi}{3} r^3 K + o(r^3)$

$\Rightarrow$ 畢氏定理不再正確！

Gauss-Bonnet 定理 (陳省身, 高維度)

$$\int_S K \, dA = 2\pi \chi(S)$$

其中，$\chi(S) \triangleq$ 点 $-$ 線 $+$ 面 $= 2-2g$

Euler number

$g=3$

我們如何知道我們所在的世界
是否是彎曲的？

Riemann 黎曼 1855 $ds^2 = \sum g_{ij}\, dx^i \otimes dx^j$

Einstein 愛因斯坦 1907～1915

$$R_{ij} - \frac{1}{2} g_{ij} = T_{ij}$$

Nash 1951

YAU 丘成桐 1976（宇宙的內在模型）

STRING THEORY（Witten …）

END