GEOMETRY 2023 (HONOR COURSE)
FINAL EXAM
A COURSE BY CHIN-LUNG WANG AT NTU

1. (a) Let $u \in C^\infty([a,b])$, $S[u] = \int_a^b L(t, u, \dot{u}, \dots, u^{(n)})\,dt$. Derive the Euler–Lagrange equation:

$$\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} + \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial L}{\partial u^{(n)}} = 0.$$

(b) Let $L = L(u, \dots, u^{(n)})$. For $i = 1, \dots, n$, we set $q_i := u^{(i-1)}$ and

$$p_i := \frac{\partial L}{\partial u^{(i)}} + \sum\nolimits_{s=1}^{n-i} (-1)^s \frac{d^s}{dt^s}\frac{\partial L}{\partial u^{(i+s)}}.$$

Assume that $p_n = \partial L/\partial \dot{q}_n$ is uniquely solved for $\dot{q}_n = u^{(n)}$. Find the "Hamiltonian function" $H = H(p_1, \dots, p_n, q_1, \dots, q_n)$ and prove

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \dots, n.$$

(You get half credits by working out the standard case $n = 1$.)

2. (a) Define the Poisson bracket $\{f, g\} := \sum_{i,j} g^{ij} f_i g_j$ on the phase space $\mathbb{R}^{2n}$ with a general non-degenerate skew-symmetric "metric" $\Omega = \sum_{i<j} g_{ij}\, dy^i \wedge dy^j$. Show that it defines a Lie algebra structure on $C^\infty(\mathbb{R}^{2n})$ if and only if Ω is symplectic, i.e. $d\Omega = 0$.

(b) Given $f_i(x, p)$, $i = 1, \dots, n$ on the (standard, abstract) phase space $\mathbb{R}^{2n}$ with linearly independent ∇f_i point-wisely and with $\{f_i, f_j\} = 0$ for all i, j. Show that the surface M defined $f_i = 0$, $i = 1, \dots, n$, is Lagrangian.

3. (a) Show that the second variation for $S[\gamma] = \frac{1}{2}\int_a^b \sum_{i,j} g_{ij}\dot{x}^i\dot{x}^j\,dt$ is

$$G_\gamma(\xi, \eta) = -\int_a^b \langle \nabla_T^2 \xi + R(\xi, T)T, \eta\rangle\,dt$$

where γ is a geodesic and $T := d/dt = \gamma' = \sum_i \dot{x}^i \partial_i$.

(b) Prove Bonnet–Myers' theorem: Given n-dimensional surface (M, g), if $R_{ij} \ge (n-1)K_0\, g_{ij}$ (as quadratic forms) for some constant $K_0 > 0$ then for any P, Q joined by a shortest geodesic γ, its length is no greater than $\pi/\sqrt{K_0}$. (You get half credits for doing $n = 2$.)

4. (a) Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^N$ and $I[f] := \int_D L(f^a, f_i^b)\,d\sigma$. The energy-momentum tensor is a tensor T_j^k satisfying $\sum_k \nabla_k T_j^k = 0$ (here $\nabla_k = \partial_k$). Show that it can be chosen by

$$T_j^k := \sum_a f_j^a \frac{\partial L}{f_k^a} - \delta_j^k L$$

Moreover, if $D \subset \mathbb{R}^{1,3}$ with $T^{ik} = g^{ij}T_j^k = O(R^{-(3+\epsilon)})$, show that the momentum (four vector) $P^i := \frac{1}{c}\int_{x^0=a} T^{ik}\,dS_k$ is well-defined and is independent of a.

Date: 08:00 – 13:00, December 22, 2023 at AMB 102. Each problem is of 20 credits. Show your answers/computations/proofs in details. You may work on each part separately by assuming other parts.

(b) For the Hilbert–Einstein action $S[g^{ij}] = \int_S R\, d\sigma = \int_S R\sqrt{|g|}\, d^n x$, show that

$$\frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{ij}} = R_{ij} - \tfrac{1}{2} R g_{ij}$$

for fast decayed g^{ij}. (You need to calculate δR^k_{lij} and $\delta\sqrt{|g|}$ in details.)

5. (a) Consider the n-dimensional graph surface S defined by $z = f(x_1, \ldots, x_n)$. Compute the Euler–Lagrange equation for the n-dimensional area functional.

(b) Let $S = r(D)$ for general $r : D \subset \mathbb{R}^m \hookrightarrow (\mathbb{R}^n, h_{kl})$ with general smooth metric h_{kl}. Under $T_P\mathbb{R}^n = T_P S \oplus^{\perp} N_P$ for $P \in S$, define the second fundamental form Π on $T_P S$ and the mean curvature vector $\vec{H}$ (both take values in N_P). Show that the "minimal surface equation" is given by $\vec{H} = 0$.

(c) Show that $\vec{H} = (\triangle r)^N$ in general. For a surface $S \subset \mathbb{R}^3$ under isothermal coordinates $d\ell^2 = \lambda(du^2 + dv^2)$, show that

$$\vec{H} = \triangle r = \frac{1}{\lambda}(r_{uu} + r_{vv}).$$

(d) Give an example so that a minimal disk might not have absolutely minimizing area among surfaces with given boundary curve $\Gamma = \partial S \subset \mathbb{R}^3$.

6. Let $G \subset \mathrm{GL}(N, \mathbb{R})$ with $\mathfrak{g} = \mathrm{Lie}\, G$. Let $\psi : U \subset \mathbb{R}^n \to \mathbb{R}^N$ and $A(x) = \sum_{i=1}^n A_i(x) dx^i \in \Lambda^1(\mathfrak{g})$. Define $\triangledown_i \psi = \partial_i \psi + A_i \psi$. $\triangledown$ is called a G connection if $\triangledown_i(g\psi) = g\triangledown_i \psi$ for $g(x) \in G$.

(a) Derive the "gauge transformation law" of $\tilde{A}$ under the "coordinate change" $\tilde{\psi} = g\psi$:

$$A \mapsto \tilde{A} = -dg\, g^{-1} + gAg^{-1}$$

and explain it using the change of frame $\Psi = \sum \psi^j e_j = \sum \tilde{\psi}^i \tilde{e}_j$ on $U \times \mathbb{R}^N$.

(b) Extend $\triangledown$ to $U \times \bigotimes^{r,s} \mathbb{R}^N$ and show that $\triangledown_i B = \partial_i B + [A_i, B]$ for $\mathfrak{g}$-valued functions $B(x)$.

(c) Extend $\triangledown$ to $d_A : \Lambda^k(\mathfrak{g}) \to \Lambda^{k+1}(\mathfrak{g})$ as in Cartan's d (where $A = 0$) and show that

$$d_A \Omega = \sum_{i<j<k} \triangledown_{[i} \Omega_{jk]}\, dx^i \wedge dx^j \wedge dx^k \qquad \text{for } k = 2.$$

Let $F \in \Lambda^2(\mathfrak{g})$ be the curvature of $\triangledown$. Prove the Bianchi identity $d_A F = 0$.

(d) For a non-degenerate Killing form $\langle\, ,\rangle$ of $\mathfrak{g}$, show that any extremal A for

$$S[A] = \int_S \langle F^{ij}, F_{ij}\rangle\, d\sigma$$

with fast decay satisfies the Yang–Mills equation $d_A^* F = 0$. Give explicit formula for $d_A^* F$.

* If you have well prepared an essential topic in the last two chapters which is not shown in the above six problems, you may choose to present it to replace one of the problems.