**GEOMETRY 2023 (HONOR COURSE)**
**MIDTERM EXAM - II**
**A COURSE BY CHIN-LUNG WANG AT NTU**

**1.** (a) Show that $L_\xi \eta = [\xi, \eta]$ for vector field $\eta$ and $L_\xi\, d = dL_\xi$ on differential forms.
(b) Let $\omega$ be a $k$ form. For vector fields $X_0, \ldots, X_k$, prove Cartan's formula:

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = {} & \sum\nolimits_{i=0}^{k} (-1)^i X_i\, \omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \\ & + \sum\nolimits_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k). \end{aligned}$$

**2.** Define $* : \Lambda^p \to \Lambda^{n-p}$ by $(*T)_{i_{p+1}\cdots i_n} = \frac{1}{p!}\sqrt{g}\, \epsilon_{i_1\cdots i_n}\, T^{i_1\cdots i_p}$ for Riemannian metric $d\ell^2$.
(a) Show that $*^2 = (-1)^{p(n-p)}$ and $T \wedge *S = \{T, S\}\, d\sigma$.
(b) On $\Omega_U^p$, forms supported in a bounded region $U$, with $\langle \omega_1, \omega_2 \rangle := \int_U \omega_1 \wedge *\omega_2$, show that the adjoint of $d$ is given by $\delta := (-1)^{np+n+1} * d *$ and $\delta^2 = 0$.
(c) Let $\triangle = d\delta + \delta d$. Show that $\triangle$ is self-adjoint and it commutes with $d$, $\delta$ and $*$.
(d) $\triangle \omega = 0$ if and only if $d\omega = 0$ and $\delta\omega = 0$. If furthermore $\omega = d\eta$ then $\omega = 0$.

**3.** For a connection $\nabla$ on $U$, its Christoffel symbol is defined by $\nabla_{\partial_i} \partial_j = \Gamma^k_{ji}\, \partial_k$.
(a) Find the transformation formula of $\Gamma^k_{ji}$ in another coordinate system.
(b) Given $n$ connections $\nabla_\alpha$ and $f_\alpha \in C^\infty(U)$, when is $\nabla := \sum_{\alpha=1}^n f_\alpha \nabla_\alpha$ a connection?
(c) Compute $\nabla^2_{X,Y} - \nabla^2_{Y,X}$ on vector fields in terms of curvature and torsion.
(d) Assume $\nabla$ is torsion free, compute a formula of $T^{i_1\ldots i_r}_{j_1\ldots j_s;pq} - T^{i_1\ldots i_r}_{j_1\ldots j_s;qp}$.

**4.** Consider a classical Lie group (i.e. matrix group) $G \subset \mathrm{GL}(n, \mathbb{F})$, $F = \mathbb{R}$ or $\mathbb{C}$.
(a) Let $X \in \mathfrak{g}$, show that $R_X$ define by $R_X(A) = -XA$ for $A \in G$ is right invariant. Also $[R_X, R_Y] = R_{[X,Y]}$ and $[L_X, R_Y] = 0$.
(b) Show that a bi-invariant metric $d\ell^2$ exists for $G \subset \mathrm{O}(n)$ or $\mathrm{U}(n)$.
(c) Under the metric $d\ell^2$ in (b), determine $\nabla^{LC}$ and all geodesics through $e \in G$.
(d) Compute the Riemann curvature tensor $R(X, Y, Z, W)$.

**5.** (a) Prove the two Bianchi identities $R^i_{[jkl]=0}$ and $R^i_{j[kl;m]=0}$.
(b) Show that $R^i_{m;i} = \frac{1}{2}\partial_m R$. If $R_{ij} = \lambda g_{ij}$, when can we deduce that $\lambda$ is a constant?
(c) For $n = 3$ show that $R_{ijkl}$ is determined by $R_{ij}$ by

$$R_{ijkl} = R_{ik}\, g_{jl} - R_{il}\, g_{ik} + R_{jl}\, g_{ik} - R_{jk}\, g_{il} + \tfrac{1}{2} R (g_{il}\, g_{jk} - g_{ik}\, g_{jl}).$$

**6.** (a) Let $\alpha$ be a piecewise smooth closed curve bounding a region $\Omega$ in a surface. Prove that $\int_\Omega K\, dA = \theta_\alpha$ where $\theta_\alpha$ is the holonomy angle along the curve.
(b) Prove the local Gauss–Bonnet Theorem

$$2\pi = \sum_{\text{outer angles}} \alpha_j + \int_{\partial\Omega} k_g\, d\ell + \int_\Omega K\, dA.$$

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*Date*: 08:30 – 12:30 or 10:00 – 14:00, November 17, 2023 at AMB 102. Each problem is of 20 credits. Show your answers/computations/proofs in details. You may work on each part separately by assuming other parts.