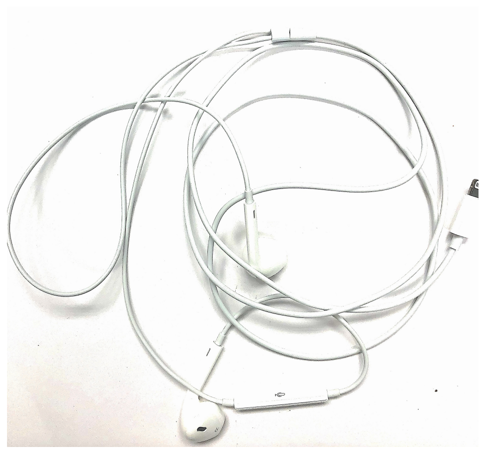
GEOMETRY 2023 (HONOR COURSE)
MIDTERM EXAM - I
A COURSE BY CHIN-LUNG WANG AT NTU

1. (a) Prove the Cauchy–Crofton formula for plane curves.
(b) Estimate the length of the following iPhone headphone (height is 15 cm).



2. (a) Compute the Frenet frame, the curvature, and the torsion of the helix

$$r(t) = (a\cos t, a\sin t, ct), \qquad a > 0, \quad c \neq 0, \quad t \in \mathbb{R}.$$

(b) Determine all space curves with constant curvature and torsion.

3. Let $S \subset \mathbb{R}^3$ be a surface parameterized by $r(u, v)$ with Gauss map $N : S \to S^2$ and fundamental forms $d\ell^2 = Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$.
(a) Show that the two principal curvatures coincide $\lambda_1 = \lambda_2 = \lambda$ at all $p \in S$ if and only if S is a portion of a sphere or a plane.
(b) Compute the matrix for $-dN_p$, $p \in S$ in the basis r_1, r_2 of T_pS, and deduce that the Gaussian curvature $K := \lambda_1\lambda_2$ equals $(LN - M^2)/(EG - F^2)$.
(c) (Gauss' Theorema Egregium) Show that K depends only on $d\ell^2$.

4. Let S be the surface of revolution given by

$$r(u, v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)), \qquad u \in [0, 2\pi].$$

(a) Calculate the two fundamental forms.
(b) Compute principal curvatures λ_1, λ_2 and K, H.
(c) If $H = 0$ on S (minimal surfaces), show that S must be the plane or the catenoid, which is defined by revolving the plane curve

$$(\phi(v), \psi(v)) = (a\cosh v, av).$$

Date: 9:00 – 12:00 or 10:10 – 13:10, October 13, 2023 at AMB 102. Each problem is of 20 credits. Show your answers/computations/proofs in details. You may work on each part separately by assuming other parts.

5. Three models of the (two dimensional) hyperbolic geometry.

(a) Lobachevsky plane: Let $L^2 \subset \mathbb{R}^{1,2}$ be the upper half of the space-like surface $t^2 - x^2 - y^2 = 1$, and $d\ell^2$ be the metric induced from $\mathbb{R}^{1,2}$. Show that

$$d\ell^2 = d\chi^2 + \sinh\chi\, d\varphi^2$$

in the pseudo-spherical coordinates (χ, φ).

(b) Poincaré model: Compute the stereographic projection $L^2 \to \mathbb{D}^2$ from the south pole $(0, 0, -1)$ onto the unit disk $\mathbb{D}^2$, and show that the induced metric is

$$d\ell^2 = \frac{4|dw|^2}{(1 - |w|^2)^2}.$$

(c) Klein model: Construct a Möbius transformation $M : \mathbb{H} \cong \mathbb{D}^2$, where $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, and show the induced metric is given by

$$d\ell^2 = \frac{dx^2 + dy^2}{y^2}.$$

(d) Determine the (direct) isometry group in one of the above three models.

(e) (Parallelism Axiom) Determine all geodesics in one of the three models. In particular show that given a point p not in a geodesic γ there are infinitely many geodesics passing through p and disjoint from γ.

6. Liouville's theorem: *A surface S with $d\ell^2 = e^{\varphi}\,|dz|^2$ and constant K is locally isometric to an open subset of S^2_R, $\mathbb{R}^2$ or L^2_R.* Prove it by the following steps. Let $\psi(z) := \varphi_{zz} - \frac{1}{2}\varphi_z^2$.

(a) Show that ψ is analytic.

(b) Under an analytic coordinate change $z = f(w)$ we get $d\ell^2 = e^{\widetilde{\varphi}}\,|dw|^2$ and the corresponding $\widetilde{\psi}(w)$. Show that $\widetilde{\varphi}(w) = \varphi(z) + \log|f'(w)|^2$ and

$$\widetilde{\psi}(w) = (f'(w))^2\psi(z) + \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)(w).$$

(c) Denote the above RHS term (the Schwarzian derivative) by $S(z; w) \equiv S(f; w)$. Show that $S(z; w) = -S(w; z)(f')^2$, and for any two independent solutions g_1, g_2 of the ODE $g''(z) + I(z)g(z) = 0$ we have $S(g_1/g_2; z) = 2I(z)$.

(d) Show that there exists analytic f such that $\widetilde{\psi} = 0$ in (b). Then conclude

$$e^{-\widetilde{\varphi}/2} = a|w|^2 + bw + \overline{b}\overline{w} + c$$

for some $a, c \in \mathbb{R}$, $b \in \mathbb{C}$.

(e) Complete the proof by a further change of coordinate.