CALCULUS 2017: FINAL EXAM

1. Sketch the graphs:

$$(a) \quad y = x^3 - 3x + 3, \qquad (b) \quad y = \ln(1 + x^2) - x.$$

2. Calculate the integrals:

$$(a) \quad \int e^{2x} \cos x \, dx, \qquad (b) \quad \int \frac{e^x}{e^{2x} + 3e^x + 2} \, dx.$$

3. Lebnitz' formula for π:

(a) Show that $\tan^{-1} b = b - \frac{b^3}{3} + \frac{b^5}{5} + \cdots + (-1)^n \frac{b^{2n+1}}{2n+1} + R_{2n+1}(b)$ where

$$R_{2n+1}(b) = (-1)^{n+1} \int_0^b \frac{x^{2n+1}}{1 + x^2} \, dx.$$

(b) Deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ by showing $\lim_{n\to\infty} R_{2n+1}(1) = 0$.

4. Find the Taylor expansion of $f(x)$ at $x = 0$:

$$(a) \quad f(x) = \ln \frac{1 + x}{1 - x}, \qquad (b) \quad f(x) = \cos \sqrt{x}, \qquad (c) \quad f(x) = \sin^{-1} x.$$

5. Interpolation for $y = f(x) = \sqrt{x}$:

(a) Find the quadratic polynomial $P_2(x)$ passing through $(4, 2)$, $(\frac{25}{4}, \frac{5}{2})$ and $(9, 3)$.

(b) For $b \in (4, 9)$ explain that there is some $\xi \in (4, 9)$ with

$$f(b) - P_2(b) = \frac{1}{3!} f'''(\xi)(b - 4)(b - \frac{25}{4})(b - 9).$$

(c) Explain $|\sqrt{5} - (2 + \frac{2}{9} + \frac{1}{99})| < \frac{1}{100}$.

6. Let $y = f(x)$ be a good function with $f(a) = 0$.

(a) Describe and derive Newton's iteration formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

(b) Suppose that $|f''(x)| \le M$ and $|f'(x)| \ge m > 0$ on an interval (c, d) containing a, and let $\alpha = \frac{M}{2m}$. Derive the error estimate

$$\alpha |x_{k+1} - a| < (\alpha |x_k - a|)^2.$$

(c) Show that $\sqrt[3]{2}$ is the limit of the sequence a_k where

$$a_0 = 2, \qquad a_{k+1} := \frac{2}{3}\Big(a_k + \frac{1}{a_k^2}\Big) \quad \text{for } k \ge 0.$$

Date: January 11, 2018, pm 3:30 – 6:30. A course by Chin-Lung Wang at NTU. Each problem is of 20 points. You may work on each part separately.