Complex Analytic and Differential Geometry Jean-Pierre Demailly

Chapter VIII : L^2 estimates on Pseudoconvex Manifolds

§11. Integrability of Almost Complex Structures

M: C^∞ manifold real dim $m=2n$ almost cpx structure $J \in \mathrm{End}(TM)$. $J: C^\infty$, $J^2 = -Id$

(M,J): almost cpx manifold. $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ $\dim_{\mathbb{C}} T^{1,0}M = \dim_{\mathbb{C}} T^{0,1}M = n$

$\Lambda T^*_{\mathbb{C}}M = \Lambda T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_k \bigoplus_{p+q=k} \Lambda^{p,q} T^*_{\mathbb{C}}M$ i-eigen space -i eigen space for $J_{\mathbb{C}} = J \otimes Id$.

↳ cotangent bundle. $\Lambda^{p,q} T^*_{\mathbb{C}}M = \Lambda^p (T^{1,0}M)^* \otimes_{\mathbb{C}} \Lambda^q (T^{0,1}M)^*$

<u>notation</u> : $C^s_{p,q}(M,E) := \{$differential forms, C^s and bidegree (p,q) on M, value in $E\}$

↳ cpx vector bundle

$\theta \cdot C^\infty(M, T^{1,0}M) \times C^\infty(M, T^{1,0}M) \longrightarrow C^\infty(M, T^{0,1}M)$ antisymmetric bilinear.

$(\xi, \eta) \longmapsto \pi^{0,1}[\xi,\eta]$

$f \in C^\infty(M,\mathbb{C})$ $[\xi, f\eta] = f[\xi,\eta] + (\xi \cdot f)\eta \in C^\infty(M, T^{1,0}M)$ $\Rightarrow \theta(\xi, f\eta) = f\,\theta(\xi,\eta)$

$\overset{''}{\xi(f\eta) - f\eta\xi} = (\xi f)\eta + f(\xi\eta - \eta\xi)$

So $\theta \in C^\infty_{2,0}(M, T^{0,1}M)$

If M: cpx analytic manifold (cpx manifold.) then $\theta = 0$. because local holo coord: $(z_1, \ldots, z_n)$

$([f_i \partial z_i, g_j \partial z_j] = f_i \partial z_i(g_j) \; \partial z_j - g_j(\partial z_j f_i) \partial z_i \in T^{1,0}(M)$ $[\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}] = 0$

<u>Def 11.4</u> $\theta \in C^\infty_{2,0}(M, T^{0,1}M)$: <u>torsion form</u> of J J is <u>integrable</u> if $\theta = 0$

<u>Example 11.5</u> M: real dim $m=2 \Rightarrow n=1$, no $(2,0)$-forms

For M: smooth oriented surface. g: Riemannian metric ⇝ $J \in \mathrm{End}(TM)$ rotation $\frac{\pi}{2}$.

change orientation $J \rightsquigarrow -J$

Conversely, given $J \Rightarrow TM$: cpx line bundle → M: oriented

g = J-invariant ⇔ J-hermitian when view TM cpx line bundle

So "conformal classes of Riemannian metric" ⟷ "almost cpx structure"

↳ given orientation

$u \in C^\infty_{p,q}(M,\mathbb{C})$ $d' = \pi^{p+1,q} \circ d$ $d'' := \pi^{p,q+1} \circ d$ $\xi \lrcorner u = u(\xi, -, \ldots, -)$

bidegree $(0,-1)$

$(\xi_1, \ldots, \xi_n)$: frame of $T^{1,0}M|_\Omega$ (section which is a basis at any point in Ω.)

$\theta = \sum_j \alpha_j \otimes \overline{\xi}_j$ $\alpha_j \in C^\infty_{2,0}(\Omega, \mathbb{C})$ ⇝ conjugate operator $\theta' u := \sum \alpha_j \wedge (\overline{\xi}_j \lrcorner u)$ bidegree $(+2,-1)$

θ', θ'' : derivation $\theta'(u \wedge v) = (\theta' u) \wedge v + (-1)^{\deg u} u \wedge (\theta' v)$. $\theta'' u := \sum_j \overline{\alpha_j} \wedge (\xi_j \lrcorner u)$ ⇝ $(-1,+2)$ $\theta'' = \overline{\theta'}$

same for θ''

(since α_j deg 2, $\xi \lrcorner -$: derivation)

<u>Prop 11.11</u> $d = d' + d'' - \theta' - \theta''$

<u>pf</u>: all derivation, only need to check deg 0 or 1. u: deg 0 $\Rightarrow \theta' u = \theta'' u = 0$ o.k.

u: 1-form ξ, η: cpx vector fields. $du(\xi,\eta) = \xi \cdot u(\eta) - \eta \cdot u(\xi) - u([\xi,\eta])$

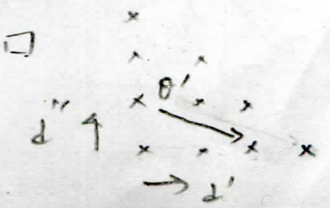
u: $(0,1)$-form ξ, η : type $(1,0) \Rightarrow \xi \cdot u(\eta) = 0 = \eta \cdot u(\xi) \Rightarrow (du)^{2,0}(\xi,\eta) = -u(\theta(\xi,\eta))$

locally

and $u(\theta(\xi,\eta)) = u(\sum_j \alpha_j(\xi,\eta)\,\overline{\xi}_j)$ $\theta' u(\xi,\eta) = \sum_j u(\overline{\xi}_j) \cdot \alpha_j(\xi,\eta)$

take bar $\Rightarrow$ this holds for $(1,0)$-form. done! $\in C^\infty(\Omega)$

$\square$



Hence. J : integrable $\overset{\text{def}}{\iff} \theta = 0 \iff \theta' = 0 = \theta''$ ($\because$ ξ_j : frame $\theta = 0 \iff \alpha_j = 0 \ \forall j$)

$\iff d = d' + d''$ $\qquad\qquad \iff \theta' = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$ $\because$ take ξ_j^*

Also, in this case $d'^2 = 0 = d''^2 \quad d'd'' + d''d' = 0$

Thm 11.8. (Newlander - Nirenberg) Every integrable almost cpx structure J on M. is defined by a unique analytic structure

pf based on [Hörmander 1966].

Def $f \in C^1(\Omega, \mathbb{C})$ $\Omega \subseteq M$. f is J-holomorphic if $d''f = 0$ $\qquad \pi^{0,1} \circ d(\overline{f_j})$

$$\begin{tikzcd} \Omega \arrow[r, "f:(f_1,\dots,f_p)"] & V \arrow[r, "h"] & \mathbb{C} \end{tikzcd}$$

$V \subseteq \mathbb{C}^p$ $\quad V$: open $\quad f_i \in C^1(\Omega, p)$

then $d''(h \circ f) = \sum_j \left(\frac{\partial h}{\partial z_j} \circ f\right) d'' f_j + \left(\frac{\partial h}{\partial \bar{z}_j} \circ f\right) \overset{\overset{\pi^{0,1}\circ d(\overline{f_j})}{\shortparallel}}{\overline{d' f_j}}$

(chain rule)

in particular $f_1, \dots, f_p$: J-holo h : holo $\Rightarrow h \circ f$: J-holo

local property

Suppose we construct holo. cpx coordinates $(z_1, \dots, z_n)$ on a nbh Ω_x of x $\forall x \in M$

$\hookrightarrow$ (all C^∞)

then change coordinate. $h : (z_k) \mapsto (w_k)$: holo in $\mathbb{C}^n$ sense. $\Rightarrow M$. has a cpx analytic altas

.since

$$\begin{tikzcd} & \Omega \subseteq M \arrow[dl, "f=(f_i)=(z_i)"'] \arrow[dr, "g=(g_k)=(w_k)"] & \\ \mathbb{C}^n \supseteq U \arrow[rr, "h"'] & & V \subseteq \mathbb{C}^n \end{tikzcd}$$

$h = g \circ f^{-1} = (h_k)$ $\quad h_k = g_k \circ f^{-1}$

$d''(h_k \circ f) = d''(g_k) = 0.$

$= \sum_j \left(\frac{\partial h_k}{\partial \bar{z}_j} \circ f\right) \overline{d' f_j}$ since $d'' f_j = 0 \ \forall j$

$d f_j \underset{\uparrow}{=} d' f_j$: basis for $T^*_{\mathbb{C}} M$ $\Rightarrow \frac{\partial h_k}{\partial \bar{z}_j} \circ f = 0 \ \forall j$

$d'' f_j = 0.$ $\qquad\qquad \Rightarrow h_k$: holo.

maximal

$\llcorner$ cpx analytic

uniqueness: If two atlas $\{(U_i, \varphi_i)\}$ $\{(V_j, \psi_j)\}$ of M, associated to same J

then they have same $T^{1,0}, T^{0,1}$ and d', d''

so $\psi_j \circ \varphi_i^{-1}$ all holo by above $\Rightarrow$. $\{(U_i, \varphi_i)\} \cup \{(V_j, \psi_j)\}$ still cpx analytic atlas

$\Rightarrow \{(U_i, \varphi_i)\} = \{(V_j, \psi_j)\}$ $\square$.

Lemma 11.10 $\forall a \in M$ $\quad s \geq 1, s \in \mathbb{N}.$ $\exists C^\infty$ cpx coordinate $(z_1, \dots, z_n)$ s.t. $d'' z_j = O(|z|^s)$ $\qquad$ (i.e. $d'' z_j = \sum f_k d\bar{z}_k$ $f_k \in O(|z|^s)$)

$1 \leq j \leq n$

pf: let $(\xi_1^*, \dots, \xi_n^*)$: basis of $\Lambda^{1,0} T^*_{\mathbb{C}} M|_a$ locally solve z_j s.t. $dz_j(a) = \xi_j^*$ (e.g. linear)

$\Rightarrow (z_1, \dots, z_n)$: coordinate (in smaller nbh). $s = 1$ o.k. $\qquad \Rightarrow d' z_j(a) = \xi_j^*$ $d'' z_j(a) = 0$

($dz_j, d\bar{z}_k$ linearly indep / $\mathbb{C}$)

if s hold. for $(z_1, \dots, z_n)$ Taylor expansion $d'' z_j = \sum_{k=1}^n P_{jk}(z, \bar{z}) \overline{d' z_k} + O(|z|^{s+1})$

$P_{jk}(z, w)$: homogeneous poly total degree s

J : integrable $\Rightarrow 0 = d''^2 z_j = \sum_{k,l} \Big(\frac{\partial P_{jk}}{\partial z_l} \underset{\hookrightarrow O(|z|^s)}{d'' z_l} \wedge \overline{d' z_k} + \frac{\partial P_{jk}}{\partial \bar{z}_l} \overline{d' z_l} \wedge \overline{d' z_k} \Big) + O(|z|^s)$

$= \sum_{k<l} \left(\frac{\partial P_{jk}}{\partial \bar{z}_l} - \frac{\partial P_{jl}}{\partial \bar{z}_k} \right) \overline{d' z_l} \wedge \overline{d' z_k} + O(|z|^s)$

$\hookrightarrow$ deg $= s-1$ $\therefore = 0$

since at a $d' z_j = dz_j$ basis of $T^{1,0}$

so near a still basis

$\overline{d'' z_j} = \sum f_j d' z_j$

$\Rightarrow d'' z_j = \sum \overline{f_j} \overline{d' z_j}$

$f_j = O(|z|^s)$

thus $\frac{\partial P_{jk}}{\partial \bar{z}_l} = \frac{\partial P_{jl}}{\partial \bar{z}_k}$ $\forall j, k, l.$

Let poly. $Q_j(z, \bar{z}) = \int_0^1 \sum_{1 \leq l \leq n} \bar{z}_l P_{jl}(z, t\bar{z}) \, dt$ $\quad$ deg $= s+1$

e.g. $z_1 z_3^3 \bar{z}_2^4 \bar{z}_5^2 \longrightarrow z_1 z_3^3 \bar{z}_2^4 \frac{\bar{z}_5^3}{7}$ for P_{j5}

then $\frac{\partial Q_j}{\partial \bar z_k} = \int_0^1 \frac{\partial}{\partial \bar z_k}\left(\sum_\ell \bar z_\ell P_{j\ell}(z,t\bar z)\right)dt = \int_0^1 \left(P_{jk} + \sum_\ell \bar z_\ell \cdot t \cdot \frac{\partial P_{j\ell}}{\partial \bar z_k}\right)(z,t\bar z)\,dt.$

$= \int_0^1 \left(P_{jk}+\sum_\ell \bar z_\ell \cdot t\, \frac{\partial P_{jk}}{\partial \bar z_\ell}\right)(z,t\bar z)\,dt = \int_0^1 \frac{d}{dt}\left(t\cdot P_{jk}(z,t\bar z).\right)dt = P_{jk}(z,\bar z)$

deg s

$\Rightarrow d''(z_j - Q_j(z,\bar z)) = \underbrace{d''z_j - \sum_k \frac{\partial Q_j}{\partial \bar z_k}\overline{d'z_k}}_{O(|z|^{s+1})} - \sum_k \frac{\partial Q_j}{\partial z_k} d''z_k = O(|z|^{s+1})$ $(s\ge1)$.

↳ $O(|z|^s)$, deg s. ↳ $O(|z|^s)$

thus $\tilde z_j = z_j - Q_j(z,\bar z)$

↳ deg $s+1\ge2$. coordinate. for $s+1$ □

↳ so $O(|z|^s)=O(|\tilde z|^s)$ if $|z|$ small.

All usually def. on cpx manifolds can extended to integrable almost cpx. manifold. e.g.

<u>Def</u>: φ is strictly plurisubharmonic if $id'd''\varphi$: positive definite (1,1)-form ⇝ $\omega = id'd''\varphi$. Kähler on (M,J)

<u>Lemma 11.11</u> $(z_1,\dots,z_n)$ coord. at $a\in M$ $d''z_j = O(|z|^s)$ $s\ge3$

then $\psi(z)=|z|^2$ $\psi_\varepsilon(z) = |z|^2 + \log(|z|^2+\varepsilon^2)$ $\varepsilon\in(0,1]$. strictly plurisubharmonic on $|z|<r_0$ ↗unif for ε

<u>pf</u> $id'd''\psi = id'd''\sum_j z_j\bar z_j = i\sum_j d'(\bar z_j d''z_j + z_j d''\bar z_j) = i\sum_j d'z_j\wedge\overline{d'z_j} + d'\bar z_j\wedge \underset{O(|z|^s)}{\underbrace{d''z_j}}$

$+ z_j d'd''\bar z_j + \bar z_j \underset{O(|z|^s)}{\underbrace{d'd''z_j}}$

note that by def it is real (1,1)-form.

Also $dz_i - d'z_i = d''z_i = O(|z|^s) \Rightarrow i\sum_j d'z_j\wedge\overline{d'z_j}$: positive near $z=0$ □ $\quad -d''\overline{d''z_j} = O(|z|^s)$

Also $id'd''\psi_\varepsilon = id'd''\psi + id'\sum_j \frac{\bar z_j d''z_j + z_j d''\bar z_j}{|z|^2+\varepsilon^2}$

$= id'd''\psi + i\sum_j \frac{d'z_j\wedge\overline{d'z_j} + d'\bar z_j\wedge \boxed{d''z_j} + \boxed{z_j d'd''\bar z_j} + \bar z_j \boxed{d'd''z_j}}{(|z|^2+\varepsilon^2)}$

$+ i\sum_j\sum_k -\frac{(\bar z_k d'z_k + \boxed{z_k d'\bar z_k})\wedge(\bar z_j \boxed{d''z_j} + z_j d''\bar z_j)}{(|z|^2+\varepsilon^2)^2}$

$= id'd''\psi + i\sum_j\frac{d'z_j\wedge\overline{d'z_j}}{|z|^2+\varepsilon^2} - i\sum_{j,k}\frac{\bar z_k z_j\, d'z_k\wedge\overline{d'z_j}}{(|z|^2+\varepsilon^2)^2} + O(|z|)$

↳ unif of ε.

clear std (1,1), real, sym. in matrix, $\frac{1}{|z|^2+\varepsilon^2}\left(I_n - \left(\frac{\bar z_k z_j}{|z|^2+\varepsilon^2}\right)\right) = \frac{1}{|z|^2+\varepsilon^2}\left(I_n - \bar\theta\theta^t\right)$

where $\theta = \begin{pmatrix} z_1\\ \vdots\\ z_n\end{pmatrix}\cdot\frac{1}{\sqrt{|z|^2+\varepsilon^2}} \Rightarrow \|\theta\| = \frac{|z|}{\sqrt{|z|^2+\varepsilon^2}} < 1$ ⇝ positive definite $\forall\varepsilon>0$

<u>pf of thm 11.8</u> notation as above. pseudoconvex open set. $\Omega = \{|z|<r\} = \{\psi(z) - r^2 < 0\}$

↳ exhaustion $r<r_0$ $-\log(r^2-\psi)$ see P.19.

Kähler metric $\omega = id'd''\psi$ $h\in\mathcal D(\Omega)$ (cpt supp C^∞) $0\le h\le1$. $h=1$ near $z=0$.

<u>Thm 6.5</u> on (0,1) forms $g_j = d''(z_j h(z)) \in C^\infty_{0,1}(\Omega,\mathbb C)$. , weight $\varphi(z) = A|z|^2 + (n+1)\log|z|^2$

$= \lim_{\varepsilon\to0} A|z|^2 + (n+1)\log(|z|^2+\varepsilon^2)$

note that $E=\mathbb C$ (trivial line bundle) $\Theta(E)=0$.

Lemma 11.11. $A\ge n+1 \Rightarrow \varphi$: plurisubharmonic. Choose A large enough s.t. $id'd''\varphi + Ricci(\omega) \ge \omega$

$\Rightarrow$ eigenvalues of $i\Theta(E) + id'd''\varphi + Ricci(\omega) \ge 1$ (w.r.t. ω), $d''g_j = 0 \Rightarrow \exists f_j$ st $d''f_j = g_j$ (by P.19 results) on Ω

but $g_j = d''z_j = O(|z|^s)$ $s\ge3$. $e^{-\varphi} = \Theta(|z|^{-2n-2})$ $\quad\int_\Omega |f_j|^2 e^{-\varphi}dV \le \int_\Omega |g_j|^2 e^{-\varphi}dV$

$\Rightarrow$ RHS conv. $\Rightarrow \int |f_j(z)|^2|z|^{-2n-2}dV$ conv. at $z=0$ f_j smooth $\Rightarrow f_j(0) = df_j(0) = 0$

take $\tilde z_j = z_j h(z) - f_j$ $1\le j\le n$ $\Rightarrow d''\tilde z_j = 0$, $d\tilde z_j(0) = dz_j(0)$ $\Rightarrow$ still coordinate □.

P.3 ($h\equiv1$ near 0)

§1. Non Bounded Operators on Hilbert Spaces

H_1, H_2 : cpx Hilbert space $T: \mathrm{Dom}\,T \overset{\subseteq H_1}{\longrightarrow} H_2$ linear. ($\mathrm{Dom}\,T$: subspace)

Assumption: T is densely defined ($\mathrm{Dom}\,T$ dense in H_1), closed ($\mathrm{Gr}\,T = \{(x, Tx) \mid x \in \mathrm{Dom}\,T\}$ (graph) closed in $H_1 \times H_2$)

(Von Neumann) adjoint $T^*: \mathrm{Dom}\,T^* \overset{\subseteq H_2}{\longrightarrow} H_1$, $y \in \mathrm{Dom}\,T^*$ iff $\mathrm{Dom}\,T \to \mathbb{C}$, $x \mapsto \langle Tx, y\rangle_2$ is bounded

$\mathrm{Dom}\,T^*$ dense $\Rightarrow$ $x \mapsto \langle Tx, y\rangle_2$ def. on H_1, by Riesz $\exists!\, T^*y \in H_1$ s.t. $\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$, $\forall x \in \mathrm{Dom}\,T^*$

$\mathrm{Gr}\,T^* \ni (T^*y, y) \Leftrightarrow \langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$ $\forall x \in \mathrm{Dom}\,T$ $\Leftrightarrow (T^*y, y) \perp (-Tx, x)$ $\Leftrightarrow (T^*y, y) \in (\mathrm{Gr}(-T))^\perp$.

$\Rightarrow \mathrm{Gr}\,T^*$: closed $\Rightarrow T^*$: closed. and $H_1 \times H_2 = \mathrm{Gr}(-T) \overset{\perp}{\oplus} \mathrm{Gr}(T^*)$

$(u, v) = (x, -Tx) + (T^*y, y)$

Take $u = 0 \Rightarrow x = -T^*y$ $v = y - Tx = y + TT^*y$

$v \in (\mathrm{Dom}\,T^*)^\perp \Rightarrow \langle v, y\rangle_2 = 0 \Rightarrow y = 0 \Rightarrow v = 0$ $\langle v, y\rangle = \langle y, y\rangle + \langle TT^*y, y\rangle = \|y\|_2^2 + \|T^*y\|_1^2$

$\Rightarrow T^*$: densely defined. ($\overline{\mathrm{Dom}\,T^*} = ((\mathrm{Dom}\,T^*)^\perp)^\perp = \{0\}^\perp = H_2$)

Thm 1.1 $T: \mathrm{Dom}\,T \overset{\subseteq H_1}{\rightarrow} H_2$ closed, densely defined, linear then so is T^*, $(T^*)^* = T$.

Also $\ker T^* = (\mathrm{Im}\,T)^\perp$. $(\ker T)^\perp = \overline{\mathrm{Im}\,T^*}$ ⇝ by taking $\perp$.

pf: by above. $\mathrm{Gr}((T^*)^*) = \mathrm{Gr}(-T^*)^\perp = F_1(\mathrm{Gr}(T^*)^\perp) = F_1(\mathrm{Gr}(-T)) = F_1F_2\,\mathrm{Gr}(T) \overset{\text{linear}}{\Rightarrow} \overset{\mathrm{Gr}(T)}{''}$ $\Rightarrow (T^*)^* = T$

$y_0 \in \ker T^* \Leftrightarrow (0, y_0) \in \mathrm{Gr}(T^*) = \mathrm{Gr}(-T)^\perp \Leftrightarrow y \in (\mathrm{Im}\,T)^\perp$ F_1 : reflection $(x, y) \mapsto (-x, y)$

$\Leftrightarrow \langle (x, -Tx), (0, y_0)\rangle_{H_1 \times H_2} = \langle -Tx, y_0\rangle_2 = 0$ $\forall x \in \mathrm{Dom}\,T$. □ F_2 '' $(x, y) \mapsto (x, -y)$

Now consider T, S closed, densely defined $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ $S \circ T = 0$

($T(\mathrm{Dom}\,T) \subseteq \ker S \subseteq \mathrm{Dom}\,S$)

Thm 1.2 $H_2 = (\ker S \cap \ker T^*) \overset{\perp}{\oplus} \overline{\mathrm{Im}\,T} \overset{\perp}{\oplus} \overline{\mathrm{Im}\,S^*}$ (1)

$\ker S = (\ker S \cap \ker T^*) \overset{\perp}{\oplus} \overline{\mathrm{Im}\,T}$ (2). ↗ or only need $|\langle v, x\rangle_2|^2 \le M(\|T^*x\|_1^2 + \|Sx\|_3^2)$ if want to solve. (**)

If $\|T^*x\|_1^2 + \|Sx\|_3^2 \ge C\|x\|_2^2$ $\forall x \in \mathrm{Dom}\,S \cap \mathrm{Dom}\,T^*$ for some $C > 0$. (*). $Tu = v$:

then $\forall v \in H_2$ $Sv = 0$. $\exists u \in H_1$ s.t. $Tu = v$ $\|u\|_1^2 \le \frac{1}{C}\|v\|_2^2$ ($\|u\|_1^2 \le M$)

in particular $\overline{\mathrm{Im}\,T} = \mathrm{Im}\,T = \ker S$, $\overline{\mathrm{Im}\,S^*} = \mathrm{Im}\,S^* = \ker T^*$.

pf: S: closed $\Rightarrow \ker S$: closed in H_2. ($y_n \in S$ $y_n \to y \Rightarrow (0, y_n) \to (0, y) \in \mathrm{Gr}(S)$)

$(\ker S)^\perp = \overline{\mathrm{Im}\,S^*} \Rightarrow H_2 = \ker S \overset{\perp}{\oplus} \overline{\mathrm{Im}\,S^*}$ $= \ker T^* \oplus \overline{\mathrm{Im}\,T}$ similarly

$S \circ T = 0 \Rightarrow \overline{\mathrm{Im}\,T} \subseteq \ker S \Rightarrow \ker S = (\ker S \cap \ker T^*) \overset{\perp}{\oplus} \overline{\mathrm{Im}\,T}$ → (1), (2) O.K.

Suppose (*) holds. $Sv = 0$ $x \in \mathrm{Dom}\,T^* \Rightarrow x = x' + x''$ $x' \in \ker S$ $x'' \in (\ker S)^\perp \subseteq (\mathrm{Im}\,T)^\perp = \ker T^*$

$x, x'' \in \mathrm{Dom}\,T^* \Rightarrow x' \in \mathrm{Dom}\,T^* \Rightarrow \langle v, x\rangle_2 = \langle v, x'\rangle_2$ ($x'' \in (\ker S)^\perp$) ($\mathrm{Im}\,T \subseteq \ker S$)

$Sx' = 0$ $T^*x'' = 0 \Rightarrow |\langle v, x\rangle_2|^2 \underset{\text{Cauchy}}{\le} \|v\|_2^2\|x'\|_2^2 \underset{(*)}{\le} \frac{1}{C}\|v\|_2^2\|T^*x'\|_1^2 = \frac{1}{C}\|v\|_2^2\|T^*x\|_1^2$

and $Sx' = 0$ $T^*x'' = 0$ ↘ const for fix v

Hence $T^*x \longmapsto \langle x, v\rangle_2$ conti. on $\mathrm{Im}\,T^* \subseteq H_1$ (We have $\ker T^* = \overline{\mathrm{Im}\,T}^\perp = (\ker S \cap \ker T^*) \overset{\perp}{\oplus} \underset{= (\ker S)^\perp}{\overline{\mathrm{Im}\,S^*}}$, $v \in \ker S$ and (**) $\Rightarrow$ well-def.)

norm $\le \frac{\|v\|_2}{\sqrt{C}}$ ↳ as required. Hahn-Banach + Riesz $\Rightarrow \exists u \in H_1$ s.t. $\langle x, v\rangle_2 = \langle T^*x, u\rangle_1$ $\forall x \in \mathrm{Dom}\,T^*$

Hence $u \in \mathrm{Dom}\,(T^*)^* = \mathrm{Dom}\,T$, $v = Tu$.

Hence $\ker S = \mathrm{Im}\,T$: closed. $\mathrm{Im}\,S^* = \ker T^*$ by consider (S^*, T^*)

($T^* \circ S^*$ well-def since $\overline{\mathrm{Im}\,S^*} = (\ker S)^\perp \subseteq (\mathrm{Im}\,T)^\perp = \ker T^*$)

$T^* \circ S^* = 0$

§2. Complete Riemannian Manifolds

(M,g) : Riemannian manifold. dim m. metrik $\sum g_{jk}(x)\, dx_j \otimes dx_k = g(x)$ $1 \le j, k \le m$

length of a path $\gamma : [a,b] \to M$. $\ell(\gamma) = \int_a^b |\gamma'(t)|_g\, dt = \int_a^b \Big(\sum_{j,k} g_{jk}(\gamma(t))\, \gamma_j'(t)\, \gamma_k'(t)\Big)^{\frac{1}{2}} dt$

geodesic distance. $\delta(x,y) = \inf_\gamma \ell(\gamma)$ $\gamma(a)=x, \gamma(b)=y$. if such γ not exists $\delta(x,y) := +\infty$

if $x \neq y \to$?

coordinate ball $\bar{B}$ $g \ge c|dx|^2$ on $\bar{B}$ $\Rightarrow$ $\delta(x,y) \ge \sqrt{c}\, r. > 0$

Also $\inf_{z \in M} \max\{\delta(x,z), \delta(y,z)\} = \frac{1}{2}\delta(x,y)$ by taking z = mid-point in γ.

$\rightsquigarrow$ geodesic metric space

$\rightsquigarrow$ inductively $\exists\, x = x_0, x_1, \dots, x_N = y$ s.t. $\delta(x_j, x_{j+1}) < \varepsilon$ $\sum \delta(x_j, x_{j+1}) < \delta(x,y) + \varepsilon$! (*) (assume connected)

Lemma 2.2 (Hopf-Rinow) (E, δ) : geodesic metric space. then (dichotomy 二分法)

" E : locally cpt, complete " $\Longleftrightarrow$ " $\bar{B}(x_0, r)$: cpt. $\forall x_0 \in E$ $r > 0$ "

pf : (b) $\Rightarrow$ (a) clear (by (*) $B(x_0, r)$ closure is $\bar{B}(x_0, r)$)

(a) $\Rightarrow$ (b). Let. $R = \sup\{r \mid \bar{B}(x_0, r)$ cpt, $r > 0\}$. for fix x locally cpt $\Rightarrow$ $R > 0$.

If $R < +\infty$. $y_\nu \in \bar{B}(x_0, R)$ fix $p \in \mathbb{N}$. Using diagonal argument

(*) $\Rightarrow$ $\exists\, z_\nu \in M$ $\delta(x_0, z_\nu) \le (1 - 2^{-p})R$ $\delta(z_\nu, y_\nu) \le \delta(x_0, y_\nu) + 2^{-p}R - \delta(x_0, z_\nu)$
$\le 2^{1-p} R$.

$\bar{B}(x_0, (1 - 2^{-p})R)$ cpt $\Rightarrow$ $\exists\, (z_{\nu(p,q)})_{q \in \mathbb{N}}$ conv. to w_p.

induction on p, $\nu(p+1, q)$: subseq. of $\nu(p,q)$. with $\delta(z_{\nu(p,q)}, w_p) \le 2^{-q}$

$\Rightarrow$ $\delta(y_{\nu(p,q)}, w_p) \le \delta(y_{\nu(p,q)}, z_{\nu(p,q)}) + \delta(z_{\nu(p,q)}, w_p) \le 2^{1-p}R + 2^{-q}$ $\forall q$

so subseq $(y_{\nu(p+1,q)})$ limit w_{p+1} have $\delta(w_{p+1}, w_p) \le 2^{1-p} R$ by $q \to \infty$

complete $\Rightarrow$ $w_p \to w \in M$. $\Rightarrow$ $\delta(y_{\nu(p,p)}, w_p) \le 2^{1-p}R + 2^{-p}$ $\to$ Cauchy seq

so $y_{\nu(p,p)} \to w$

Hence $\boxed{\bar{B}(x_0, R) : \text{cpt.}}$

locally cpt $\Rightarrow$ $\bar{B}(x_0, R) \subseteq \bigcup_{j=1}^{N} B(y_j, \frac{\varepsilon_{y_j}}{3})$ $y_j \in \bar{B}(x_0, R)$ $\bar{B}(y_j, \frac{\varepsilon_{y_j}}{2})$: cpt.

$\varepsilon := \min \varepsilon_{y_j}$ $\Rightarrow$ $\bar{B}(x_0, R + \frac{\varepsilon}{2}) \subseteq \bigcup \bar{B}(y_j, \varepsilon_{y_j})$: cpt $\rightarrow\!\!\times$ $\square$

($\forall z \in \bar{B}(x_0, R + \frac{\varepsilon}{2})$ by (*) $\exists\, y \in M$ $\delta(y, z) \le \frac{2\varepsilon}{3} \le \frac{2\varepsilon_{y_j}}{3}$ $\forall j$

thus $y \in \bar{B}(y_j, \frac{\varepsilon_{y_j}}{3})$ for some j o.k.) $\delta(x,y) \le \delta(x,z) - \delta(y,z) + \frac{\varepsilon}{3}$
$\le R \Rightarrow y \in \bar{B}(x_0, R)$

Def 2.3 (a) (M,g) Riemannian manifold. (M,g) is complete if (M, δ) : complete as metric space

(b) $\psi : M \to \mathbb{R}$ conti is exhaustive if $\forall c \in \mathbb{R}$. $M_c = \{x \in M \mid \psi(x) < c\} = \psi^{-1}(-\infty, c)$.

is relatively cpt in M

(c) $(K_\nu)_{\nu \in \mathbb{N}}$ of cpt set. in M is exhaustive if $M = \bigcup K_\nu$ $K_\nu \subseteq \operatorname{int}(K_{\nu+1})$ $\forall \nu$

Lemma 2.4 following are equivalent. (a) (M,g) Complete (b) $\exists\, \psi \in C^\infty(M,\mathbb{R})$ exhaustive st. $|d\psi|_g \le 1$

(c) $\exists\, (K_\nu)_{\nu\in\mathbb{N}}$ exhaustive seq of cpt subset $\psi_\nu \in C^\infty(M,\mathbb{R})$ s.t. $\psi_\nu \equiv 1$ in nbh of K_ν
$\operatorname{Supp}\psi_\nu \subseteq \operatorname{int}(K_{\nu+1})$

Pf: (a) ⇒ (b). WLOG M: connected pick $x_0 \in M$ $\psi_0(x) = \frac{1}{2}\delta(x_0,x)$ $0 \le \psi_\nu \le 1$ $|d\psi_\nu| \le 2^{-\nu}$

⇒ ψ_0: Lipschitz with const $\frac{1}{2}$ (triangular inequality)

by Rademacher's thm. ψ_0 diff a.e. $|d\psi_0|_g \le \frac{1}{2}$. find smoothing ψ of ψ_0 s.t. $|d\psi|_g \le 1$

by lemma 2.2 $\overline{B}(x_0,r)$ cpt $\forall r>0$ so $\psi^{-1}(-\infty,c)$ relatively cpt. □ $|\psi - \psi_0| \le 1$

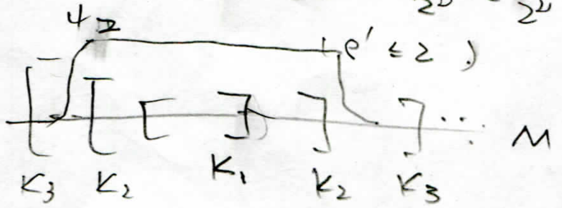
(b) ⇒ (c). choose $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ st. $\rho \equiv 1$ on $(-\infty, 1.1]$. $\rho = 0$ on $[1.9, +\infty)$

then $K_\nu = \{x \in M \mid \psi(x) \le 2^{\nu+1}\}$ exhausion. $0 \le \rho' \le 2$ on $[1,2]$

$\psi_\nu(x) := \rho(2^{-\nu-1}\psi(x))$ $|d\psi_\nu| = |\rho'(2^{-\nu-1}\psi(x)) \cdot 2^{-\nu-1} d\psi| \le \frac{|d\psi|}{2^\nu} \le \frac{1}{2^\nu}$

(c) ⇒ (b) set $\psi = \sum 2^{\nu-1}(1-\psi_\nu)$

on K_n $\infty > \sum_{\nu \le n-1} 2^{\nu-1} \ge \psi \rightsquigarrow C^\infty$



Also on $K_{n+2} \setminus K_n$ $d\psi_\nu = 0$ except ψ_n, ψ_{n+1}. ($\psi_\nu \equiv 0$ for $\nu \le n-1$)
$\psi_\nu \equiv 1$ for $\nu \ge n+2$

so $|d\psi| \le |2^{n-1} d\psi_n| + |2^n d\psi_{n+1}| \le 1$

(b) ⇒ (a) $|d\psi|_g \le 1 \Rightarrow |\psi(x) - \psi(y)| = |\int_\gamma d\psi| \le \int_\gamma |d\psi|_g |\gamma'|_g \le \int_\gamma |\gamma'|_g = \ell(\gamma)$

so $|\psi(x) - \psi(y)| \le \delta(x,y)$ $\forall x,y \in M$ ⇒ $\overline{B}(x_0,R) \subseteq \psi^{-1}(-\infty, \psi(x_0) + R + 1)$
$|\psi(y)| \le |\psi(x)| + \delta(x,y)$
relatively cpt. □

§3. L^2 Hodge Theory on Complete Riemannian Manifolds

(M,g): Riemannian manifold F_1, F_2: hermitian C^∞-vector bundle. on M.

$P: C^\infty(M,F_1) \to C^\infty(M,F_2)$ differtial operator, smooth. coeff.

$\rightsquigarrow$ $\tilde{P}: L^2(M,F_1) \to L^2(M,F_2)$ non bounded operator.

$u \in L^2(M,F_1) \subseteq L^1_{loc}(M,F_1)$ $\rightsquigarrow$ $\tilde{P}u$ exists in distribution sense. → $u \in \operatorname{Dom}\tilde{P}$ if $\tilde{P}u \in L^2(M,F_2)$

$\mathcal{D}(M,F_1)$ = cpt supp section $\subseteq \operatorname{Dom} P$ ⇒ densely defined
↳ dense in $L^2(M,F_1)$

Also $\operatorname{Gr}\tilde{P}$: closed if $u_\nu \to u$ in $L^2(M,F_1)$ $\tilde{P}u_\nu \to v \in L^2(M,F_2)$

by def φ: C^∞ cpt supp $\langle \tilde{P}u_\nu, \varphi \rangle = \langle u_\nu, P^*\varphi \rangle \to \langle u, P^*\varphi \rangle$
$= \langle \tilde{P}u, \varphi \rangle$

so $\tilde{P}u_\nu \to \tilde{P}u$ in weak topology ⇒ $v = \tilde{P}u$. $(u,v) \in \operatorname{Gr}\tilde{P}$

By §1. $\tilde{P}$ has closed, densely def. $(\tilde{P})^*$. $\langle Pu, v \rangle = \langle u, P^*v \rangle$ for $u \in C^\infty(M,F_1)$ $v \in C^\infty(M,F_2)$
$\operatorname{supp} u \cap \operatorname{supp} v \subset\subset M$

<u>Rmk</u>: We have $P^*: C^\infty(M,F_2) \to C^\infty(M,F_1)$ and extend $\widetilde{P^*}: L^2(M,F_2) \to L^2(M,F_1)$

but in general $\widetilde{P^*} \ne (\tilde{P})^*$ $u \in \operatorname{Dom}((\tilde{P})^*) \Leftrightarrow \exists\, v \in L^2(M,F_1)$ $\langle u, \tilde{P}f \rangle = \langle v, f \rangle$
$\forall f \in \operatorname{Dom}\tilde{P}$

$u \in \operatorname{Dom}(\widetilde{P^*}) \Leftrightarrow \exists\, v \in L^2(M,F_1)$ $\langle u, \tilde{P}f \rangle = \langle v, f \rangle$
$\forall f \in \mathcal{D}(M,F_1)$

$\operatorname{Dom}(\tilde{P})^* \subseteq \operatorname{Dom}(\widetilde{P^*})$ but may be strict.

Example 3.1 $P = \frac{d}{dx} : L^2(0,1) \to L^2(0,1)$ (use Lebesgue measure). $(F_1 = F_2 = \mathbb{C})$.

$\operatorname{Dom}\tilde{P} = \{ f \in L^2 \mid f$ have L^2 derivatives $\}$. $\tilde{P}$: elliptic. deg 1 $\tilde{P}f \in L^2 = H_0 \overset{\text{Regularity thm}}{\Rightarrow} f \in H_1$ (Sobolev space) $\Rightarrow f \in C^0$ since $0 + \frac{m}{2} < 1$ $m = 1$ $(\mathbb{R}^1)$

Actually in Sobolev lemma we find $f_j \in C_0^0(\mathbb{R}^m)$ s.t. $f_j \to f$. by Sobolev lemma. so f : Conti. on $[0,1]$

integration by part : for $f \in C_0^\infty(0,1)$, $u \in C^\infty(0,1)$. $\langle P^* u, f \rangle = \langle u, Pf \rangle = \int_0^1 u \overline{\frac{df}{dx}} \underset{q}{=} \int_0^1 -\frac{du}{dx} \bar{f}$ f cpt supp

so $P^* u = -\frac{du}{dx}$ $P^* = -\frac{d}{dx} = -P$

$u \in \operatorname{Dom}(\tilde{P})^*$ iff. $\operatorname{Dom}\tilde{P} \to \mathbb{C}$, $f \mapsto \langle Pf, u \rangle$ is conti. but. $\langle Pf, u \rangle = \int_0^1 f \overline{\left(-\frac{du}{dx}\right)} + f(1)\overline{u(1)} - f(0)\overline{u(0)}$

Conti. w.r.t. L^2-norm $\Rightarrow u(1) = u(0) = 0$ ($\because$ we can find $|f-g|_{L^2}$ small but $|f(1) - g(1)|$ big or $|f(0) - g(0)|$ big)

so $\operatorname{Dom}(\tilde{P^*}) = \operatorname{Dom}\tilde{P} \supsetneq \operatorname{Dom}(\tilde{P})^* = \{ u \in \operatorname{Dom}\tilde{P} \mid u(0) = u(1) = 1 \}$ $\square$.

chap V. §2. Linear connection

Def : A (linear) connection D on bundle E. is linear diff. operator of order 1 on $C^\infty_\bullet(M,E) = \bigoplus_{p=0}^{m} C^\infty_p(M,E)$

and satisfy $D : C^\infty_q(M,E) \to C^\infty_{q+1}(M,E)$

$f \wedge s \mapsto df \wedge s + (-1)^p f \wedge Ds$ $\forall f \in C^\infty_p(M,\mathbb{C})$, $s \in C^\infty_q(M,E)$

local trivialization $\theta : E|_\Omega \to \Omega \times \mathbb{C}^r$ $(e_1, \ldots, e_r)$: frame $s = \sum \sigma_\lambda \otimes e_\lambda \in C^\infty_q(\Omega, E)$

then $Ds = \sum d\sigma_\lambda \otimes e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda$ write $De_\lambda = \sum_\mu a_{\mu\lambda} \otimes e_\mu$ $\sigma_\lambda \in C^\infty_q(\Omega, \mathbb{C})$

then $Ds \underset{\theta}{\equiv} d\sigma + A \wedge \sigma$ $A = (a_{\lambda\mu}) \in C^\infty_1(\Omega, \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ $a_{\lambda\mu} \in C^\infty_1(\Omega, \mathbb{C})$

$(-1)^p \sigma_\lambda \wedge a_{\mu\lambda} = a_{\mu\lambda} \wedge \sigma_\lambda$). A : connection form of D associated to θ.

Also let $\delta = D^* = (-1)^{mp+1} * D *$ formal adjoint $\Delta = D\delta + \delta D$ Laplacian when E : hermitian manifold (It seems that we don't need Chern connection here)

All extend to $L^2_\bullet(M,E) = \bigoplus_{p=0}^{m} L^2_p(M,E)$ M : dim $= m$ / $\mathbb{R}$, closed, densely defined

Thm 3.2 Assume (M, g) is complete. then

(a) $D_\bullet(M,E)$ Dense in $\operatorname{Dom}D$, $\operatorname{Dom}\delta$ $\operatorname{Dom}D \cap \operatorname{Dom}\delta$ w.r.t. $u \mapsto \|u\| + \|Du\|$, $u \mapsto \|u\| + \|\delta u\|$, $u \mapsto \|u\| + \|Du\| + \|\delta u\|$. resp.

(b) $D^* = \delta$, $\delta^* = D$ in Von Neumann's sense.

(c). $\langle u, \Delta u \rangle = \|Du\|^2 + \|\delta u\|^2$ $\forall u \in \operatorname{Dom}\Delta$

so $\operatorname{Dom}\Delta \subseteq \operatorname{Dom}D \cap \operatorname{Dom}\delta$, $\operatorname{Ker}\Delta = \operatorname{Ker}D \cap \operatorname{Ker}\delta$. Δ : self-adjoint.

(d) If $D^2 = 0$ then $L^2_\bullet(M,E) = H^\bullet(M,E) \oplus \overline{\operatorname{Im}D} \oplus \overline{\operatorname{Im}\delta}$

$\operatorname{Ker}D = H^\bullet(M,E) \oplus \overline{\operatorname{Im}D}$

$H^\bullet(M,E) = \{ u \in L^2_\bullet(M,E) \mid \Delta u = 0 \} \subseteq C^\infty_\bullet(M,E)$ by Hodge theory Δ : elliptic. harmonic forms

Pf : for $u \in \operatorname{Dom}D$ $u, Du \in L^2_\bullet(M,E)$ (ψ_ν) as in lemma 2.4 (c) $\Rightarrow \psi_\nu u \to u$ in $L^2_\bullet(M,E)$ by DCT

Also $D(\psi_\nu u) = \psi_\nu Du + d\psi_\nu \wedge u$. Also $|d\psi_\nu \wedge u| \le |d\psi_\nu||u| \le 2^{-\nu}|u| \to 0$ as $\nu \to \infty$

Hence $D(\psi_\nu u) \to Du$. So replace u by $\psi_\nu u$. u : cpt supp.

finite partition of unity on supp u $\Rightarrow$ Assume. supp $u \subseteq$ a coordinate chart. E : trivial on it on this chart say $D = d + A\wedge\bullet$ (ρ_ε) : fanily of smoothing kernels.

so $u * \rho_\varepsilon \in D_0(M,E) \to u$ in $L^2(M,E)$ $D(u*\rho_\varepsilon) - (Du)*\rho_\varepsilon = A\wedge(u*\rho_\varepsilon) - (A\wedge u)*\rho_\varepsilon$
(convolution)

since $d(u*\rho_\varepsilon) = (du)*\rho_\varepsilon$.

Also $Du*\rho_\varepsilon \to Du$ in $L^2(M,E)$ $A \in \mathcal{C}^\infty_1(\Omega, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ u : cpt supp.

so $A\wedge(u*\rho_\varepsilon) \to A\wedge u$ ($u*\rho_\varepsilon$: unif cpt supp)

Hence. $D(u*\rho_\varepsilon) \to Du$. $(A\wedge u)*\rho_\varepsilon \to A\wedge u$.

And thus $D_0(M,E)$ dense in Dom D

For $\delta = (-1)^{mp+1} * D *$ $\delta(\psi_\nu u) = (-1)^{mp+1} * D(\psi_\nu(*u)). = (-1)^{mp+1} *(d\psi_\nu(*u)) + (-1)^{mp+1} * \psi_\nu(D(*u))$

↑ Hodge star

$= (-1)^{mp+1} * (d\psi_\nu\ *u)) + \psi_\nu \delta u$

↘ 0.   ↘ δu.

so still can assume supp $u \subseteq$ a chart on M.   For (ρ_ε) estimate. need below lemma.

<u>Lemma 3.3</u> (K.O. Friedrichs). $Pf = \sum a_k \frac{\partial f}{\partial x_k} + bf$. diff. operator of order 1 on $\Omega \subseteq \mathbb{R}^n$

then for any $v \in L^2(\mathbb{R}^n)$ supp $v \subseteq \Omega$, cpt. $a_k \in C^1(\Omega)$ $b \in C^0(\Omega)$

we have $\lim_{\varepsilon\to 0} \| P(v*\rho_\varepsilon) - (Pv)*\rho_\varepsilon \|_{L^2} = 0$

pf: It suffice to consider $P = a\frac{\partial}{\partial x_k}$. $\left(\| b\cdot(v*\rho_\varepsilon) - (bv)*\rho_\varepsilon \|^2_{L^2} = \int_{\mathbb{R}^2} \left| \int_{B_\varepsilon(0)} (b(x) - b(x-y))\, v(x-y)\rho_\varepsilon(y) \right|^2 dx \right.$

↳ linear

if $v \in C^1 \Rightarrow$ o.k. by right. $\le \int_{\mathbb{R}^2} \varepsilon' \left(\int_{B_\varepsilon(0)} |v(x-y)| \rho_\varepsilon(y) \right)^2 dx \to 0$ by unif conti. of b

$\left(\frac{\partial}{\partial x_k}(v*\rho_\varepsilon) = (\frac{\partial}{\partial x_k} v)*\rho_\varepsilon \right)$. $\left. = \varepsilon' \cdot \| |v| * \rho_\varepsilon \|_{L^2}. \right)$ (take $\rho_\varepsilon \geq 0$)

First we show $\| P(v*\rho_\varepsilon) - (Pv)*\rho_\varepsilon \|_{L^2} \le C \|v\|_{L^2}$

by integration by part $W_\varepsilon := P(v*\rho_\varepsilon) - (Pv)*\rho_\varepsilon = \int_{\mathbb{R}^n} (a(x) - a(x-\varepsilon y)) \frac{\partial v}{\partial x_k}(x-\varepsilon y)\rho(y)\, dy$.

($\frac{\partial v}{\partial x_k}$ exists in distribution)

$= \int_{\mathbb{R}^n} (a(x) - a(x-\varepsilon y))\, v(x-\varepsilon y) \frac{1}{\varepsilon}\frac{\partial \rho}{\partial y_k}(y)\, dy + \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} a(x-\varepsilon(y))\, v(x-\varepsilon y)\rho(y)\, dy$

↑ $\frac{\partial v}{\partial x_k}(x-\varepsilon y) = \frac{-\partial v}{\partial \varepsilon y_k}(x-\varepsilon y)$

say $|\partial a| \le C_0$ near supp v.   then $|W_\varepsilon(x)| \le C_0 \int_{\mathbb{R}^n} |v(x-\varepsilon y)| \left(|y|. |\partial_k \rho(y)| + |\rho(y)| \right) dy$

by Minkowski's inequality $\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$. $\|W_\varepsilon\|_{L^2} \le C_0 \|v\|_{L^2} \cdot \int_{\mathbb{R}^n} (|y| |\partial_k \rho(y)| + |\rho(y)|)\, dx \le C\|v\|_{L^2}$
(Young's convolution thm)   (ρ : cpt supp)

For general v, given $\tilde\varepsilon > 0$ take $\underset{\in C^1}{\tilde v}$ s.t. $\|v - \tilde v\|_{L^2} \le \tilde\varepsilon$.

then $\limsup_{\varepsilon\to 0} \| P(v*\rho_\varepsilon) - (Pv)*\rho_\varepsilon \|_{L^2} \le \limsup_{\varepsilon\to 0} \| P((v-\tilde v)*\rho_\varepsilon) - P(v-\tilde v)*\rho_\varepsilon \|_{L^2}$
$+ \| P(\tilde v * \rho_\varepsilon) - P(\tilde v)*\rho_\varepsilon \|_{L^2}$.

$\tilde\varepsilon$: arbitrary $\Rightarrow$ done! □. $\le C \cdot \|v - \tilde v\|_{L^2} \le C \cdot \tilde\varepsilon$.

For Dom $D \cap$ Dom δ. clear by above. argument. □

(b) only need $\langle Du, v\rangle = \langle u, \delta v\rangle$ $\forall u \in$ Dom D $v \in$ Dom δ.   (so Dom δ = Dom$(\widetilde{D^*}) \subseteq$ Dom D^*)
but this holds for $u, v \in D_0(M,E)$ by def. of δ. by (a) + taking limit
ok.

(c) let $u\in \mathrm{Dom}\,\Delta$ Δ : elliptic operator, order $=2$ $\Rightarrow$ $u\in W^2_\bullet(M,E,loc)$ (Sobolev space, by Gårding's inequality)

in particular $Du,\ \delta u\in L^2(M,E,loc)$. $(\Delta u\in W^0_\bullet(M,E,loc)=L^2(M,E,loc))$

, we can use integration by part after multiply ψ_ν cpt-supp)

Hence $\|\psi_\nu Du\|^2+\|\psi_\nu\delta u\|^2=\langle\psi_\nu^2 Du,Du\rangle+\langle u,D(\psi_\nu^2\delta u)\rangle$ (i.e. use $D^*=\delta$)

$(\because \delta^*=D)$.

$=\langle \underset{\sim}{D}(\psi_\nu^2 u)-2\psi_\nu\cdot d\psi_\nu\wedge u,Du\rangle+\langle \underset{\sim}{u},\underline{\psi_\nu^2 D\delta u}+2\psi_\nu d\psi_\nu\wedge\delta u\rangle$

$=\underset{\sim}{\langle\psi_\nu^2 u,\delta Du\rangle}+\underline{\langle\psi_\nu^2 u,D\delta u\rangle}-2\langle d\psi_\nu\wedge u,\psi_\nu Du\rangle+2\langle\psi_\nu u,d\psi_\nu\wedge\delta u\rangle$

$\leq\langle\psi_\nu^2 u,\Delta u\rangle+2\cdot 2^{-\nu}\|u\|\,\|\psi_\nu Du\|+2\cdot 2^{-\nu}\|u\|\,\|\delta u\|$

$\leq\langle\psi_\nu^2 u,\Delta u\rangle+2^{-\nu}(\|\psi_\nu Du\|^2+\|\psi_\nu\delta u\|^2+2\|u\|^2)$. $(2|a|\cdot|b|\leq|a|^2+|b|^2)$

Hence $\|\psi_\nu Du\|^2+\|\psi_\nu\delta u\|^2\leq\frac{1}{1-2^{-\nu}}(\langle\psi_\nu^2 u,\Delta u\rangle+2^{1-\nu}\|u\|^2)$

note that $\psi_\nu\nearrow$ by def. ($\mathrm{supp}\,\psi_\nu\subseteq\mathrm{int}(K_{\nu+1})$ $K_{\nu+1}\subseteq K_{\nu+2}\subseteq\ldots$).

$\psi_\nu\equiv 1$ on K_ν.

so by MCT $\|Du\|^2+\|\delta u\|^2\leq\langle u,\Delta u\rangle$ (we know $\psi_\nu u\to u$ in L^2 by DCT since $u\in L^2$.

Hence $Du,\ \delta u\in L^2_\bullet(M,E)$

Now for $u,v\in\mathrm{Dom}\,\Delta$ $\psi_\nu u$: cpt supp $\Rightarrow$ $\langle\psi_\nu u,\Delta v\rangle\overset{\overset{\Delta=D\delta+\delta D\quad D^*=\delta}{\downarrow}}{=}\langle D\psi_\nu u,Dv\rangle+\langle\delta\psi_\nu u,\delta v\rangle$

In (a) pf $\psi_\nu u\to u$, $D(\psi_\nu u)\to Du$ $\delta(\psi_\nu u)\to\delta u$ in L^2. Hence $\langle u,\Delta v\rangle$

$=\langle Du,Dv\rangle+\langle\delta u,\delta v\rangle$

so Δ : self-adjoint.

$\underset{\substack{\uparrow\\\text{by symmetry}}}{=}\langle\Delta u,v\rangle$.

(c) by Thm1.2. $H_1\xrightarrow{D}H_2\xrightarrow{D}H_3$ $H_1=H_2=H_3=L^2_\bullet(M,E)$.

$D^2=0$ and $\ker D\cap\ker D^*\underset{\substack{\uparrow\\(b)\ D^*=\delta}}{=}\ker D\cap\ker\delta\underset{\substack{\uparrow\\\because(c)}}{=}\ker\Delta$

Chap V §3. Curvature tensor.

If we use Chern connection, $D'^2=D''^2=0$ $\Rightarrow$ same results $\square$

$D:E\to E$ $\quad\begin{matrix}E\\\downarrow\\M\end{matrix}$ vector bundle D: Connection local trivialization $\theta.\ E|_\Omega\to\Omega\times\mathbb{C}^r$

we have $D^2s\underset{\theta}{\simeq}(d+A\wedge)(d\sigma+A\wedge\sigma)=\cancel{d^2\sigma}+dA\wedge\sigma-\underline{A\wedge d\sigma}+\underline{A\wedge d\sigma}+A\wedge A\wedge d\sigma$

$=(dA+A\wedge A)\wedge d\sigma.$ $\quad(A\in C^\infty_1(\Omega,\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)))$

$\Theta(D)\in C^\infty_2(M,\mathrm{Hom}(E,E))$ <u>curvature tensor</u>. of D s.t. $D^2s=\Theta(D)\wedge s$

locally $\Theta(D)\underset{\theta}{\simeq}dA+A\wedge A$.

chap V §10 Connection of Type (1,0) and (0,1) over Complex manifold

<u>Def</u>: (notation as above) A <u>connection of type (1,0)</u> on E is a differential operator D' or order 1 on $C^\infty_{\cdot,\cdot}(X,E)$ st. $D':C^\infty_{p,q}(X,E)\to C^\infty_{p+1,q}(X,E)$ $D'(f\wedge s)=d'f\wedge s+(-1)^{\deg f}f\wedge D's$

same def. for type (0,1) D'' $\quad f\in C^\infty_{p_1,q_1}(X,\mathbb{C})$ $s\in C^\infty_{p_2,q_2}(X,E)$

locally $D's\underset{\theta}{\simeq}d'\sigma+A'\wedge\sigma$ $D''s\underset{\theta}{\simeq}d''\sigma+A''\wedge\sigma$ $A'\in C^\infty_{1,0}(\Omega,\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r))$

$A''\in C^\infty_{0,1}(\Omega,\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r))$

D : connection $\exists!\ \underset{(1,0)}{D'},\underset{(0,1)}{D''}$ s.t. $D=D'+D''$ conversely $D'+D''$: connection in d sense

$(\because d=d'+d'')$

chap 5. §7. Hermition Vector Bundles and Connections

$\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ complex vector bundle. E is **hermition** if. a positive definite. hermitian form $|\cdot|^2$ on each fiber E_x and $\begin{matrix} E \to \mathbb{R}_{\geq 0} \\ \xi \mapsto |\xi|^2 \end{matrix}$ is smooth.

$\theta : E|_\Omega \to \Omega \times \mathbb{C}^r$ trividization $(e_1, .., e_r)$: frame of $E|_\Omega$. $\rightsquigarrow (h_{\lambda\mu})$ C^∞ coeft. $\langle e_\lambda, e_\mu \rangle = h_{\lambda\mu}$

we define $C_p^\infty(M,E) \times C_q^\infty(M,E) \to C_{p+q}^\infty(M, \mathbb{C})$ sesquilinear (i.e. linear in 1^{st} component
$(s,t) \longmapsto \{s,t\}$ anti-linear in 2^{nd} component)

$s = \sum \sigma_\lambda \otimes e_\lambda \quad t = \sum \tau_\mu \otimes e_\mu \quad \{s,t\} := \sum_{\lambda,\mu} \sigma_\lambda \wedge \overline{\tau_\mu} \langle e_\lambda, e_\mu \rangle$

connection D is **hermitian** if $d\{s,t\} = \{Ds, t\} + (-1)^p \{s, Dt\}$ $s \in C_p^\infty(M,E)$

choose $(e_1, .., e_r)$ orthnomal frame of $E|_\Omega$ write $\theta(s) = \sigma = (\sigma_\lambda)$ $\theta(t) = \tau = (\tau_\lambda)$

$\Rightarrow \{s,t\} = \sum \sigma_\lambda \wedge \overline{\tau_\lambda} \quad d\{s,t\} = \sum d\sigma_\lambda \wedge \overline{\tau_\lambda} + (-1)^p \sum \sigma_\lambda \wedge d\overline{\tau_\lambda} = \{d\sigma, \tau\} + (-1)^p \{\sigma, d\tau\}$

$D|_\Omega$ is hermitian iff. $\{A \wedge \sigma, \tau\} + (-1)^p \{\sigma, A \wedge \tau\} = \{(A + A^*) \wedge \sigma, \tau\} = 0$. iff $A^* = -A$

$(\because \{d\sigma, \tau\} = \{D\sigma, \tau\} - \{A \wedge \sigma, \tau\}$. Also. $\{\sigma, A \wedge \tau\} = \{\sigma \wedge A^*, \tau\}.)$

(back to §10) Assume θ : isometry (i.e. orthnormal frame). then D : hermitian $\Leftrightarrow$ $A^* = -A$.
$A = A' + A''$ $(1,0)$ and $(0,1)$ so D : hermitian iff $A' = -(A'')^*$. thus below prop.

Prop 10.3 D_0'' : $(0,1)$-connection. on hermitian bundle $\pi : E \to M$ then $\exists!$ hermitian connection $D = D' + D''$ s.t. $D'' = D_0''$.

chap 5 §12. Chern Connection

$\pi : E \to X$ hermitian holomorphic vector bundle rank r.

Def 12.1. The unique hermitian connection D s.t. $D'' = d''$ is called the Chern connection of E.

$\Theta(E) := \Theta(D)$ Chern curvature tensor of E

On local holomorphic trividization $\theta : E|_\Omega \to \Omega \times \mathbb{C}^r$ $H := (h_{\lambda\mu})$ hermitian matrix, C^∞ coefficient

so. $\theta(s) = \sigma$ $\theta(t) = \tau$ $\rightarrow \{s,t\} = \sum h_{\lambda\mu} \sigma_\lambda \wedge \overline{\tau_\mu} =: \overset{\text{transpose}}{\sigma^t} \wedge H \overline{\tau}$

$$\{Ds, t\} + (-1)^{\deg s} \{s, Dt\} = d\{s,t\} = (d\sigma)^t \wedge H \overline{\tau} + (-1)^{\deg \sigma} \sigma^t \wedge (\overset{\overset{d'H + \overline{d'\overline{H}}}{\parallel}}{dH} \wedge \overline{\tau} + H d\overline{\tau})$$

$$= (d\sigma + \overline{H}^{-1} d'\overline{H} \wedge \sigma)^t \wedge H \overline{\tau} + (-1)^{\deg \sigma} \sigma^t \wedge H \overline{(d\tau + \overline{H}^{-1} d'\overline{H} \wedge \tau)}$$

$$\left(\begin{aligned} &(-1)^{\deg \sigma} \sigma^t \wedge d''H \wedge \overline{\tau} = (-1)^{\deg \sigma} \sigma^t \wedge \underset{\overset{\uparrow}{1\text{-form}}}{d''H} \cdot \underset{\overset{\uparrow}{0\text{ form.}}}{H^{-1}} \wedge H \overline{\tau} \\ &= (-1)^{\deg \sigma} \sigma^t \wedge (\overline{H}^{-1} d'\overline{H})^t \wedge H \overline{\tau} \\ &= \overline{H}^{-1} d'\overline{H} \wedge \sigma \cdot \wedge H \overline{\tau} \qquad (\because \overline{H}^t = H) \end{aligned} \right)$$

thus Chern connection D coincides with $Ds \simeq_\theta d\sigma + \overline{H}^{-1} d'\overline{H} \sigma$.

$D' \simeq_\theta d' + \overline{H}^{-1} d'\overline{H} \wedge \bullet = \overline{H}^{-1} d'(\overline{H} \bullet)$ $D'' = d''$.

so $D''^2 = 0$ $D'^2 = \overline{H}^{-1} d'^2 (\overline{H} \cdot) = 0$ $(d'^2 = 0)$ $\rightsquigarrow D^2 = D'D'' + D''D'$ $\rightsquigarrow \Theta(E)$ type $(1,1)$

$d'd'' + d''d' = 0$ $\rightsquigarrow (D'D'' + D''D')s \simeq_\theta \overline{H}^{-1} d'\overline{H} \wedge d''\sigma + d''(\overline{H}^{-1} d'\overline{H} \wedge \sigma) = d''(\overline{H}^{-1} d'\overline{H}) \wedge \sigma$

Thm 12.4 The Chern curvature tensor $i\Theta(E) \in C_{1,1}^\infty(X, \mathrm{Herm}(E,E))$
$\theta : E|_\Omega \to \Omega \times \mathbb{C}^r$ holo. trivialization H : hermitian matrix rep. metric
then $i\Theta(E) = i d''(\overline{H}^{-1} d'\overline{H})$ on Ω.

$(e_1,\dots,e_r)$: C^∞ orthonormal frame over coordinate patch $\Omega \subset X$ cpx coordinate $(z_1,\dots,z_n)$

then $i\Theta(E) = i \sum_{\substack{1\le j,k\le n\\ 1\le \lambda,\mu\le r}} c_{jk\lambda\mu}\, dz_j \wedge d\bar z_k \otimes e_\lambda^* \otimes e_\mu$. $c_{jk\lambda\mu} \in \mathbb{C}$. hermitian ⇒ $\bar c_{jk\lambda\mu} = c_{kj\mu\lambda}$

Example 12.6 $r = \mathrm{rank}\, E = 1$ ⇝ $H = e^{-\varphi} > 0$ $\varphi \in C^\infty(\Omega, \mathbb{R})$. so $D' = d' - d'\varphi \wedge \cdot = e^{\varphi} d(e^{-\varphi}\cdot)$

$i\Theta(E) = i d'd''\varphi$ on Ω. ⇝ closed real $(1,1)$-form on X

$(i\Theta(E) = i d''(e^{\varphi} d' e^{-\varphi}) = i d''(e^{\varphi}(-e^{-\varphi}) d'\varphi) = i d'd''\varphi$. $\varphi = \bar\varphi$ ⇒ $\overline{i\Theta(E)} = i\Theta(E)$ ⇒ real $)$

Rmk In general we can't find local frames $(e_1,\dots,e_r)$ simultaneously holo. and orthonormal.

if we can find it → $H = (\delta_{\lambda\mu})$ → $\Theta(E) = 0$.

Conversely $\Theta(E) = 0$ ⇒ we can find orthonormal parallel frames (e_λ) ⇒ $D''e_\lambda = 0$. ⇒ holo.

" Chern curvature tensor is the obstruction to the existence of orthnormal holo. frames "

Prop 12.10: $\forall x_0 \in X$ coordinate $(z_j)_{1\le j\le n}$ at x_0 $\exists$ holo frame $(e_\lambda)_{1\le\lambda\le r}$ in nbd of x_0

s.t. $\langle e_\lambda(z), e_\mu(z)\rangle = \delta_{\lambda\mu} - \sum_{1\le j,k\le n} c_{jk\lambda\mu} z_j \bar z_k + O(|z|^3)$

and $(c_{jk\lambda\mu})$: coeff. of Chern curvature tensor $\Theta(E)_{x_0}$. (e_λ) : normal coordinate frame at x_0

pf: (h_λ) : holo. frame of E. replace (h_λ) by suitable const. coeff. linear combination

WLOG $(h_\lambda(x_0))$: orthnormal basis of E_{x_0} (Gram-Schmidt)

say $\langle h_\lambda(z), h_\mu(z)\rangle = \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu} z_j + a'_{j\lambda\mu} \bar z_j) + O(|z|^2)$. hermitian metric ⇒ $a_{j\lambda\mu} z_j = \bar a_{j\mu\lambda}$

set $g_\lambda(z) = h_\lambda(z) - \sum_{j,\mu} a_{j\lambda\mu} z_j h_\mu(z)$. ⇝ holo.

then $\langle g_\lambda(z), g_\mu(z)\rangle = \langle h_\lambda(z), h_\mu(z)\rangle - \langle h_\lambda(z), \sum_{j,\alpha} a_{j\mu\alpha} z_j h_\alpha(z)\rangle$
$- \langle \sum_{k,\beta} a_{k\lambda\beta} z_k h_\beta(z), h_\mu(z)\rangle + O(|z|^2)$

$$= \delta_{\lambda\mu} + \sum_j \underline{a_{j\lambda\mu} z_j} + \utilde{a'_{j\lambda\mu}} \bar z_j - \utilde{\sum_{j,\alpha} \delta_{\lambda\alpha} \overline{a_{j\mu\alpha} z_j}} - \underline{\sum_{k,\beta} \delta_{\beta\mu} a_{k\lambda\beta} z_k} + O(|z|^2)$$

$$= \delta_{\lambda\mu} + O(|z|^2) = \delta_{\lambda\mu} + \sum_{j,k} (a_{jk\lambda\mu} z_j \bar z_k + a'_{jk\lambda\mu} z_j z_k + a''_{jk\lambda\mu} \bar z_j \bar z_k) + O(|z|^3)$$

hermitian ⇒ $\overline{a_{jk\lambda\mu}} = a_{kj\mu\lambda}$ $\overline{a'_{jk\lambda\mu}} = a''_{jk\mu\lambda}$

now take $e_\lambda(z) = g_\lambda(z) - \sum_{j,k,\mu} a'_{jk\lambda\mu} z_j z_k g_\mu(z)$: holo

$$\langle e_\lambda(z), e_\mu(z)\rangle = \Big[\Big(\delta_{\lambda\mu} + \sum_{j,k} (a_{jk\lambda\mu} z_j \bar z_k + \utilde{a'_{jk\lambda\mu} z_j z_k} + \underline{a''_{jk\lambda\mu} \bar z_j \bar z_k}) \Big) \Big]$$
$$- \underline{\overline{\sum_{j,k,\alpha} a'_{jk\alpha\mu} z_j z_k}\, \delta_{\lambda\alpha}} - \utilde{\sum_{j,k,\alpha} a'_{jk\lambda\alpha} z_j z_k \delta_{\alpha\mu}} + O(|z|^3)$$
$$= \delta_{\lambda\mu} + \sum_{j,k} a_{jk\lambda\mu} z_j \bar z_k + O(|z|^3)$$

$d'\langle e_\lambda, e_\mu\rangle = \{D'e_\lambda, e_\mu\}$ (∵ e_μ holo) $\quad (d\langle e_\lambda, e_\mu\rangle = d\{e_\lambda, e_\mu\} = \{De_\lambda, e_\mu\} + \{e_\lambda, De_\mu\}$
$= \underset{d'\langle e_\lambda, e_\mu\rangle}{(\{D'e_\lambda, e_\mu\} + \{e_\lambda, D''e_\mu\})} + \underset{d''\langle e_\lambda, e_\mu\rangle}{(\{D''e_\lambda, e_\mu\} + \{e_\lambda, D'e_\mu\})})$

$= \sum_{j,k} a_{jk\lambda\mu} \bar z_k dz_j + O(|z|^2)$

$$\Theta(E) \cdot e_\lambda = \underset{(D'^2 = 0,\ D''e_\lambda = 0)}{D''(D'e_\lambda)} = D''\Big(\sum_{j,k,\mu} a_{jk\lambda\mu} \bar z_k dz_j \otimes e_\mu + O(|z|^2)\Big) = \sum_{j,k,\mu} a_{jk\lambda\mu} d\bar z_k \wedge dz_j \otimes e_\mu + O(|z|)$$

Hence by def. $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$ $\square$.

Chap VII §1. Bochner-Kodaira-Nakano Identity

(X,ω): hermitian manifold $\dim_{\mathbb{C}} X = n$ E: hermitian holo. vector bundle, rank r / X

$D = D' + D''$ chern connection $\delta = \delta' + \delta''$ extend L, Λ to $\Lambda^{p,q}T^*X \otimes E$ by identity on E.

$$\overset{\text{"}\,d^*\text{"}}{\delta} \quad \overset{\text{"}\,d'^*\text{"}}{} \quad \overset{\text{"}\,d''^*\text{"}}{}$$

Thm 1.1. $\tau := [\underset{\substack{\uparrow\\(-1,-1)}}{\Lambda}, \underset{\substack{\uparrow\\(2,1)}}{d'\omega}]$ operator of type $(1,0)$ on $C^\infty_{\bullet,\bullet}(X,E)$ then

(a) $[\delta'', L] = i(D'_E + \tau)$

(b) $[\delta'_E, L] = -i(D''_E + \overline{\tau})$

(c) $[\Lambda, D''_E] = -i(\delta'_E + \tau^*)$

(d) $[\Lambda, D'_E] = i(\delta''_E + \overline{\tau^*})$

pf: (a) ⇒ (b) by take "bar" (a) ⇒ (c), (b) ⇒ (d) by take adjoint.

Fix a point $x_0 \in X$ coordinate system $z = (z_1, .., z_n)$.

Prop 12.10 ⇒ $\exists$ normal coordinate frame (e_λ). For section $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda \in C^\infty_{p,q}(X,E)$

then $D_E s = \sum_\lambda d\sigma_\lambda \otimes e_\lambda + O(|z|)$, $\delta''_E s = \sum_\lambda \delta''\sigma_\lambda \otimes e_\lambda + O(|z|)$, ... ($\because$ holo + $\langle e_\lambda, e_\mu\rangle = \delta_{\lambda\mu} + O(|z|^2)$)

so we may assume $E = \mathbb{C}$. Following proof in chap VI. §6. Commutator Relations

let $(z_j)_{1\le j\le n}$ cpx coordinates system at $x_0 \in X$ s.t. $(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})$ orthonormal basis of $T_{x_0}X$.

Consider $\omega_0 = i\sum_j dz_j \wedge d\overline{z}_j$ ⇒ $\omega = \omega_0 + \gamma$ $\gamma = O(|z|)$, (1,1) - form. for $\omega(x_0)$.

and denote $\langle\cdot,\cdot\rangle_0$ L_0, Λ_0, δ'_0, δ''_0 from ω_0. $dV_0 := \omega_0^n / 2^n n!$

Lemma 6.9 $u, v \in C^\infty(X, \Lambda^{p,q}T^*_X)$. then $\langle u,v\rangle dV = \langle u - [\gamma, \Lambda_0]u, v\rangle_0 dV_0 + O(|z|^2)$

pf: let $\gamma = i\sum_j \gamma_j \xi_j^* \wedge \overline{\xi_j^*}$ $\gamma_1 \le \dots \le \gamma_n$. diagonalization of (1,1) - form $\gamma(z)$ w.r.t. orthnormal basis (ξ_j) of T_zX w.r.t. $\omega_0(z)$

i.e. $\omega = \omega_0 + \gamma = i\sum \lambda_j \xi_j^* \wedge \overline{\xi_j}^*$ $\lambda_j = 1 + \gamma_j$ $\gamma_j = O(|z|)$ ($\because \omega$. diagonalizable ω_0: identity matrix.)

set $\xi_J^* = \xi_{j_1}^* \wedge \dots \wedge \xi_{j_p}^*$ if $J = \{j_1 < \dots < j_p\}$ $\lambda_J = \lambda_{j_1} \cdots \lambda_{j_p}$. $u = \sum u_{JK} \xi_J^* \wedge \overline{\xi_K^*}$ $|J| = p$

note that $\langle \xi_j^*, \xi_j^* \rangle = \lambda_j^{-1}$ w.r.t. ω. $v = \sum v_{JK} \xi_J^* \wedge \overline{\xi_K^*}$ $|K| = q$.

so $\langle u,v\rangle dV = \sum_{J,K} \lambda_J^{-1}\lambda_K^{-1} u_{J,K} \overline{v}_{J,K} \lambda_1 \cdots \lambda_n dV_0 = \sum_{J,K} (1 - \underset{\substack{\uparrow\\\lambda_J^{-1}}}{\sum_{j\in J}\gamma_j} - \underset{\substack{\uparrow\\\lambda_K^{-1}}}{\sum_{j\in K}\gamma_j} + \underset{\substack{\uparrow\\\lambda_1\cdots\lambda_n}}{\sum_{1\le j\le n}\gamma_j}) u_{J,K}\overline{v_{J,K}} \lambda_1 \cdots \lambda_n dV_0 + O(|z|^2)$

by Brute-force we get the result. $\square$ (Prop 5.8) in P.18 of this handout. since $\frac{1}{1+x} = 1 - x + x^2 \dots$

Lemma 6.10 $\delta'' = \delta''_0 + [\Lambda_0, [\delta''_0, \gamma]]$. at x_0.

pf: $\because \delta'' = d''^*$ order 1 Lemma 6.9 ⇒ δ'' = formal adjoint w.r.t. $\langle u,v\rangle_1 = \int_X \langle u - [\gamma,\Lambda_0]u, v\rangle_0 dV_0$ of d''. at x_0.

$u \in C^\infty(X, \Lambda^{p,q}T^*_X)$. $v \in C^\infty(X, \Lambda^{p,q-1}T^*_X)$ cpt supp. then

$\langle u, d''v\rangle = \int_X \langle d_0''^* u - d_0''^*[\gamma,\Lambda_0]u, v\rangle_0 dV_0$.

so $d''^* u = d_0''^* u - d_0''^*[\gamma,\Lambda_0]u = d_0''^* u - [d_0''^*, [\gamma,\Lambda_0]]u$. since $\omega = \omega_0$ at x_0 $\gamma(x_0) = 0$.

i.e. $d''^* = d_0''^* - [\underset{\substack{\uparrow\\\deg -1}}{d_0''^*}, [\underset{\substack{\uparrow\\\deg 2}}{\gamma}, \underset{\substack{\uparrow\\\deg -2}}{\Lambda_0}]] \underset{\substack{\uparrow\\\text{Jacobi identity}}}{=} d_0''^* + [\Lambda_0, [d_0''^*, \gamma]]$ $\square$

since $[\Lambda_0, d_0''^*] = [d'', L_0]^* = 0$

$(d''(\omega_0 \wedge -) - \omega_0 d''(\cdot) = d''\omega_0 \wedge -, d''\omega_0 = 0)$

by Lemma 6.10 $L = L_0 + \gamma$ ⇒ $[L, d''^*] = [L_0, d_0''^*] + [L_0, [\Lambda_0, [\delta''_0, \gamma]]] + [\gamma, d_0''^*]$.

$+ \cancel{[\gamma, [\Lambda_0, [\delta''_0, \gamma]]]}$ at x_0.

$\because \gamma(x_0) = 0$

by Jcobian identety $[L_0,[\Lambda_0, C]] = -[\Lambda_0,[C, L_0]] - [C,[L_0,\Lambda_0]]$

$C=[d_0''^*,\gamma]$, deg 1, deg 2 deg -2

Kähler identity $\qquad d'\omega_0=0$

Also $[C, L_0] = [\underset{\uparrow\ \text{deg }1}{[d_0''^*,\gamma]}, \underset{\uparrow\ \text{deg }2}{L_0}] = -[\underset{\uparrow\ \text{deg}2}{L_0},[\underset{\uparrow\ \text{deg}-1}{d_0''^*},\underset{\nwarrow\ \text{deg }2}{\gamma}]] = [\gamma,[L_0,d_0''^*]] = [\underset{\uparrow\ \text{deg }2}{\gamma}, \underset{\uparrow\ \text{deg }1}{-id'}] = id'\gamma = id'\omega$

$\because [\gamma, L_0]=0$. (both 2-form)

C : type $(0,-1)+(1,1)=(1,0) \Rightarrow [C,[\underset{\uparrow\ \text{deg }0}{L_0,\Lambda_0}]] = C\cdot(p+q-n) - (p+1+q-n)C = -C$
$= -[d_0''^*,\gamma]$

In conclusion

$[L_0,[\Lambda_0,[d_0''^*,\gamma]]] = -[\Lambda_0, id'\omega] + [\underset{\uparrow\ \text{deg}1}{d_0''^*}, \underset{\uparrow\ \text{deg }2}{\gamma}]$ (since $[L_0,\Lambda_0]u = (p+q-n)u$ for $u\in\Lambda^{p,q}T_X^*$ from 3/12 class; sl_2-rep)

so $[L, d''^*] = [L_0, d_0''^*] - [\Lambda_0, id'\omega]$

$= -id' - i[\Lambda, d'\omega]$ $\quad(\omega=\omega_0$ at $x_0)$

$= -i(d'+\tau)$ $\quad$ by Kähler identity $\square$.

(back to Chap VII)

Thm1.2 $\Delta'' = \Delta' + [i\Theta(E),\Lambda] + [D',\tau^*] - [D'',\bar\tau^*]$

pf: $\delta'' = -i[\Lambda, D_E'] - \bar\tau^*$ by (d) $\Rightarrow \Delta'' = D''\delta''+\delta''D'' = [\underset{\uparrow\ \text{deg }1}{D''}, \underset{\uparrow\ \text{deg}-1}{\delta''}] = -i[D'',[\Lambda,D']] - [D'',\bar\tau^*]$

Also $[\underset{\uparrow\ \text{deg }1}{D''},[\underset{\uparrow\ \text{deg}2}{\Lambda},\underset{\uparrow\ \text{deg}1}{D'}]] \underset{\uparrow\ \text{Jacobi identity}}{=} [\Lambda,[D',D'']] + [D',[D'',\Lambda]] \underset{\uparrow\ (c)}{=} [\Lambda,\Theta(E)] + i[D',\delta'+\tau^*]$

$\Theta(E) = D^2 = D'D''+D''D'$

so $\Delta'' = -i[\underset{\uparrow\ \text{deg}-2}{\Lambda},\underset{\uparrow\ \text{deg }2}{\Theta(E)}] + \overset{\Delta'}{\overset{\parallel}{[D',\delta']}} + [D',\tau^*] - [D'',\bar\tau^*]$ $\square$

Cor ([Akizuki-Nakano 1955]) ω: Kähler then $\Delta'' = \Delta' + [i\Theta(E),\Lambda]$ $\quad(d'\omega=0\Rightarrow\tau\equiv0)$

Thm1.4 ([Demailly 1985]) $\Delta_\tau' := [D'+\tau, \delta'+\tau^*]$ positive, formally self-adjoint, same principal part

Also $\Delta'' = \Delta_\tau' + [i\Theta(E),\Lambda] + T_\omega \quad T_\omega = [\Lambda,[\Lambda,\frac{i}{2}d'd''\omega]] - [d'\omega,(d'\omega)^*]$ as Δ'.

pf: $\because (D'+\tau)^* = \delta'+\tau^*$ $\therefore$ self-adjoint. $\langle\Delta_\tau' u,u\rangle = \|D'u+\tau u\|^2 + \|\delta'u+\tau^*u\|^2 \geq 0$ if u: cpt supp.

principal part $[D',\delta'] = \Delta'$ only depends on X.

Lemma1.5 (a) $[L,\tau] = 3d'\omega$ (b) $[\Lambda,\tau] = -2i\bar\tau^*$

pf: (a). ω: 2-form $\Rightarrow [L,d'\omega]=0$. so $[L,\tau] = [\underset{\text{deg}2}{L},[\underset{\text{deg}-2}{\Lambda},\underset{\text{deg}3}{d'\omega}]] \overset{\text{Jacobi identity}}{=} -[\underset{\uparrow\ \text{deg }3}{d'\omega},[L,\Lambda]] \underset{\uparrow}{=} 3d'\omega$.

use $[L,\Lambda]u = (p+q-n)u$

(b). $[\Lambda,\tau]$

$\underset{\uparrow\ \text{(a). in Thm1.1}}{=} -i[\Lambda,[\delta'',L]] - [\Lambda,D'] \underset{\uparrow\ \text{(d) in Thm1.1}}{=} -i([\Lambda,[\delta'',L]] + \delta'' + \bar\tau^*)$

Also $[\underset{\uparrow\ \text{deg}-2}{\Lambda},[\underset{\uparrow\ \text{deg}-1}{\delta''},\underset{\uparrow\ \text{deg }2}{L}]] = -[L,[\Lambda,\delta'']] - [\delta'',[L,\Lambda]] \underset{\uparrow}{=} -[[d'',L],\Lambda]^* - \delta''$

still use $[L,\Lambda]u = (p+q-n)u$.

$= -[\underset{\uparrow\ \text{deg}3}{d''\omega},\underset{\uparrow\ \text{deg }2}{\Lambda}]^* - \delta'' = \bar\tau^* - \delta''$ $\quad(\Lambda$: real$)$ $\quad$ so $[\Lambda,\tau] = -2i\bar\tau^*$ $\square$.

Lemma1.6 (a) $[D',\bar\tau^*] = -[D',\delta''] = [\tau,\delta'']$ (b) $-[D'',\bar\tau^*] = [\tau,\delta'+\tau^*] + T_\omega$

pf: (a) Jacobi identity $-[\underset{\uparrow\ \text{deg}1}{D'},[\underset{\uparrow\ \text{deg}2}{\Lambda},\underset{\uparrow\ \text{deg}1}{D'}]] + [D',[D',\Lambda]] + [\Lambda,\underset{D'^2=0}{\cancel{[D',D']}}] = 0 \Rightarrow -2[D',[D',\Lambda]] = 0$

so Thm1.1 (d) get

$[D',\bar\tau^*] = -[D',\delta'']$

Similarly $[\underset{\substack{\uparrow\\ \text{deg-1}}}{\delta''}, [\underset{\substack{\uparrow\\ \text{deg-1}}}{\delta''}, \underset{\substack{\uparrow\\ \text{deg 2}}}{L}]] - [\delta'', [L, \delta'']] + [L, \cancel{[\delta'', \delta'']}] = 0 \Rightarrow [\delta'', [\delta'', L]] = 0$

so by Thm 1 (a) O.K.

(b) by Lemma 1.5 (b) $\bar{\tau}^* = \frac{i}{2}[\Lambda, \tau] \Rightarrow [D'', \bar{\tau}^*] = \frac{i}{2}[D'', [\Lambda, \tau]]$

by Jacobi identity $[\underset{\substack{\uparrow\\ \text{deg 1}}}{D''}, [\underset{\substack{\uparrow\\ \text{deg-2}}}{\Lambda}, \underset{\substack{\uparrow\\ \text{deg 1}}}{\tau}]] = [\Lambda, [\tau, D'']] + [\tau, [D'', \Lambda]]$ (*)

and $[\underset{\substack{\uparrow\\ \text{deg1}}}{\tau}, \underset{\substack{\uparrow\\ \text{deg1}}}{D''}] = [D'', \tau] \underset{\substack{\uparrow\\ \text{def}}}{=} [\underset{\substack{\uparrow\\ \text{deg1}}}{D''}, [\underset{\substack{\uparrow\\ \text{deg-2}}}{\Lambda}, \underset{\substack{\uparrow\\ \text{dog3}}}{d'\omega}]] = [\Lambda, [d'\omega, D'']] + [d'\omega, \underset{\substack{\|\\ A := i(\delta'_E + \tau^*)}}{[D'', \Lambda]}] = [\Lambda, d''d'\omega] + [d'\omega, A]$

so $[\Lambda, [\tau, D'']] = [\Lambda, [\Lambda, d''d'\omega]] + [\Lambda, [d'\omega, A]]$. (**)

$[\underset{\substack{\uparrow\\ \text{deg-2}}}{\Lambda}, [\underset{\substack{\uparrow\\ \text{deg3}}}{d'\omega}, \underset{\substack{\uparrow\\ \text{deg-1}}}{A}]] = [A, \underset{\substack{\|\\ \tau}}{[\Lambda, d'\omega]}] - [d'\omega, [A, \Lambda]] = [\tau, A] + [d'\omega, [\Lambda, A]]$.

$[\Lambda, A] = [\underset{\substack{\uparrow\\ \text{deg-2}}}{\Lambda}, [\underset{\substack{\uparrow\\ \text{deg1}}}{D''}, \Lambda]] \underset{\substack{\uparrow\\ \text{Thm 1.1 (c)}}}{=} i[\Lambda, \delta' + \tau^*] = i[D' + \tau, L]^*$

by Lemma 1.5 (b) $[\tau, L] = -3d'\omega$. also $[D', L] = d'\omega$ by def.

so $[\Lambda, A] = -2i(d'\omega)^*$. , $[\Lambda, [d'\omega, A]] = [\tau, [D'', \Lambda]] + 2i[d'\omega, (d'\omega)^*]$ (***)

$[D'', [\Lambda, \tau]] \underset{\substack{\uparrow\\ (*)+(**)+(***)}}{=} \underbrace{[\Lambda, [\Lambda, d''d'\omega]]}_{(d''d' = -d'd'')} + 2[\tau, [D'', \Lambda]] \underbrace{- 2i[d'\omega, (d'\omega)^*]}_{2i\,T_\omega} = 2i(T_\omega + [\tau, \delta' + \tau^*])$

So $[D'', \bar{\tau}^*] = \frac{i}{2}[D'', [\Lambda, \tau]] = -T_\omega - [\tau, \delta' + \tau^*]$ □

pf of Thm1.4 by Lemma 1.6 (b) $\Delta'' \underset{\substack{\uparrow\\ \text{Thm1.2}}}{=} \Delta' + [i\Theta(E), \Lambda] + [D', \tau^*] - [D'', \bar{\tau}^*]$.

$= [i\Theta(E), \Lambda] + [D', \delta'] + [D', \tau^*] + [\tau, \delta' + \tau^*] + T_\omega = \Delta_\tau + [i\Theta(E), \Lambda] + T_\omega$ □.

If ω: Kähler $\tau = T_\omega = 0$ so by Lemma 1.6 (a) $[D', \delta''] = 0$. take adjoint get $[D'', \delta'] = 0$.

Hence $\Delta = [D' + D'', \delta' + \delta''] = \Delta' + \Delta''$ In general. Lemm 1.6 (a) $[D' + \tau, \delta''] = 0$

Prop1.16 $\Delta_\tau := [D + \tau, \delta + \tau^*] = [(D' + \tau) + D'', (\delta' + \tau^*) + \delta''] = \Delta_\tau + \Delta''$

Bochner-Kodaira-Nakano inequality : $\|D''u\|^2 + \|\delta''u\|^2 \geq \int_X \langle([i\Theta(E), \Lambda] + T_\omega)u, u\rangle \, dV$

Since LHS $= \langle \Delta''u, u\rangle = \langle \Delta'_\tau, u\rangle +$ RHS and $\langle \Delta'_\tau, u\rangle \geq 0$. if u: cpt. supp.

Chapter VIII §4. General Estimate for d'' on Hermitian manifolds

(X, ω): complete hermitian manifold E: hermitian holo. vector bundle. rank r / X.

$D = D' + D''$ $\delta = D^* = \delta' + \delta''$ Chern connection. $A_{E,\omega} = [i\Theta(E), \Lambda] + T_\omega = \Delta'' - \Delta'_\tau$, hermitian

Assume $A_{E,\omega}$ semi-positive on $\Lambda^{p,q}T^*_X \otimes E$. then by Thm3.2 (a) (p.7) ($\because$ RHS is) ($D_{p,q}(X,E)$ dense)

$\|D''u\|^2 + \|\delta''u\|^2 \geq \int_X \langle A_{E,\omega}u, u\rangle \, dV$ $\forall u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$ (*) and above

Assume $g \in L^2_{p,q}(X, E)$ st. $D''g = 0$ and for a.e. $x \in X$ $\exists \alpha = \alpha(x) \in [0, \infty)$ s.t. $|\langle g(x), u\rangle|^2 \leq \alpha \langle A_{E,\omega}u, u\rangle$ $\forall u \in (\Lambda^{p,q}T^*_X \otimes E)_x$.

Denote $\langle A^{-1}_{E,\omega} g(x), g(x)\rangle =$ min of such α

since if $A_{E,\omega}$ invertible. $\frac{|\langle g(x), u\rangle|^2}{|\langle A_{E,\omega}u, u\rangle|} = \frac{|\langle A^{-\frac{1}{2}}_{E,\omega} g(x), A^{\frac{1}{2}}_{E,\omega} u\rangle|^2}{\|A^{\frac{1}{2}}_{E,\omega} u\|^2} \underset{\substack{\uparrow\\ \text{Cauchy}}}{\leq} \|A^{-\frac{1}{2}}_{E,\omega} g(x)\|^2 = \langle A^{-1}_{E,\omega} g(x), g(x)\rangle$

(we have assumed $A_{E,\omega}$ semi-positive

Thm 4.5 (X,ω) complete. $A_{E,\omega} \geq 0$ in bidegree (p,q) then $\forall g \in L^2_{p,q}(X,E)$ $\int_X \langle A_{E,\omega}^{-1} g, g\rangle dV < \infty$ $D''g = 0$

$\exists f \in L^2_{p,q-1}(X,E)$ s.t. $D''f = g$ $\|f\|^2 \leq \int_X \langle A_{E,\omega}^{-1} g, g\rangle dV$.

pf: $\forall u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$ $|\langle u,g\rangle|^2 = |\int_X \langle u,g\rangle dV|^2 \leq \left(\int_X \langle A_{E,\omega}^{-1} g, g\rangle^{\frac{1}{2}} \langle A_{E,\omega} u, u\rangle^{\frac{1}{2}} dV\right)^2$

$$\underset{\text{Hölder}}{\leq} \left(\int_X \langle A_{E,\omega}^{-1} g, g\rangle dV\right) \cdot \left(\int_X \overset{\geq 0}{\langle A_{E,\omega} u, u\rangle}\right) dV$$

$$\leq C \cdot (\|D''u\|^2 + \|\delta'' u\|^2) \text{ by } (*) \qquad C = \int_X \langle A_{E,\omega}^{-1} g, g\rangle dV$$

repeat the pf of Thm 1.2. For $u \in \operatorname{Dom} \delta''$ $u = u_1 + u_2$ $u_1 \in \ker D''$ $u_2 \in (\ker D'')^\perp = \overline{\operatorname{Im} \delta''} \subseteq \ker \delta''$

so $D''u_1 = 0 = \delta'' u_2$ $g \in \ker D'' \Rightarrow |\langle u,g\rangle|^2 = |\langle u_1, g\rangle|^2 \leq C \|\delta'' u_1\|^2 = C\|\delta'' u\|^2$ (Thm 3.2, p.4)

Hahn-Banach Thm: extend $L^2_{p,q-1}(X,E) \ni \delta'' u \mapsto \langle u, g\rangle$ to $v \mapsto \langle v, f\rangle$ $f \in L^2_{p,q-1}(X,E)$

(still by above, $\langle \ker \delta'' \cap \ker D'', g\rangle = 0$ $\Rightarrow \langle \ker \delta'', g\rangle = 0$ so well def) $\therefore |f| \leq C^{\frac{1}{2}}$.

Hence $\langle u, g\rangle = \langle \delta'' u, f\rangle$ $\forall u \in \operatorname{Dom} \delta''$ i.e. $D''f = g$ □

Rmk 4.6 we may replace f by proj. on $\underset{\overline{\operatorname{Im}\delta''}}{\underset{\parallel}{(\ker D'')^\perp}}$ which is unique $(D''f_1 = g = D''f_2 \Rightarrow f_1 - f_2 \in \ker D'')$ and thus minimal L^2-norm

so $\delta'' f = 0 \Rightarrow \Delta'' f = \delta'' g$. if $g \in C^\infty_{p,q}(X,E)$ Δ'': elliptic $\Rightarrow f \in C^\infty_{p,q-1}(X,E)$

Rmk 4.8. If $A_{E,\omega}$ positive-definite $\lambda(x) > 0$: smallest eigenvalue of $A_{E,\omega}$ at x. λ: conti.

Also $\int_X \langle A_{E,\omega}^{-1} g, g\rangle dV \leq \int_X \lambda(x)^{-1} |g(x)|^2 dV$.

For example ω = Kähler, complete. $E >_m 0$ $p = n$ $q \geq 1$ $m \geq \min\{n-q+1, r\}$

§5. Estimates on Weakly Pseudoconvex Manifolds

Def 5.1 X: complex manifold. X is weakly pseudoconvex if $\exists$ exhaustion function $\psi \in C^\infty(X, \mathbb{R})$ s.t. $id'd''\psi \geq 0$ i.e. ψ is plurisubharmonic.

e.g. X: cpt take $\psi \equiv 0$. $\psi^{-1}(-\infty, c) = X$: cpt. $\Rightarrow$ exhaustion, and $id'd''\psi \geq 0$.

Thm 5.2 (X,ω): weakly pseudoconvex, Kähler then $\exists$ complete Kähler metric $\hat\omega$ on X.

pf: Let $\psi \in C^\infty(X, \mathbb{R})$ exhaustive plurisubharmonic. note that $\psi(-\infty, c)$ relatively cpt $\Rightarrow \psi \geq -M > -\infty$ for some $M \in \mathbb{R}$

replace ψ by $\psi + m$ we may assume $\psi \geq 0$

Take $\hat\omega = \omega + id'd''(\psi^2)$ Kähler $(d\hat\omega = \overset{0}{\cancel{d\omega}} + i\,d\,\overset{0}{\cancel{d'd''}}(\psi^2))$

Also $\hat\omega = \omega + id'(2\psi d''\psi) = \omega + \underset{\geq 0}{\underline{2i\psi d'd''\psi}} + 2i\,d'\psi \wedge d''\psi \geq \omega + 2i\,d'\psi \wedge d''\psi$

$d\psi = d'\psi + d''\psi$ real $\Rightarrow |d\psi|_{\hat\omega} = \sqrt{2}|d'\psi|_{\hat\omega} \leq \sqrt{2}|d'\psi|_\omega = |d\psi|_\omega \underset{\uparrow}{\leq} 1$.

so by Lemma 2.4. $(X, \hat\omega)$ is complete. $\because$ exhaustion

(Note that if ω, induces $g = (g_{ij})$ on $TM|_\Omega$. $g^{-1} = (g^{ij})$ on $T^*M|_\Omega$. write $\sigma = \sqrt{2}d'\psi$ on Ω

$\omega_1 = \omega + 2i\,d'\psi \wedge d''\psi$ induces $g_1 = g + \sigma\sigma^*$ $g_1^{-1} = g^{-1} - \dfrac{g^{-1}\sigma\sigma^* g^{-1}}{1 + \sigma^* g^{-1}\sigma} = g^{-1} - \dfrac{g^{-1}\sigma\sigma^* g^{-1}}{1 + |\sigma|_\omega}$ (column vector)

in particular $|\sigma|_{\omega_1} = |\sigma|_\omega - \dfrac{|\sigma|_\omega^2}{1 + |\sigma|_\omega} = \dfrac{|\sigma|_\omega}{1 + |\sigma|_\omega} < |\sigma|_\omega$ (Sherman-Morrison formula)

so $|d'\psi|_{\hat\omega} \leq |d'\psi|_{\omega_1} < |d'\psi|_\omega$)

More generally. set $\hat{\omega} = \omega + id'd''(\chi \circ \psi)$ χ : convex increasing function $\Rightarrow \chi' \geq 0$ $\chi'' \geq 0$, $\hat{\omega}$ Kähler

then $\hat{\omega} = \omega + i\, d'(\chi' \circ \psi\, d''\psi) = \omega + i \underbrace{(\chi' \circ \psi)}_{\geq 0} d'd''\psi + i(\chi'' \circ \psi)\, d'\psi \wedge d''\psi \geq i\, d'\rho\circ\psi \wedge d''\rho\circ\psi + \omega$

We have $|d'(\rho\circ\psi)|_{\hat{\omega}} \leq |d'(\rho\circ\psi)|_{\omega} = |\overset{(\sqrt{\chi''\circ\psi})^2}{\chi''}\, d'\psi| \leq M$ if $|\chi''| \leq M$ $\rho = \int_0^t \sqrt{\chi''(u)}\, du$

$\rho' = \sqrt{\chi''}$

for exhaustin $\frac{\rho\circ\psi}{M}$, need. $\lim_{t\to\infty} \rho(t) = \infty$ i.e. $\int_0^\infty \sqrt{\chi''(u)}\, du = \infty$ e.g. take $\chi(t) = t - \log t$ $t \geq 1$

(need $\rho^{-1}(-\infty, c) \neq \mathbb{R}$ $\forall c < \infty$). $\chi' = 1 - \frac{1}{t}$ $\chi'' = \frac{1}{t^2}$

§6. Hörmander's Estimates for non-Complete Kähler Metrics

Thm 6.1 $(X, \hat{\omega})$ complete Kähler manifold. ω : Kähler metric (may non complete)

$E \downarrow X$ m-semi-positive vector bundle (defined below). $g \in L^2_{n,q}(X,E)$ $D''g = 0$. $\int_X \langle A_q^{-1} g, g\rangle dV < \infty$ w.r.t. ω.

$A_q = A_{E,\omega} = [i\Theta(E), \Lambda] = i\Theta(E) \wedge \Lambda$ on bidegree (n,q) $q \geq 1$

(assume $m \geq \min\{n-q+1, r\}$. so A_q semi-positive) ($\because \Theta(E)$ (1,1) $\therefore$ no such term)

then $\exists f \in L^2_{n,q-1}(X,E)$ s.t. $D''f = g$ $\|f\|^2 \leq \int_X \langle A_q^{-1} g, g\rangle dV$.

Def : (a) $i\Theta(E)$ corresponding to hermitian form $\theta_E \in TX \otimes E$. locally $(e_1, \dots, e_r)$ orthnormal frame.

(chap VII, §6, §7). $i\Theta(E) = i \sum c_{jk\lambda\mu}\, dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$ $\overline{c_{jk\lambda\mu}} = c_{kj\mu\lambda}$ $(z_1, \dots, z_n)$ cpx coordinate

$\theta(E) := \sum c_{jk\lambda\mu} (dz_j \otimes e_\lambda^*) \otimes (d\bar{z}_k \otimes e_\mu^*)$ $\Rightarrow \theta_E(u,u) = \sum c_{jk\lambda\mu}(x)\, u_{j\lambda}\, \bar{u}_{k\mu}$

(b). T, E : cpx vector space dim n, r resp. Θ : hermitian form on $T \otimes E$.

(1) $u \in T \otimes E$ rank m if m smallest ≥ 0 integer s.t. $u = \sum_{j=1}^m \xi_j \otimes s_j$ $\xi_j \in T$ $s_j \in E$.

(2) Θ <u>m-positive</u> if $\Theta(u,u) > 0$ (resp. $\Theta(u,u) \leq 0$) $\forall u \in T \otimes E$ rank $\leq m$ $u \neq 0$

(resp. <u>m-semi-negative</u>)

In this case we write $\Theta >_m 0$. (resp. $\Theta \leq_m 0$.)

(c) E is m-positive if $\theta_E >_m 0$. $dz_{[n]} := dz_1 \wedge \dots \wedge dz_n$

Lemma 7.2 (chap VII). $E >_m 0$ then $[i\Theta(E), \Lambda]$ positive-definite on $\Lambda^{n,q} T^*X \otimes E$ $q \geq 1$

$m \geq \min\{n-q+1, r\}$

pf: note that $\Lambda = L^* = (\omega \wedge -)^*$ $\omega, \Theta(E)$: (1,1)-type $\Rightarrow [i\Theta(E), \Lambda]\, dz_{[n]} \wedge d\bar{z}_J = \sum_{j,k} \bigcirc \cdot dz_{[n]}\, d\bar{z}_{Jj\backslash k}$

in particular $\langle [i\Theta(E), \Lambda]\, dz_{[n]} \wedge d\bar{z}_J, d\bar{z}_{[n]} \wedge d\bar{z}_K \rangle$ if $|J \cap K| \leq q-2$. (so $Jj\backslash k \neq K$.)

Hence $u = \sum_{|J|=q-1,k} c_J\, dz_{[n]} \wedge d\bar{z}_J \otimes e_k$ $\langle [i\Theta(E), \Lambda]u, u\rangle = \sum_{|S|=q-1} \langle [i\Theta(E), \Lambda] \sum_{j,k} c_{Sj}\, dz_{[n]} \wedge d\bar{z}_{Sj} \otimes e_k, \sum_{j,k} c_{Sj}\, dz_{[n]} \wedge d\bar{z}_{Sj} \otimes e_k \rangle$

and for fix S, $|S| = q-1$ only involve $dz_{[n]} \wedge d\bar{z}_{Sj} \otimes e_k$ $1 \leq k \leq r$, $1 \leq j \leq n$ $j \notin |S| \Rightarrow$ rank. $\leq \min\{r, n-q+1\}$

(back to chap VIII §6.) ☐

Lemma 6.3 ω, γ. hermitian metric on X $\gamma \geq \omega$. For any $u \in \Lambda^{n,q} T^*X \otimes E$ $q \geq 1$ we have

$|u|^2_\gamma dV_\gamma \leq |u|^2 dV$, $\langle A_{q,\gamma}^{-1} u, u\rangle_\gamma dV_\gamma \leq \langle A_q^{-1} u, u\rangle dV$.

$|\cdot|_\gamma$. dV_γ, $A_{q,\gamma}^{-1}$: from γ. (prove later)

pf of Thm 6.1 : For $\varepsilon > 0$ $\omega_\varepsilon := \omega + \varepsilon\hat{\omega}$: Kähler, complete (ψ : from $\hat{\omega}$ $|d\sqrt{\varepsilon}\psi|_{\omega_\varepsilon} \leq \sqrt{\varepsilon}|d\psi|_{\varepsilon\hat{\omega}} \leq 1$

$|\cdot|_\varepsilon$. dV_ε, $A_{q,\varepsilon}^{-1}$: from ω_ε by Lemma 6.3 $|u|^2_\varepsilon dV_\varepsilon \leq |u|^2 dV$ $\langle A_{q,\varepsilon}^{-1} u, u\rangle_\varepsilon dV_\varepsilon \leq \langle A_q^{-1} u, u\rangle dV$

$\Downarrow$

$g \in L^2$ w.r.t. ω_ε $\forall u \in \Lambda^{n,q} T_X^* \otimes E$

by Thm4.5 $\exists f_\varepsilon \in \underset{\hookrightarrow \text{ depends on } \varepsilon.}{L^2_{n,q-1}(X,E)}$ s.t. $D''f_\varepsilon = g$. $\int_X |f_\varepsilon|^2_\varepsilon dV_\varepsilon \le \int_X \langle A_{q,\varepsilon}^{-1} g, g\rangle_\varepsilon dV_\varepsilon \le$

(D'' only depends on hermitian structure of E, not X). $\le \int_X \langle A_q^{-1} g, g\rangle dV < \infty$.

Note that X admit. exhaustion function (from $\hat{\omega}$) $\Rightarrow$ X admit. cpt set exhaustion. $\cup K_n = X$

on cpt set. (f_ε) : bdd. in L^2-norm (of ω). $K_n \subseteq K^\circ_{n+1}$

$L^2(K)$ separable since K : second-countable. $\Rightarrow$ $\exists$ weakly convergent subseq $f_{\varepsilon_\nu} \to f$ in L^2_{loc} K_n : cpt.

(by using diagonal argument)

for cpt supp u. $\langle D''f, u\rangle = \langle f, -D''u\rangle = \lim_{\nu\to\infty} \langle f_{\varepsilon_\nu}, -D''u\rangle = \lim_{\nu\to\infty} \langle D''f_{\varepsilon_\nu}, u\rangle = \lim_{\nu\to\infty}\langle g, u\rangle$
$= \langle g, u\rangle$

so $D''f = g$ Also on cpt set as $\varepsilon \to 0$ we have $\|f\|_{L^2} \le \liminf \|f_{\varepsilon_\nu}\|_{L^2} \le \int_X \langle A_q^{-1} g, g\rangle dV < \infty$

so this also holds for L^2-norm on X. □.

($f \in H$: Hilbert space $|f|_H = |\langle \cdot, f\rangle|_{H^*}$ so $f_n \to f$ weakly $\Rightarrow$ $\underset{|f|_{H^*}}{\underset{\shortparallel}{|\langle\cdot, f\rangle|_{H^*}}} \le \liminf \underset{|f_n|_{H^*}}{\underset{\shortparallel}{|\langle \cdot, f_n\rangle|_{H^*}}}$)

<u>pf of lemma 6.3</u> Take $(z_1, \dots, z_n)$ orthnormal coordinate s.t. $\omega = i\sum dz_i \wedge d\bar{z}_j$ $\gamma = i\sum \gamma_j \, dz_j \wedge d\bar{z}_j$ at x_0

γ_j : eigenvalues of γ $\Rightarrow$ $\gamma_j \ge 1$. ($\gamma_j \to 1$ as $\gamma \to \omega$.)

$|dz_j|^2_\gamma = \gamma_j^{-1}$ $|dz_K|^2_\gamma = \gamma_K^{-1}$ K: multi-index $\gamma_K := \prod_{j\in K} \gamma_j$ (diagonalize ω at first, then do it on γ) $[n] = \{1, \dots, n\}$

For $u = \sum u_{K,\lambda}\, dz_{[n]} \wedge d\bar{z}_K \otimes e_\lambda$ $|K| = q$ $1 \le \lambda \le r$ $\Rightarrow$ $|u|^2_\gamma = \sum_{K,\lambda} \gamma_{[n]}^{-1} \gamma_K^{-1} |u_{K,\lambda}|^2$ $dV_\gamma = \gamma_{[n]} dV$

so $|u|^2_\gamma dV_\gamma = \sum \gamma_K^{-1} |u_{K,\lambda}|^2 dV \le |u|^2 dV$. Also, on cpt set $\gamma_j \to 1$ unif. $\Rightarrow \int |u|^2_\gamma dV_\gamma \to \int |u|^2 dV$

$$\Lambda_\gamma u = \sum_{|I|=q-1} \sum_{j,\lambda} i(-1)^{n+j-1} \gamma_j^{-1} u_{jI,\lambda} (\widehat{dz_j}) \wedge d\bar{z}_I \otimes e_\lambda \qquad \widehat{dz_j} = dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

since $\Lambda_\gamma = -*^{-1} L_\gamma *$ Also $(* = *_\gamma)$ $(*^{-1} \gamma_\lambda^{-1} dz_\lambda \wedge d\bar{z}_\lambda *)(dz_\lambda \wedge d\bar{z}_\lambda \wedge \sigma) = *^{-1} \gamma_\lambda dz_\lambda \wedge d\bar{z}_\lambda \wedge \tau = -\gamma_\lambda^{-1} \sigma$

$\Lambda u_J dz_{[n]} \wedge d\bar{z}_J \otimes e_\lambda = \sum_j i(-1)^{n+j-1} \gamma_j^{-1} u_{jI,\lambda} (\widehat{dz_j}) \wedge dz_I \otimes e_\lambda$ $\overline{dz_\lambda \wedge d\bar{z}_\lambda \wedge \sigma \wedge \bar{\tau} = \gamma_\lambda^{-2} |\sigma|^2 dV_\gamma}$ $|\tau| = |\sigma| \gamma_\lambda^{-1}$
$-\gamma_\lambda^{-1} \sigma \wedge \gamma_\lambda dz_\lambda \wedge d\bar{z}_\lambda \wedge \tau) = \gamma_\lambda^{-2} |\sigma| dV_\gamma$

where $u_J dz_{[n]} \wedge d\bar{z}_J = u_{jI,\lambda} dz_{[n]} \wedge d\bar{z}_j \wedge d\bar{z}_I$
$= u_{jI,\lambda} (-1)^{n+j} dz_j \wedge d\bar{z}_j (\widehat{dz_j}) \wedge dz_I$

$i\Theta(E) = i\sum c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_\mu$

Now. $A_{q,\gamma} u = i\Theta(E) \Lambda u = \sum_{|I|=q-1} \sum_{j,k,\lambda,\mu} (-1)^{n+j} \gamma_j^{-1} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \wedge (\widehat{dz_j}) \wedge d\bar{z}_I \otimes e_\mu$. I: unique s.t. jI same orientation as J.

$= \sum_{|I|=q-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} c_{jk\lambda\mu} dz_{[n]} \wedge d\bar{z}_{kI} \otimes e_\mu$

$\langle A_{q,\gamma} u, u\rangle_\gamma = \sum_{|I|=q-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} c_{jk\lambda\mu} \underset{\overline{u_{kI,\mu}}}{u_{jI,\lambda}} \langle dz_{[n]} \wedge d\bar{z}_{kI} \otimes e_\mu, dz_{[n]} \wedge d\bar{z}_{kI} \otimes e_\mu \rangle$

$= \gamma_{[n]}^{-1} \sum_{|I|=q-1} \gamma_I^{-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{jI,\lambda} \overline{u_{kI,\mu}} \gamma_j^{-1} \gamma_k^{-1}$

$\underset{\substack{\uparrow \\ \gamma_j \ge 1\ \forall j}}{\ge} \gamma_{[n]}^{-1} \sum_{|I|=q-1} \gamma_I^{-2} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{jI,\lambda} \overline{u_{kI,\mu}} \gamma_j^{-1} \gamma_k^{-1} = \gamma_{[n]} \langle A_q S_\gamma u, S_\gamma u \rangle$

where $S_\gamma u = \sum_K (\gamma_{[n]} \gamma_K)^{-1} u_{K,\lambda} dz_{[n]} \wedge d\bar{z}_K \otimes e_\lambda$.

So $|\langle u, v\rangle_\gamma|^2 = |\langle u, S_\gamma v\rangle|^2 \underset{\substack{\uparrow \\ \text{def of } \langle A_q^{-1} u, u\rangle}}{\le} \langle A_q^{-1} u, u\rangle \langle A_q S_\gamma v, S_\gamma v\rangle \le \gamma_{[n]}^{-1} \langle A_q^{-1} u, u\rangle \langle A_{q,\gamma} v, v\rangle_\gamma$.

take $v = A_{q,\gamma}^{-1} u$ $\Rightarrow$ $\langle A_{q,\gamma}^{-1} u, u\rangle_\gamma \le \gamma_{[n]}^{-1} \langle A_q^{-1} u, u\rangle$ Also $dV_\gamma = \gamma_{[n]} dV$. o.k. □

Prop 5.8 (in chap VI §5) $\omega = i\sum \xi_j^*\wedge\overline{\xi}_j^*$ $\gamma = i\sum_j \gamma_j \xi_j^*\wedge\overline{\xi}_j^*$ $\gamma_j\in\mathbb{R}$ then. for $u=\sum u_{J,K}\,\xi_J^*\wedge\overline{\xi}_K^*$.

$$[\gamma,\Lambda]u = \sum_{J,K}\Big(\sum_{j\in J}\gamma_j + \sum_{j\in K}\gamma_j - \sum_{1\le j\le n}\gamma_j\Big)u_{J,K}\,\xi_J^*\wedge\overline{\xi}_K^* \quad \text{by brute-force.}$$

Def (chap VI. 3,7) : $\theta\in T_M$. $u\in\Lambda^p T_M^*$ **Contraction** $\theta\lrcorner u\in\Lambda^{p-1}T_M^*$ $\theta\lrcorner u(\eta_1,\dots,\eta_{p-1}) = u(\theta,\eta_1,\dots,\eta_{p-1})$ $\eta_j\in T_M$

This is bilinear, and for basis (ξ_j) (maybe not orthogonal) $\xi_l\lrcorner(\xi_l^*\wedge\sigma)=\sigma \Rightarrow \xi_l\lrcorner\,\xi_{i_1}^*\wedge\cdots\wedge\xi_{i_p}^* = \begin{cases}0 & l\notin\{i_1,\dots,i_p\}\\ (-1)^{k-1}\xi_{i_1}^*\wedge\cdots\wedge\widehat{\xi_{i_k}^*}\wedge\cdots\wedge\xi_{i_p}^* & i_k=l.\end{cases}$

$\theta\lrcorner-$: derivation i.e. $\theta\lrcorner(u\wedge v) = (\theta\lrcorner u)\wedge v + (-1)^{\deg u}\,u\wedge(\theta\lrcorner v)$

since this is 3-linear only need $\theta = dz_i$ $u=dz_i\wedge\sigma$ or $\theta=dz_i$ $v=dz_i\wedge\sigma$ o.k.

Also if ξ_i : orthnormal then $*^{-1}(\xi_i^*\wedge -)*\sigma = (-1)^{\deg\sigma-1}\,\overline{\xi}_i\lrcorner\sigma$. $(\sigma\perp\overline{\xi}_i^*)$. since WLOG $\sigma=\overline{\xi}_i^*\wedge\tau$.

Also note that $*^2 = (-1)^{k(2m-k)} = (-1)^k$. on k-form. $\dim_{\mathbb{R}}X = 2m$.

$\overline{\xi}_i^*\wedge\tau\wedge\overline{*\sigma} = |\sigma|^2*1$

$\Rightarrow (-1)^{\deg\tau}\tau\wedge(\overline{\xi_i^*\wedge*\overline{\sigma}}) = |\sigma|^2*1$

$= (-1)^{\deg\tau}\tau\wedge\overline{\xi_i^*\wedge*\sigma}$

<u>pf of prop 5.8</u> by def $\Lambda u = i\sum_j -*^{-1}(\xi_j^*\wedge -)*\,*^{-1}(\overline{\xi}_j^*\wedge -)*u$

$= -i\sum_j \overline{\xi}_j\lrcorner(\xi_j\lrcorner u)$

So if $u=\sum u_{J,K}\,\xi_J^*\wedge\overline{\xi}_K^*$ type (p,q).

$\Lambda u = -i\sum_{J,K,l} u_{J,K}\;\overline{\xi}_l\lrcorner\big((\xi_l\lrcorner\xi_J^*)\wedge\overline{\xi}_K^*\big) = i(-1)^p\sum_{J,K,l}u_{J,K}(\xi_l\lrcorner\xi_J^*)\wedge(\overline{\xi}_l\lrcorner\overline{\xi}_K^*)$

$\gamma\wedge u = i(-1)^p\sum_{J,K,m}\gamma_m u_{J,K}\,\xi_m^*\wedge\xi_J^*\wedge\overline{\xi}_m^*\wedge\overline{\xi}_K^*$

$(\overline{\xi}_m\lrcorner\overline{\xi}_l^*)\wedge\overline{\xi}_K^* - \overline{\xi}_l^*\wedge(\overline{\xi}_m\lrcorner\overline{\xi}_K^*)$

$\|$

so $[\gamma,\Lambda]u = \sum_{J,K,l,m}u_{J,K}\gamma_m\Big[(\xi_l^*\wedge(\xi_m\lrcorner\xi_J^*))\wedge(\overline{\xi}_l^*\wedge(\overline{\xi}_m\lrcorner\overline{\xi}_K^*)) - (\xi_m\lrcorner(\xi_l^*\wedge\xi_J^*))\wedge(\overline{\xi}_m\lrcorner(\overline{\xi}_l^*\wedge\overline{\xi}_K^*))\Big]$

$\|$

$(\xi_m\lrcorner\xi_l^*)\wedge\xi_J^* - \xi_l^*\wedge(\xi_m\lrcorner\xi_J^*)$

$= \sum_{J,K,m}\gamma_m u_{J,K}\Big[(\xi_m^*\wedge(\xi_m\lrcorner\xi_J^*)\wedge\overline{\xi}_K^*) + \xi_J^*\wedge\overline{\xi}_m^*\wedge(\overline{\xi}_m\lrcorner\overline{\xi}_K^*) - \xi_J^*\wedge\overline{\xi}_K^*\Big]$

cancel!

$= \sum_{J,K}\Big(\sum_{m\in J}\gamma_m + \sum_{m\in K}\gamma_m - \sum_{1\le m\le n}\gamma_m\Big)u_{J,K}\,\xi_J^*\wedge\overline{\xi}_K^*$ since $\xi_m^*\wedge(\xi_m\lrcorner\xi_J^*) = \begin{cases}\xi_J^* & m\in J\\ 0 & m\notin J.\end{cases}$ □

(back to chap VIII, §6)

If E semi-positive i.e. $0\le\lambda_1(x)\le\cdots\le\lambda_n(x)$ eigenvalues of $i\Theta(E)_x$ w.r.t. ω_x

then $\langle A_q u,u\rangle = \langle[\gamma,\Lambda]u,u\rangle = \sum_{J,K}\sum_{m\in K}\gamma_m|u_{JK}|^2 \ge \sum_{J,K}(\lambda_1+\cdots+\lambda_q)|u_{JK}|^2$

$u=\sum_{J,K}\xi_J^*\wedge\overline{\xi}_K^*\cdot u_{JK}$ (n,q)-form.

$= (\lambda_1+\cdots+\lambda_q)|u|^2$ by above prop 5.8.

so we have $|\langle g,u\rangle|^2\le|g|^2|u|^2\le\dfrac{|g|^2}{\lambda_1+\cdots+\lambda_q}\cdot\langle A_q u,u\rangle$. by def. $\langle A_q^{-1}g,g\rangle\le\dfrac{|g|^2}{\lambda_1+\cdots+\lambda_q}$

we have $\int_X\langle A_q^{-1}g,g\rangle\,dV \le \int_X\dfrac{|g|^2}{\lambda_1+\cdots+\lambda_q}dV \overset{}{=}\alpha$ (see P.14 in this handout)

<u>Thm 6.5</u> (X,ω) : weakly pseudoconvex Kähler manifold. E : hermitian line bundle on X. $\varphi\in C^\infty(X,\mathbb{R})$ weight function.

Assume. $i\Theta(E)+id'd''\varphi$ eigenvalues $0\le\lambda_1\le\cdots\le\lambda_n$ (w.r.t. ω).

then for any form g, type (n,q) $q\ge1$. L^2_{loc} (resp. C^∞) coeff. $D''g=0$. $\int\dfrac{|g|^2}{\lambda_1+\cdots+\lambda_q}e^{-\varphi}dV<\infty$

we can find L^2_{loc} (resp. C^∞) form f of type $(n,q-1)$ s.t. $D''f=g$.

and $\int_X|f|^2e^{-\varphi}dV \le \int_X\dfrac{|g|^2}{\lambda_1+\cdots+\lambda_q}e^{-\varphi}dV$

<u>pf</u>: Note that if on $E_k|_\Omega$ its metric is H_k $k=1,2$ then chern curvature $i\Theta(E_k) = id''(\overline{H}_k^{-1}d'\overline{H}_k)$.

So for $E_1\otimes E_2$ $i\Theta(E_1\otimes E_2) = id''((\overline{H}_1\otimes\overline{H}_2)^{-1}d'(\overline{H}_1\otimes\overline{H}_2)) = id''(\overline{H}_1^{-1}\otimes\overline{H}_2^{-1}(d'\overline{H}_1\otimes\overline{H}_2 + \overline{H}_1\otimes d'\overline{H}_2))$

$= id''(\overline{H}_1^{-1}d'\overline{H}_1)\otimes Id + i\,Id\otimes d''(\overline{H}_2^{-1}d'\overline{H}_2) = i\Theta(E_1)\otimes Id + Id\otimes i\Theta(E_2)$

in particular E and $(X\times\mathbb{C}, e^{-\varphi})$ we have $E_\varphi = E\otimes(X\times\mathbb{C})$ (metric multi. by $e^{-\varphi}$)
↳ trivial line bundle.

$i\Theta(E_\varphi) = i\Theta(E) + id'd''\varphi$ (see P.11 example) we have $g\in L^2_{n,q,loc}(X, E_\varphi)$

Exhaust X by relatively cpt weakly pseudoconvex domains $X_c = \{x\in X \mid \psi(x)<c\}$ $\psi\in C^\infty(X,\mathbb{R})$
Plurisubharmonic exhaustion function

X_c still Kähler, and is weakly pseudoconvex since $-\log(c-\psi)$: exhausion.

(exhaustion since log ↗. Also $id'd''(-\log(c-\psi)) = id'\frac{d''\psi}{c-\psi} = \frac{id'd''\psi}{c-\psi} + \frac{id'\psi\wedge d''\psi}{(c-\psi)^2} \geq 0$).

($d'\psi\wedge d''\psi = d'\psi\wedge\overline{d'\psi}$)

so $g\in L^2_{n,q}(X_c, E_\varphi|_{X_c})$. Also wealy pseudoconvex + Kähler ⇒ ∃ complete Kähler metric. (Thm 5.2, P.15)

so by Thm 6.1 E_φ : semi-positive. ∃ f_c on X_c s.t. $D''f_c = g$ on X_c.

and $\int_{X_c}|f_c|^2e^{-\varphi}dV \leq \int_{X_c}\langle A_q^{-1}g,g\rangle e^{-\varphi}dV \leq \int_X \frac{|g|^2}{\lambda_1+\cdots+\lambda_q}e^{-\varphi}dV$ by above observation

so unif. L^2 bound ⇒ ∃ weak sol. f. and it is the desired sol. (and C^∞ if g is, by Rmk 6.6 (P.15)) □

For general (p,q)-forms. note that $\Lambda^pT_X^* \simeq \Lambda^{n-p}T_X\otimes\Lambda^nT_X^*$ by contraction n-forms by (n-p) vectors

so $\Lambda^{p,q}T_X^*\otimes E \simeq \Lambda^{n,q}T_X^*\otimes F$ $F = E\otimes\Lambda^{n-p}T_X$ and D'' same.

Def : $\mathrm{Ricci}(\omega) := i\Theta(\Lambda^nT_X)$ Ricci curvature of ω : hermitian metric on T_X.

by Example 12.6 (P.11) locally coordinate $(z_1,\dots,z_n)$ holo. n-form $dz_1\wedge\cdots\wedge dz_n$ section of $\Lambda^nT_X^*$

so $\mathrm{Ricci}(\omega) = id'd''(-\log|dz_1\wedge\cdots\wedge dz_n|^2_\omega) = -id'd''\log\det(\omega_{jk})$

so for (0,q)-form $\Lambda^{0,q}T_X^*\otimes E \simeq \Lambda^{n,q}T_X^*\otimes F$ $F = E\otimes\Lambda^nT_X$.

$i\Theta(F) = i\Theta(E) + \mathrm{Ricci}(\omega)$ ⇝ need consider eigenvalues of $i\Theta(E)+\mathrm{Ricci}(\omega)+id'd''\varphi$ □.