THE CALDERÓN–ZYGMUND INEQUALITY AND PARAMETER DEPENDENCE OF THE BELTRAMI EQUATION

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1. Introduction

Let μ be a complex measurable function on $\mathbb{C}$ with $\|\mu\|_\infty \leq k < 1$. We ask whether there exists a quasiconformal mapping f with dilation $\mu_f = \mu$. In other words, we are looking for solutions to the Beltrami equation

$$f_{\bar{z}} = \mu f_z. \tag{1}$$

The solution f should be a homeomorphism with locally integrable distributional derivatives. In order to solve this equation, we define two integral operators P and T. The operator P is defined on functions $h \in L^p$ with $p > 2$ by

$$Ph(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z - \zeta} - \frac{1}{z} \right) dx\, dy.$$

It is shown that Ph is continuous and that the operator P is continuous with Hölder constant $1 - 2/p$, that is,

$$|Ph(\zeta_1) - Ph(\zeta_2)| \leq K_p \|h\|_p |\zeta_1 - \zeta_2|^{1-2/p},$$

where K_p is a constant depending only on p. The second operator T is defined only for compactly supported functions $h \in C_0^2$, by

$$Th(\zeta) = \lim_{\epsilon \to 0} -\frac{1}{\pi} \int_{|z-\zeta|>\epsilon} \frac{h(z)}{(z - \zeta)^2}\, dx\, dy.$$

The operators P and T then satisfy the following relations.

Lemma 1.1. *For $h \in C_0^2$, Th is continuously differentiable (i.e., C^1). Moreover, we have*

$$(Ph)_{\bar{z}} = h, \quad (Ph)_z = Th, \tag{2}$$

and

$$\int |Th|^2\, dx\, dy = \int |h|^2\, dx\, dy. \tag{3}$$

The isometric property of T allows us to extend T to L^2 by continuity since C_0^2 in dense in L^2, but it is difficult to extend P in a similar manner so that Lemma 1.1 still holds for $h \in L^2$. Nevertheless, Calderón and Zygmund showed that the isometric relation (3) can be replaced by

$$\|Th\|_p \leq C_p \|h\|_p \tag{4}$$

for any $p \geq 2$. In addition, the constant C_p tends to 1 as $p \to 2$. This enables us to extend T to L^p. In particular, for $p > 2$ the differential relations (2) of P is well-defined and hold in the distributional sense by approximating $h \in L^p$ using $h_n \in C_0^2$. We shall use a fixed exponent $p > 2$ so that $kC_p < 1$.

Now, if μ has compact support, then the Beltrami equation (1) can be uniquely solved by expressing the solution in terms of T and P. Such solution is called a normal solution.

Theorem 1.2. *If μ has compact support, there exists a unique solution f of (1) such that $f(0) = 0$ and $f_z - 1 \in L^p$.*

The solution f is a homeomorphism if we first assume that μ has distributional derivative $\mu_z \in L^p$, $p > 2$. Then, by choosing a sequence $\mu_n \in C^1$ with $\mu_n \to \mu$ almost everywhere, we can obtain a sequence of normal solutions f_n which converges to the solution f with complex dilation μ. More generally, the assumption that μ is compactly supported can be removed, and we have the following theorem.

Theorem 1.3. *For any measurable μ with $\|\mu\|_\infty < 1$, there exists a unique normalized quasiconformal mapping f^μ with complex dilation μ that leaves 0, 1, and ∞ fixed.*

The first part of this report is dedicated to prove the Calderón–Zygmund inequality (4). After that, we will discuss the behavior of the solution f^μ as we vary the parameter μ.

2. The Calderón–Zygmund Inequality

Let us first consider a one-dimensional analog to the problem.

Lemma 2.1. *For $f \in C_0^1(\mathbb{R})$, define*

$$Hf(\xi) = \text{pr. v.}\, \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x - \xi}\, dx.$$

Then $\|Hf\|_p \leq A_p \|f\|_p$ for $p \geq 2$ with $A_2 = 1$.

Proof. Let

$$F(\zeta) = u + iv = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x - \zeta}\, dx,$$

where $\zeta = \xi + i\eta$ is defined on the upper half-plane. Then F is by definition analytic. Since the imaginary part

$$v(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta}{(x - \xi)^2 + \eta^2} f(x)\, dx$$

is the Poisson integral, we have $v(\xi, 0) = f(\xi)$. On the other hand, for the real part we write

$$\begin{aligned} u(\xi, \eta) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x - \xi}{(x - \xi)^2 + \eta^2} f(x)\, dx \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{f(\xi + x) - f(\xi - x)}{x} \frac{x^2}{x^2 + \eta^2}\, dx. \end{aligned}$$

Then since $f \in C_0^1$,

$$\begin{aligned} |u(\xi, \eta) - Hf(\xi)| &\leq \frac{1}{\pi} \int_0^{\infty} \left| \frac{f(\xi + x) - f(\xi - x)}{x} \right| \left| \frac{x^2}{x^2 + \eta^2} - 1 \right| dx \\ &\leq C \int_0^{\infty} \frac{\eta^2}{x^2 + \eta^2}\, dx \to 0 \quad \text{as} \quad \eta \to 0^+. \end{aligned}$$

Now, noting that $F = u + iv$ is analytic, by a direct calculation we see that

$$\begin{aligned}\Delta|u|^p &= p(p-1)|u|^{p-2}(u_\xi^2 + u_\eta^2)\\ \Delta|v|^p &= p(p-1)|v|^{p-2}(u_\xi^2 + u_\eta^2)\\ \Delta|F|^p &= p^2|F|^{p-2}(u_\xi^2 + u_\eta^2)\end{aligned}$$

holds almost everywhere. Thus, we find that

$$\Delta\left(|F|^p - \frac{p}{p-1}|u|^p\right) = p^2(|F|^{p-2} - |u|^{p-2})(u_\xi^2 + u_\eta^2) \geq 0.$$

Let $g = |F|^p - \frac{p}{p-1}|u|^p$. Applying Stokes theorem to a semicircle S_R, we have

$$0 \leq \int_{S_R} \Delta g = \int_{A_R} g_\zeta(\zeta)\, d|\zeta| - \int_{-R}^{R} g_\eta(\zeta)\, d\xi,$$

where A_R is the arc of S_R. Since

$$\frac{\partial F}{\partial \zeta} = \int_{-\infty}^{\infty} \frac{f(x)}{(x-\zeta)^2}\, dx,$$

we have $g_\zeta(\zeta) = O(|\zeta|^{-2})$. It follows that by taking $R \to \infty$,

$$\frac{\partial}{\partial \eta}\int_{-\infty}^{\infty} g(\zeta)\, d\xi \leq 0,$$

from which we see that $\int_{-\infty}^{\infty} g(\zeta)\, d\xi$ is decreasing in η. Also, it is easy to see that this integral vanishes as $\eta \to \infty$, so we can deduce that

$$\int_{-\infty}^{\infty} |F(\xi + i\eta)|^p\, d\xi \geq \frac{p}{p-1}\int_{-\infty}^{\infty} |u(\xi + i\eta)|^p\, d\xi$$

for every $\eta > 0$. Now, observe that

$$\begin{aligned}\left(\frac{p}{p-1}\right)^{2/p} \|u^2\|_{p/2} &\leq \left(\int |F|^p\, d\xi\right)^{2/p}\\ &\leq \|u^2 + v^2\|_{p/2}\\ &\leq \|u^2\|_{p/2} + \|v^2\|_{p/2}.\end{aligned}$$

Thus,

$$\int |u|^p\, d\xi \leq \left(\left(\frac{p}{p-1}\right)^{2/p} - 1\right)^{-p/2} \int |v|^p\, d\xi,$$

and the proof concludes by letting $\eta \to 0$. $\square$

We proceed to prove the Calderón–Zygmund inequality. Recall that

$$Th = \lim_{\epsilon \to 0} -\frac{1}{\pi}\int_{|z-\zeta|>\epsilon} \frac{h(z)}{(z-\zeta)^2}\, dx\, dy$$

is defined for $h \in C_0^2$. We will extend T to L^p for $p \geq 2$ so that T satisfies

$$\|Th\|_p \leq C_p\|h\|_p$$

and $C_p \to 1$ as $p \to 2$.

Define the operator

$$T^* f(\zeta) = \frac{1}{2\pi}\int f(z+\zeta)\frac{dx\, dy}{z|z|}, \quad f \in C_0^2.$$

Let $z = re^{i\theta}$. Then we have

$$T^*f(\zeta) = \frac{1}{2}\int_0^{\pi}\left(\frac{1}{\pi}\int_0^{\infty}\frac{f(\zeta + re^{i\theta}) - f(\zeta - re^{i\theta})}{r}\,dr\right)e^{-i\theta}\,d\theta.$$

Taking the (two-dimensional) p-norm, we see that

$$\|T^*f\|_p \le \frac{\pi}{2}\max_{\theta\in[0,\pi]}\left\|\frac{1}{\pi}\int_0^{\infty}\frac{f(\zeta + re^{i\theta}) - f(\zeta - re^{i\theta})}{r}\,dr\right\|_p$$

Now, write $\zeta = \xi + i\eta$ and let $g_\eta^\theta(\xi) = f(e^{i\theta}(\xi + i\eta))$. Note that the norm on the right does not change if we replace ζ with $\zeta e^{i\theta}$, in which case the integral becomes

$$\frac{1}{\pi}\int_0^{\infty}\frac{f(e^{i\theta}(\zeta + r)) - f(e^{i\theta}(\zeta - r))}{r}\,dr = \frac{1}{\pi}\int_0^{\infty}\frac{g_\eta^\theta(\xi + r) - g_\eta^\theta(\xi - r)}{r}\,dr = Hg_\xi^\theta(\xi).$$

Therefore, we can use Lemma 2.1 to obtain

$$\begin{aligned}
\|Hg_\eta^\theta(\xi)\|_p^p &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Hg_\eta^\theta(\xi)|^p\,d\xi\,d\eta \\
&\le \int_{-\infty}^{\infty}A_p^p\left(\int_{-\infty}^{\infty}|g_\eta^\theta(\xi)|^p\,d\xi\right)d\eta \\
&= A_p^p\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(e^{i\theta}\zeta)|^p\,d\xi\,d\eta = A_p^p\|f\|_p,
\end{aligned}$$

and we find that $\|T^*f\|_p \le \frac{\pi}{2}A_p\|f\|_p$.

Now, by continuity we extend T^* to L^p. If we can show that $Tf = -T^*T^*f$ for $f \in C_0^2$, then T is natrually extended to L^p and the proof is completed. Note that $\frac{\partial}{\partial z}(1/|z|) = -1/2z|z|$. Recall that integration by parts gives

$$\begin{aligned}
\int_{\mathbb{C}} f\frac{\partial g}{\partial z}\,dx\,dy &= \lim_{R\to\infty}\frac{i}{2}\int_{|z|<R} f\frac{\partial g}{\partial z}\,dz\wedge d\bar{z} \\
&= \lim_{R\to\infty}\frac{i}{2}\int_{|z|<R} f\,d(g\,d\bar{z}) \\
&= \lim_{R\to\infty}\frac{i}{2}\left(\int_{|z|=R} fg\,d\bar{z} - \int_{|z|<R}\frac{\partial f}{\partial z}g\,dz\wedge d\bar{z}\right) \\
&= \lim_{R\to\infty}\frac{i}{2}\int_{|z|=R} fg\,d\bar{z} - \int_{\mathbb{C}}\frac{\partial f}{\partial z}g\,dx\,dy
\end{aligned}$$

whenever it makes sense. If $f \in C_0^2$, we obtain

$$\begin{aligned}
T^*f(\zeta) &= -\frac{1}{\pi}\int f(z+\zeta)\frac{\partial}{\partial z}\frac{1}{|z|}\,dx\,dy \\
&= \frac{1}{\pi}\int f_z(z+\zeta)\frac{1}{|z|}\,dx\,dy \\
&= \frac{1}{\pi}\frac{\partial}{\partial\zeta}\int f(z)\frac{dx\,dy}{|z-\zeta|} \\
&= \frac{1}{\pi}\frac{\partial}{\partial\zeta}\int f(z)\left(\frac{1}{|z-\zeta|} - \frac{1}{|z|}\right)dx\,dy.
\end{aligned} \tag{5}$$

For any test function ϕ, it follows that

$$\int T^* f(\zeta)\phi(\zeta)\, d\xi\, d\eta = -\frac{1}{\pi}\int f(z)\left(\frac{1}{|z-\zeta|} - \frac{1}{|z|}\right)\phi_\zeta(\zeta)\, dx\, dy\, d\xi\, d\eta.$$

This remains true for $f \in L^p$ since the integral on the right is absolutely convergent. In other words, (5) holds for $f \in L^p$ in the distributional sense. Applying T^* to (5) again, we have

$$\begin{aligned}
T^*T^*f(w) &= \frac{1}{\pi}\frac{\partial}{\partial w}\int T^*f(\zeta)\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta \\
&= \frac{1}{\pi^2}\frac{\partial}{\partial w}\int\left(\int \frac{f_z(z)\, dx\, dy}{|z-\zeta|}\right)\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta \\
&= \frac{1}{\pi^2}\frac{\partial}{\partial w}\int f_z(z)\left(\int \frac{1}{|z-\zeta|}\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta\right) dx\, dy \\
&= -\frac{1}{\pi^2}\frac{\partial}{\partial w}\int f(z)\frac{\partial}{\partial z}\left(\int \frac{1}{|z-\zeta|}\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta\right) dx\, dy.
\end{aligned} \tag{6}$$

We try to compute the integrand

$$\frac{\partial}{\partial z}\lim_{R\to\infty}\int_{|\zeta-w|<R}\frac{1}{|z-\zeta|}\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta. \tag{7}$$

The differentiation and limit can be interchanged, since

$$\frac{\partial}{\partial z}\int_{|\zeta-w|<R}\frac{1}{|z-\zeta|}\left(\frac{1}{|\zeta-w|} - \frac{1}{|\zeta|}\right) d\xi\, d\eta$$

converges to zero as $R \to \infty$ unifomly on compact sets. Moreover, we may replace (7) with

$$\lim_{R\to\infty}\frac{\partial}{\partial z}\int_{|\zeta-w|<R}\frac{d\xi\, d\eta}{|z-\zeta||\zeta-w|} - \int_{|\zeta|<R}\frac{d\xi\, d\eta}{|\zeta||z-\zeta|},$$

for their difference converges to zero uniformly as $R \to \infty$. By a change of variable $\zeta \mapsto \zeta|z-w| + w$, the first term becomes

$$\begin{aligned}
\frac{\partial}{\partial z}\int_{|\zeta-w|<R}\frac{d\xi\, d\eta}{|z-\zeta||\zeta-w|} &= \frac{\partial}{\partial z}\int_{|\zeta|<R/|z-w|}\frac{d\xi\, d\eta}{|1-\zeta||\zeta|} \\
&= \frac{\partial}{\partial z}\int_0^{R/|z-w|}\int_0^{2\pi}\frac{dr\, d\theta}{1-re^{i\theta}} \\
&= -\frac{R}{2(z-w)|z-w|}\int_0^{2\pi}\frac{d\theta}{|1-Re^{i\theta}/|z-w||},
\end{aligned}$$

which is easily seen to tend to $-\pi/(z-w)$ as $R \to \infty$. Similarly, the second term converges to $-\pi/z$. Putting them back into (6), we conclude that

$$\begin{aligned}
T^*T^*f(w) &= \frac{1}{\pi}\frac{\partial}{\partial w}\int f(z)\left(\frac{1}{z-w} - \frac{1}{z}\right) dx\, dy \\
&= -\frac{\partial}{\partial w}P(w) = -Tf(w).
\end{aligned}$$

This completes the proof of (4).

To see that $C_p \to 1$ as $p \to 2$, we show more generally that $\log C_p$ is a convex function, and then the assertion follows by continuity since from (3) we know that $C_p = 2$ when $p = 2$. This result is also called the Riesz–Thorin convexity theorem.

Theorem 2.2 (Riesz–Thorin)**.** *The best constant C_p is such that $\log C_p$ is a convex function of $1/p$.*

We first prove an easy lemma.

Lemma 2.3. *Suppose $1 \le p, q \le \infty$ with $1/p + 1/q = 1$. If $f \in L^p$, then*

$$\|f\|_p = \sup_{\|g\|_q = 1} \int fg.$$

Proof. We follow the proof in [SS11]. If $f = 0$ then there is nothing to prove, so we may assume $f \neq 0$. By Hölder's inequality, we have

$$\int fg \le \|f\|_p \|g\|_q.$$

Taking the supremum over $\|g\|_q \le 1$ we have one side of the inequality. To prove the reverse inequality, consider the sign function

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If $p = 1$ and $q = \infty$, we may take $g(x) = \operatorname{sgn} f(x)$. Then clearly $\|g\|_\infty = 1$ and $\int fg = \|f\|_1$. If $1 < p, q < \infty$, then we set

$$g(x) = \frac{|f(x)|^{p-1}}{\|f\|_p^{p-1}} \operatorname{sgn} f(x).$$

Observe that

$$\|g\|_q^q = \int \frac{|f(x)|^{q(p-1)}}{\|f\|_p^{q(p-1)}} = 1$$

since $q(p-1) = p$, and that $\int fg = \|f\|_p$. Finally, if $p = \infty$ and $q = 1$, let $\epsilon > 0$, and E a set of finite positive measure on which $|f(x)| \ge \|f\|_\infty - \epsilon$ (whose existence is from the defintion of ∞-norm). Take

$$g(x) = \frac{\chi_E(x)}{\mu(E)} \operatorname{sgn} f(x),$$

where χ_E is the characteristic function of E. Then we see that $\|g\|_1 = 1$, and also

$$\int fg = \frac{1}{\mu(E)} \int_E |f| \ge \|f\|_\infty - \epsilon$$

This completes the proof. $\square$

Proof of Theorem 2.2. Let $p_1 = 1/\alpha_1$ and $p_2 = 1/\alpha_2$ with $p_1, p_2 \ge 2$. Assume that

$$\|Tf\|_{1/\alpha_1} \le C_1 \|f\|_{1/\alpha_1},$$
$$\|Tf\|_{1/\alpha_2} \le C_2 \|f\|_{1/\alpha_2}.$$

If $\alpha = (1-t)\alpha_1 + t\alpha_2$ for $0 \le t \le 1$, we show that

$$\|Tf\|_{1/\alpha} \le C_1^{1-t} C_2^t \|f\|_{1/\alpha}.$$

We write α' for the conjugate exponent of α, i.e., $\alpha + \alpha' = 1$. For fixed f, g and any complex ζ, define

$$F(\zeta) = |f|^{\alpha(\zeta)/\alpha} \frac{f}{|f|},$$
$$G(\zeta) = |g|^{\alpha(\zeta)'/\alpha'} \frac{g}{|g|},$$

where $\alpha(\zeta) = (1-\zeta)\alpha_1 + \zeta\alpha_2$ and $\alpha(\zeta)' = 1 - \alpha(\zeta)$, with the knowledge that $F(\zeta) = 0$ and $G(\zeta) = 0$ whenever $f = 0$ and $g = 0$. Note that $F(t) = f$ and $G(t) = g$ for real $0 \le t \le 1$. Set

$$\phi(\zeta) = \int TF(\zeta) \cdot G(\zeta)\, dx\, dy.$$

Notice that ζ is viewed as a parameter and $F(\zeta), G(\zeta)$ are functions of $z = x + iy$. Since simple functions with compact support are dense in any L^p, we may assume that f, g are such functions. It follows that F, G are also simple, and we may write $F(\zeta) = \sum_i F_i \chi_i$ and $G(\zeta) = \sum_j G_j \chi_j^*$. Thus,

$$\phi(\zeta) = \sum_{i,j} F_i G_j \int T\chi_i \cdot \chi_j^*\, dx\, dy,$$

from which we see that ϕ is an exponential polynomial of the form $\phi(\zeta) = \sum_i a_i e^{\lambda_i \zeta}$ with λ_i real. Therefore ϕ is bounded if $\xi = \operatorname{Re}\zeta$ is bounded. Consider now the special cases $\xi = 0$ and $\xi = 1$. If $\xi = 0$ then $\operatorname{Re}\alpha(\zeta) = \alpha_1$ and hence $|F(\zeta)| = |f|^{\alpha_1/\alpha}$ and $|G(\zeta)| = |g|^{\alpha_1'/\alpha'}$. It follows that

$$\|F(\zeta)\|_{1/\alpha_1} = (\|f\|_{1/\alpha})^{\alpha_1/\alpha},$$
$$\|G(\zeta)\|_{1/\alpha_1'} = (\|g\|_{1/\alpha'})^{\alpha_1'/\alpha'} = 1.$$

We may assume the normalization $\|f\|_{1/\alpha} = 1$ for simplicity, and we obtain

$$|\phi(\zeta)| \le \|TF(\zeta)\|_{1/\alpha_1} \|G(\zeta)\|_{1/\alpha_1'} \le C_1.$$

A similar argument shows that $|\phi(\zeta)| \le C_2$ when $\xi = 1$. Hence we conclude that

$$\log|\phi(\zeta)| - (1-\xi)\log C_1 - \xi \log C_2 \le 0$$

on the boundary of the strip $0 \le \xi \le 1$. Since the function on the left is subharmonic, the maximum principle applies and thus the inequality holds in the strip. The theorem follows by taking $\zeta = t$. □

3. Parameter Dependence of the Beltrami Equation

Recall that f^μ is the solution of the Beltrami equation with fixpoints at $0, 1, \infty$. We start with two lemmata.

Lemma 3.1 (Cauchy–Pompeiu formula). *Let $D \subseteq \mathbb{C}$ be a region and let $f\colon \overline{D} \to \mathbb{C}$ be a C^1 function. Then for any $\zeta \in U$,*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)\, dz}{z - \zeta} - \frac{1}{\pi}\int_D \frac{f_{\bar z}(z)}{z - \zeta}\, dx\, dy.$$

Proof. For $\epsilon > 0$ let $\Delta_\epsilon(\zeta)$ denote the disk with radius ϵ at ζ. Then by Stokes theorem we have

$$\int_{\partial D} \frac{f(z)\,dz}{z-\zeta} - \int_{\partial\Delta_\epsilon(\zeta)} \frac{f(z)\,dz}{z-\zeta} = \int_{D\setminus\Delta_\epsilon(\zeta)} d\left(\frac{f(z)\,dz}{z-\zeta}\right) = -\int_{D\setminus\Delta_\epsilon(\zeta)} \frac{f_{\bar z}(z)}{z-\zeta}\,dz\wedge d\bar z.$$

Since

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\partial\Delta_\epsilon(\zeta)} \frac{f(z)\,dz}{z-\zeta} = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})\,d\theta = f(z)$$

by continuity, the result follows by letting $\epsilon \to 0$. □

Lemma 3.2. *If $k = \|\mu\|_\infty \to 0$, then $\|f^\mu_z - 1\|_{1,p} \to 0$ for all p, where $\|\cdot\|_{R,p}$ denotes the p-norm over $|z| \le R$.*

Proof. We first show the case where μ has compact support. Let F^μ be the normal solution obtained in Theorem 1.2, in whose proof we recall that $h = F^\mu_z - 1$ is obtained from

$$h = T(\mu h) + T\mu.$$

This implies that $\|h\|_p \le C\|\mu\|_p \to 0$ as long as $kC_p < 1$. Since $f^\mu = F^\mu/F^\mu(1)$ is just a normalization, and $F^\mu(1) \to 1$, the assertion is clear for compactly supported μ.

Let $\check f(z) = 1/f(1/z)$. We show that $\|\check f^\mu_z - 1\|_{1,p} \to 0$ as well when μ has compact support. Again it suffices to prove the corresponding statement for the normal solution $\check F^\mu$. Observe that

$$\int_{|z|<1} |\check F_z - 1|^p\,dx\,dy = \int_{|z|>1}\left|\frac{z^2F_z(z)}{F(z)^2} - 1\right|^p \frac{dx\,dy}{|z|^4}.$$

For $1 < |z| < R$, the integral can be written as

$$\int_{1<|z|<R}\left|\frac{z^2(F_z(z)-1)}{F(z)} + \frac{z^2}{F(z)^2} - 1\right|^p \frac{dx\,dy}{|z|^4},$$

which converges to zero since $\|F_z - 1\|_{R,p} \to 0$ and $F(z) \to z$ uniformly. It is also easy to see that the integral vanishes for $|z| > R$.

In the general case we write $f = \check g \circ h$ with $\mu_h = \mu_f$ inside the unit disk, and h analytic outside. Note that μ_g and μ_h are both bounded by k and have compact support. Since

$$f_z = (\check g_z \circ h)h_z + (\check g_{\bar z}\circ h)\overline{h}_z = (\check g_z\circ h)h_z,$$

in the unit disk, we have

$$\|f_z - 1\|_{1,p} \le \|((\check g_z - 1)\circ h)h_z\|_{1,p} + \|h_z - 1\|_{1,p}.$$

The second term tends to zero by our previous case. For the first term on the right, we obtain by a change of variable

$$\begin{aligned}
\|((\check g_z - 1)\circ h)h_z\|^p_{1,p} &= \int_{|z|<1} |(\check g_z - 1)\circ h|^p |h_z|^p\,dx\,dy \\
&\le \frac{1}{1-k^2}\int_{h(\{|z|<1\})} |\check g_z - 1|^p |hc_z \circ h^{-1}|^{p-2}\,dx\,dy \\
&\le \frac{1}{1-k^2}\left(\int_{h(\{|z|<1\})} |\check g_z - 1|^{2p}\int_{|z|<1} |h_z|^{2(p-2)}\right)^{1/2} \to 0,
\end{aligned}$$

where we used the fact that the Jacobian determinant is $J(h) = |h_z|^2(1-\mu^2)$. Note that the region $h(\{|z|<1\})$ may be slightly different from the unit disk, but this does not change the result as can be seen from the proof for $\check{f}$. This completes the proof. □

Now, we assume that μ depends on a parameter t in the form

$$\mu(z,t) = t\nu(z) + t\epsilon(z,t),$$

where $\nu, \epsilon \in L^\infty$ and $\|\epsilon(z,t)\|_\infty \to 0$ as $t \to 0$. We will derive a first-order approximation for $f^\mu = f(z,t)$ by finding a formula for the t-derivative $\dot{f} = \partial f/\partial t$ at $t=0$. For $|\zeta|<1$, we write

$$f(\zeta) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)\,dz}{z-\zeta} - \frac{1}{\pi}\int_{|z|<1} \frac{f_{\bar{z}}(z)}{z-\zeta}\,dx\,dy. \tag{8}$$

Replacing z with $1/z$, the first integral on the right becomes

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=1} \frac{f(1/z)\,dz}{z(1-z\zeta)} &= \frac{1}{2\pi i}\int_{|z|=1} f(1/z)\left(\frac{1}{z} + \zeta + \frac{z\zeta^2}{1-z\zeta}\right)dz \\
&= A + B\zeta + \frac{\zeta^2}{2\pi i}\int_{|z|=1} \frac{\check{f}(z)^{-1}z}{1-z\zeta}\,dz \\
&= A + B\zeta - \frac{\zeta^2}{\pi}\int_{|z|<1} \frac{\check{f}_{\bar{z}}(z)z}{\check{f}(z)^2(1-z\zeta)}\,dx\,dy,
\end{aligned}$$

where the convergence at the last line holds for t sufficiently small so that $\|\check{\mu}\|_\infty = K < 2$ since $|\check{f}(z)| > C|z|^K$ for $|z| = \delta$ small and

$$\int_{|z|=\delta} \frac{|\check{f}(z)^{-1}||z|}{|1-z\zeta|}|dz| < C'\delta^{2-K} \to 0 \quad \text{as} \quad \delta \to 0.$$

The constants A and B can be solved by the normalization $f(0)=0$, $f(1)=1$, and we obtain

$$\begin{aligned}
A &= \frac{1}{\pi}\int_{|z|<1} \frac{f_{\bar{z}}(z)}{z}\,dx\,dy, \\
B &= \zeta + \frac{1}{\pi}\int_{|z|<1} f_{\bar{z}}(z)\left(\frac{1}{z-1} - \frac{1}{z}\right)dx\,dy + \frac{1}{\pi}\int_{|z|<1} \frac{\check{f}_{\bar{z}}(z)z}{\check{f}(z)^2(1-z)}\,dx\,dy.
\end{aligned}$$

Thus (8) becomes

$$\begin{aligned}
f(\zeta) = \zeta &- \frac{1}{\pi}\int_{|z|<1} f_{\bar{z}}(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right)dx\,dy \\
&- \frac{1}{\pi}\int_{|z|<1} \frac{\check{f}_{\bar{z}}(z)}{\check{f}(z)^2}\left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right)dx\,dy.
\end{aligned}$$

We write $f_{\bar{z}} = \mu f_z = \mu(f_z - 1) + \mu$ and use a corresponding expression for $\check{f}_{\bar{z}}$ with $\check{\mu}(z) = (z/\bar{z})^2\mu(1/z)$. Since $\|f_{\bar{z}} - 1\|_{1,p} \to 0$ and $\|\check{f}_{\bar{z}} - 1\|_{1,p} \to 0$ by Lemma 3.2, and also $\mu/t \to \nu$ as $t \to \infty$, we have

$$\begin{aligned}
\dot{f}(\zeta) &= \lim_{t\to 0} \frac{f(\zeta)-\zeta}{t} \\
&= -\frac{1}{\pi}\int_{|z|<1} \nu(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right) dx\,dy \\
&\quad -\frac{1}{\pi}\int_{|z|<1} \nu(1/z)\frac{1}{\bar{z}^2}\left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right) dx\,dy,
\end{aligned}$$

where we used the fact that $z^2/\check{f}(z)^2 \to 1$. Note that the convergence is uniform in a compact subset of $|\zeta| < 1$. Now, observe that under inversion $z \mapsto 1/z$, the integrand of the second integral is the same as that in the first integral, and the domain becomes $|z| > 1$. Hence we find that

$$\dot{f}(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(z) R(z,\zeta)\, dx\,dy,$$

where

$$R(z,\zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)}.$$

We now expand $\mu(t)$ about an arbitrary t_0, that is, we assume

$$\mu(t) = \mu(t_0) + \nu(t_0)(t-t_0) + o(t-t_0).$$

Consider

$$f^{\mu(t)} = f^{\lambda} \circ f^{\mu(t_0)},$$

where

$$\lambda = \lambda(t) = \left(\frac{\mu(t_0)}{1-\mu(t)\overline{\mu}(t_0)} \cdot \frac{f_z^{\mu_0}}{\overline{f}_z^{\mu_0}}\right) \circ (f^{\mu_0})^{-1}$$

by the composition of quasiconformal mappings. It is clear that $\lambda(t) = (t-t_0)\dot{\lambda}(t_0) + o(t-t_0)$ with

$$\dot{\lambda}(t_0) = \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f}_{\bar{z}}^{\mu_0}}\right) \circ (f^{\mu_0})^{-1}.$$

It then follows that

$$\begin{aligned}
\frac{\partial f}{\partial t}(\zeta, t_0) &= \dot{f} \circ f^{\mu_0} \\
&= -\frac{1}{\pi}\int \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\overline{f}_{\bar{z}}^{\mu_0}}\right) \circ (f^{\mu_0})^{-1} R(z, f^{\mu_0}(\zeta))\, dx\,dy \\
&= -\frac{1}{\pi}\int \nu(t_0, z)(f_z^{\mu_0})^2 R(f^{\mu_0}(z), f^{\mu_0}(\zeta))\, dx\,dy.
\end{aligned}$$

This is the general perturbation formula. Our result is summarized as follows.

Theorem 3.3. *Suppose*

$$\mu(t+s)(z) = \mu(t)(z) + s\nu(t)(z) + s\epsilon(s,t)(z)$$

with $\nu(t), \nu(t), \epsilon(s,t) \in L^\infty$, $\|\mu(t)\|_\infty < 1$, *and* $\|\epsilon(s,t)\|_\infty \to 0$ *as* $s \to 0$. *Then*

$$f^{\mu(t+s)}(\zeta) = f^{\mu(t)}(\zeta) + s\dot{f}(\zeta,t) + o(s)$$

uniformly on compact sets, where

$$\dot{f}(\zeta,t) = -\frac{1}{\pi}\int \nu(t)(z)(f_z^{\mu(t)}(z))^2 R(f^{\mu(t)}(z), f^{\mu(t)}(\zeta))\,dx\,dy.$$

References

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