TOPICS IN COMPLEX GEOMETRY 2025
MIDTERM EXAM

1. Let $g \in C_0^\infty(\mathbb{C})$. (a) Solve $\partial f/\partial \bar{z} = g$ and (b) show that it has a solution $f \in C_0^\infty(\mathbb{C})$ if and only if $\int_{\mathbb{C}} z^n g(z)\, dz \wedge d\bar{z} = 0$ for all $n \in \mathbb{N}$.

2. Let X be a compact Riemann surface, $\mathscr{D}$ the sheaf of meromorphic 1-forms η with $\operatorname{res}_x \eta = 0$, $\forall x \in X$. Show that $H^1(X, \mathbb{C}) \cong \mathscr{D}(X)/d\mathscr{M}(X)$.

3. Prove Abel's theorem on a compact Riemann surface X: a divisor D with $\deg D = 0$ is principal if and only if there is a one chain c with $D = \partial c$ such that $\int_c \omega = 0$ for all $\omega \in \Omega(X)$.

4. Prove Riemann's extension theorem: Let $f \in \mathscr{O}(U)$ with U open in $\mathbb{C}^n$, and let $g \in \mathscr{O}(U \setminus Z(f))$. If g is locally bounded near $Z(f)$ then g extends uniquely to $\tilde{g} \in \mathscr{O}(U)$.

5. Compute the Kodaira dimension of a smooth hypersurface $X \subset \mathbb{P}^4$ of degree d in the following 3 cases: $d = 4, 5, 6$.

6. 6. Let (X, g) be a Kähler manifold of dimension $n > 1$. (a) Show that if $e^f g$ is another Kähler metric, then f is a constant function. (b) Let α be a closed (p, q)-form primitive at every point. Show that α is harmonic.

7. Prove the Hodge decomposition theorem on compact Riemannian manifolds by assuming the PDE results on compactness and regularity. Namely show that $\dim \mathbb{H}^p < \infty$ and $A^p = \mathbb{H}^p \oplus^\perp \triangle A^p$.

* You may replace ONE problem by presenting a major topic or the proof of a major theorem you have well prepared but not given above.

** Each problem deserves 15 pts. Very good/complete answer to a problem might receive bonus points.

Date: April 9, am 10:10 - 13:10 at Room 101. An course by Chin-Lung Wang at NTU..