Non-compact Riemann Surfaces

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0 Introduction

In this note, we will first introduce the *Runge Approximation Theorem*, as the most important technic using in the researches of non-compact Riemann surfaces. Then as an application, we can use it to prove that any holomorphic vector bundles on a non-compact Riemann surface is trivial. Which is however, a highly non-trivial result.

In Forster's book [1] and this note, we say X is a Riemann surface means it is a Hausdorff space together with a maximal complex atlas Σ. Unlike the usual definition of manifolds, we do not assume it to be second countable. But there is a theorem assert that these condition are enough to make X be second countable.

Theorem 0.0.1 (Radó). *Every Riemann surface X is second countable.*

Although we will not prove or apply this theorem in this note, it's still very important in the theory of Riemann surfaces. By this theorem any construction of gluing local complex charts is automatically a Riemann surface, even though the number of charts is uncountable.

Next, we will focus on our main theorem.

1 The Runge Approximation Theorem

Before we start to prove the theorem, there are two lemmata we must handle first, the finiteness theorem (2.1.3) and Weyl's Lemma (1.1.7). Forster's book [1] gives a proof of the finiteness theorem for only trivial bundle. But in order to prepare for the proof of triviality of vector bundles, we may apply the general version for any holomorphic vector bundles, which we will prove in 2.1. Thus, we only need to prove Weyl's Lemma as the setup.

1.1 Weyl's Lemma

Recall the definition of distributions:

Definition 1.1.1. Suppose X is a open subset of $\mathbb{C}$, a *distribution* on X is a continuous linear mapping

$$T : C_0^\infty(X) \to \mathbb{C}, \quad f \mapsto T[f].$$

Saying that T is continuous is in the sense that if f_j converges uniformly to f in any order, then $T[f_j]$ converges to $T[f]$. Let $\mathcal{D}'(X)$ denotes the vector space of all distributions on X.

Definition 1.1.2. The *differentiation* of a distribution T is defined by

$$(D^\alpha T)[f] = (-1)^{|\alpha|}T[D^\alpha f].$$

There is a natural inclusion $C_0^\infty(X) \hookrightarrow \mathcal{D}'(X)$ given by

$$h \mapsto T_h, \quad T_h[f] \coloneqq \iint_X h(z)f(z)\,dx\,dy.$$

One can check easily that

$$(D^\alpha T_h)[f] = T_{D^\alpha h}[f].$$

There are some basic properties of distributions:

Proposition 1.1.3. *Suppose given an open subset $X \subset \mathbb{C}$, a compact subset $K \subset X$ and an open interval $I \subset \mathbb{R}$. Suppose $g : X \times I \to \mathbb{C}$ is an infinitely (real) differentiable function with* Supp$(g) \subset K \times I$ *and T is a distribution on X. Then the function $t \mapsto T_z[g(z,\, t)]$ is infinitely differentiable on I and satisfies*

$$\frac{d}{dt}T_z[g(z,\, t)] = T_z\left[\frac{\partial g(z,\, t)}{dt}\right]. \tag{1}$$

The subscript z indicates that T operates on $g(z, t)$ as a function of z.

Proof

It suffices to prove (1), since repeated application of this result will show the infinite differentiability with respect to t. For fixed $t \in I$ and sufficiently small $h \neq 0$ let

$$f_h(z) \coloneqq \frac{1}{h}(g(z,\, t+h) - g(z,\, t)).$$

Then $f_h \in C_0^\infty(X)$ and

$$f_h \to \frac{\partial g(\cdot,\, t)}{\partial t} \text{ as } h \to 0 \text{ uniformly in any order.}$$

Hence, because T is linear and continuous,

$$\frac{d}{dt}T_z[g(z,\, t)] = \lim_{h\to 0}\frac{1}{h}(T_z[g(z,\, t+h)] - T_z[g(z,\, t)]) = \lim_{h\to 0} T_z[f_h] = T_z\left[\frac{\partial g(z,\, t)}{\partial t}\right].$$

■

Proposition 1.1.4. *Suppose X, Y are open subsets of $\mathbb{C}$ and $K \subset X$, $L \subset Y$ are compact subsets. Further suppose $g : X \times Y \to \mathbb{C}$ is an infinitely (real) differentiable function with* $\mathrm{Supp}(g) \subset K \times L$*. Then for any distribution T on X*

$$T_z\left[\iint\limits_Y g(z,\, \zeta)\, d\xi\, d\eta\right] = \iint\limits_Y T_z[g(z,\, \zeta)]\, d\xi\, d\eta.$$

Proof

It follows from (1.1.3) that $T_z[g(z,\, \zeta)]$ is infinitely differentiable with respect to ζ. Thus the integral on the right hand side is well-defined. Suppose $R \subset \mathbb{C}$ is a rectangle with sides parallel to the axes which contains L. Then the function $g(z,\, \zeta)$ extends as zero to $K \times R$. For every integer $n > 0$ partition R into n^2 subrectangles $R_{n\nu}, \nu = 1,\, \ldots,\, n^2$, by subdividing the sides into n equal parts. Choose a point $\zeta_{n\nu}$ in each $R_{n\nu}$. Let A be the area of R. Then the Riemann sums

$$G_n(z) := \frac{A}{n^2}\sum_{\nu=1}^{n^2} g(z,\, \zeta_{n\nu})$$

converges as $n \to \infty$ to the integral $\iint_Y g(z,\, \zeta)\, d\xi\, d\eta$ uniformly in any order. Thus from the continuity of T it follows that

$$\iint\limits_Y T_z[g(z,\, \zeta)]\, d\xi\, d\eta = \lim_{n\to\infty}\frac{A}{n^2}\sum_{\nu=1}^{n^2} T_z[g(z,\, \zeta_{n\nu})] = \lim_{n\to\infty} T_z[G_n] = T_z\left[\iint\limits_Y g(z,\, \zeta)\, d\xi\, d\eta\right].$$

■

Definition 1.1.5. For $\epsilon > 0$ denote by $D(z,\, \epsilon)$ the open disk with center z and radius ϵ and by $\overline{D}(z,\, \epsilon)$ its closure. If $U \subset \mathbb{C}$ is an open set, then

$$U^{(\epsilon)} := \{z \in U : \overline{D}(z,\, \epsilon) \subset U\}$$

is also open.

Choose a smooth function ρ on $\mathbb{C}$ with the following properties:

i. $\mathrm{Supp}(\rho) \subset D(0,\, 1)$,

ii. ρ is invariant under rotations, i.e., $\rho(z) = \rho(|z|)$ for every $z \in \mathbb{C}$,

iii. $\iint_{\mathbb{C}} \rho(z)\, dx\, dy = 1$.

Set

$$\rho(z) = \frac{1}{\epsilon^2}\rho\left(\frac{z}{\epsilon}\right).$$

Then $\mathrm{Supp}(\rho_\epsilon) \subset D(0,\, \epsilon)$ and

$$\iint\limits_{\mathbb{C}} \rho_\epsilon(z)\, dx\, dy = 1.$$

Given a continuous function $f : U \to \mathbb{C}$, define a new function

$$(\mathrm{sm}_\epsilon f)(z) := \iint\limits_{U} \rho_\epsilon(z-\zeta) f(\zeta)\, d\xi\, d\eta = \iint\limits_{|\zeta|<\epsilon} \rho_\epsilon(\zeta) f(z+\zeta)\, d\xi\, d\eta, \quad z \in U^{(\epsilon)}.$$

Clearly $\mathrm{sm}_\epsilon f$ is a smooth function on $U^{(\epsilon)}$, since one can differentiate under the integral. The function $\mathrm{sm}_\epsilon f$ is called a *smoothing* of f.

Proposition 1.1.6. *Suppose $U \subset \mathbb{C}$ is open, $f \in C^\infty(U)$ and $\epsilon > 0$. The followings are true.*

i. For every α, $D^\alpha(\mathrm{sm}_\epsilon f) = \mathrm{sm}_\epsilon(D^\alpha f)$.

ii. If $z \in U^{(\epsilon)}$ and f is harmonic on $D(z,\, \epsilon)$, then $(\mathrm{sm}_\epsilon f)(z) = f(z)$.

Proof

i. is trivial. *ii.* If f is harmonic on $D(z,\, \epsilon)$, then for every $r \in [0,\, \epsilon)$ it satisfies the Mean Value Principle

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta.$$

Thus

$$\begin{aligned}(\mathrm{sm}_\epsilon f)(z) &= \iint\limits_{|\zeta|<\epsilon} \rho_\epsilon(\zeta) f(z+\zeta)\, d\zeta\, d\eta = \int_0^{2\pi}\int_0^{\epsilon} \rho_\epsilon(r) f(z+re^{i\theta}) r\, dr\, d\theta \\ &= 2\pi f(z)\int_0^{\epsilon} \rho_\epsilon(r) r\, dr = f(z).\end{aligned}$$

■

Theorem 1.1.7 (Weyl's Lemma)**.** *Suppose U is an open set in $\mathbb{C}$ and T is a distribution on U with $\Delta T = 0$. Then T is a smooth function. In other words, if $T[\Delta\varphi] = 0$ for every $\varphi \in C_0^\infty(U)$, then there exists a function $h \in C^\infty(U)$ with $\Delta h = 0$ and*

$$T[f] = \iint\limits_{U} h(z) f(z)\, dz\, dy, \quad \textit{for every } f \in C_0^\infty(U).$$

Proof

Suppose $\epsilon > 0$ is arbitrary. For $z \in U^{(\epsilon)}$ the function $\zeta \mapsto \rho_\epsilon(\zeta - z)$ has compact support in U. Hence

$$h_\epsilon(z) \coloneqq T_\zeta[\rho_\epsilon(\zeta - z)]$$

is defined. By (1.1.3) the function h_ϵ is smooth on $U^{(\epsilon)}$. For every function $f \in C_0^\infty(U^{(\epsilon)})$, the function $\mathrm{sm}_\epsilon f$ has compact support in $U^{(\epsilon)}$ and by (1.1.4) one has

$$T[\mathrm{sm}_\epsilon f] = T\left[\iint\limits_{U^{(\epsilon)}} \rho_\epsilon(\zeta - z) f(z)\, dx\, dy\right] = \iint\limits_{U^{(\epsilon)}} h_\epsilon(z) f(z)\, dx\, dy.$$

By the Doubeault Lemma there exists a function $\psi \in C^\infty(\mathbb{C})$ with $\Delta\psi = f$. The function ψ is harmonic on $V \coloneqq \mathbb{C} \setminus \mathrm{Supp}(f)$. Thus by (1.1.6 *ii.*)

$$\psi = \mathrm{sm}_\epsilon \psi \quad \text{on } V^{(\epsilon)}.$$

Hence $\varphi \coloneqq \psi - \mathrm{sm}_\epsilon\psi$ has compact support in U and by (1.1.6 *i.*) satisfies

$$\Delta\varphi = \Delta(\psi - \mathrm{sm}_\epsilon\psi) = \Delta\psi - \mathrm{sm}_\epsilon\Delta\psi = f - \mathrm{sm}_\epsilon f.$$

Since $\Delta T = 0$, one has

$$T[f] = T[\mathrm{sm}_\epsilon f] = \iint\limits_{U^{(\epsilon)}} h_\epsilon(z) f(z)\, dx\, dy. = T_{h_\epsilon}[f]$$

operates on $f \in C_0^\infty(U^{(\epsilon)})$. $\Delta h_\epsilon = 0$ is from $\Delta T = \Delta T_{h_\epsilon} = T_{\Delta h_\epsilon} = 0$.
For $\epsilon' > \epsilon$, the distributions T_{h_ϵ} and $T_{h_{\epsilon'}}$ operate identically on the space $C_0^\infty(U^{(\epsilon')})$. Thus, $h_\epsilon|_{U^{(\epsilon')}} = h_{\epsilon'}$ and

$$h(z) \coloneqq h_\epsilon(z), \quad \text{for } z \in U^{(\epsilon)}$$

is a well-defined smooth function on U which we want.

■

Corollary 1.1.8. *Suppose T is a distribution on the open set $U \subset \mathbb{C}$ with $(\partial T/\partial \overline{z}) = 0$. Then T is a holomorphic function on U.*

Proof

Since

$$\Delta T = 4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \overline{z}} T\right) = 0,$$

one can see that $T \in C^\infty(U)$. Because $(\partial T/\partial \overline{z}) = 0$, T is holomorphic.

■

1.2 Doubeault's Lemma on Relatively Compact Open Subset

Lemma 1.2.1. *Suppose Y is a relatively compact open subset of a non-compact Riemann surface X. Then there exists a holomorphic function $f : Y \to \mathbb{C}$ which is not constant on any connected component of Y.*

Proof

Choose a domain Y' such that $Y \Subset Y' \Subset X$ and a point $a \in Y' \setminus Y$. Since X is non-compact and connected, $Y' \setminus Y$ is not empty. Now apply (2.2.6) to Y' and a.

■

Lemma 1.2.2. *Suppose X is a non-compact Riemann surface and $Y \Subset Y \subset X$ are open subsets. Then*

$$L \coloneqq \mathrm{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0.$$

Proof

By (2.1.3) we already know that $\dim L = k < \infty$. Choose cohomology classes $\xi_1, \ldots, \xi_k \in H^1(Y', \mathcal{O})$ such that their restrictions to Y span the vector space L. According to (1.2.1) we may choose a function $f \in \mathcal{O}(Y')$ which is not constant on any connected component of Y'. Since $H^1(Y', \mathcal{O})$ is in a natural way a module over $\mathcal{O}(Y')$, the products $f\xi_j \in H^1(Y', \mathcal{O})$ are defined. By the choice of the ξ_j there exists constants $c_{jl} \in \mathbb{C}$ such that

$$f\xi_j = \sum_{l=1}^{k} c_{jl}\xi_l \quad \text{on } Y \quad \text{for } j = 1, \ldots, k.$$

Set

$$F \coloneqq \det(f\delta_{jl} - c_{jl}).$$

Then F is a holomorphic function on Y' which is not identically zero on any connected component of Y'. It follows that

$$F\xi_j|_Y = \xi_j - \sum_{l=1}^{k} c_{jl}\xi_l = 0 \quad \text{for } j = 1, \ldots, k.$$

An arbitrary cohomology class $\zeta \in H^1(Y', \mathcal{O})$ can be represented by a cocycle $(f_{jl}) \in Z^1(\mathfrak{U}, \mathcal{O})$, where $\mathfrak{U} = \{U_j\}$ is an open covering of Y' such that each zero of F contained in at most one U_j, Thus for $j \neq l$ one has $F|_{U_j \cap U_l} \in \mathcal{O}^*(U_j \cap U_l)$. Hence there exists a cocycle $(g_{jl}) \in Z^1(\mathfrak{U}, \mathcal{O})$ such that $f_{jl} = Fg_{jl}$. Let $\xi \in H^1(Y', \mathcal{O})$ be the cohomology class of (g_{jl}). Then $\zeta = F\xi$. Hence on gets $\zeta|_Y = F\xi|_Y = 0$.

■

Theorem 1.2.3. *Suppose X is a non-compact Riemann surface and $Y \Subset Y' \subset X$ are open subsets. Then for every differential form $\omega \in \bigwedge^{0,1}(Y')$ there exists a function $f \in C^\infty(Y)$ such that $\overline{\partial} f = \omega|_Y$.*

Proof

The problem has a solution locally. Thus there exist an open covering $\mathfrak{U} = \{U_j\}$ of Y' and functions $f_j \in C^\infty(U_j)$ such that $\overline{\partial} f_i = \omega|_{U_j}$. The differences $f_j - f_l$ are holomorphic on $U_j \cap U_l$ and define a cocycle in $Z^1(\mathfrak{U}, \mathcal{O})$. By (1.2.2) this cocycle is cohomologous to zero on Y and thus there exist holomorphic functions $g_j \in \mathcal{O}(U_j \cap Y)$ such that

$$f_j - f_l = g_j - g_l \quad \text{on } U_j \cap U_l \cap Y.$$

Hence there exists a function $f \in C^\infty(Y)$ such that

$$f = f_j - g_j \quad \text{on } U_j \cap Y.$$

But then the function f satisfies the equation $\overline{\partial} f = \omega|_Y$.

■

1.3 Proof of the Runge Approximation Theorem

Definition 1.3.1. Suppose X is a Riemann surface. For any subset $Y \subset X$ let $h(Y)$ denote the union of Y with all the relatively compact connected components of $X \setminus Y$. An open subset $Y \subset X$ is called *Runge* if $Y = h(Y)$, i.e., if none of the connected components of Y is compact.

There are some basic properties of the hull operator.

Proposition 1.3.2. *Suppose Y and Z are subsets of a Riemann surface X, the followings are true.*

i. $h(h(Y)) = h(Y)$.

ii. $Y \subset Z \Rightarrow h(Y) \subset h(Z)$.

iii. Y *is closed* $\Rightarrow h(Y)$ *is closed.*

iv. Y *is compact* $\Rightarrow h(Y)$ *is compact.*

Lemma 1.3.3. *Suppose K_1 and K_2 are compact subsets of a Riemann surface X with $K_1 \subset \operatorname{int}(K_2)$ and $K_2 = h(K_2)$. Then there exists an open subset Y of X which is Runge and satisfies $K_1 \subset Y \subset K_2$. Moreover one may choose Y so that its boundary is regular.*

Proof

Given $x \in \partial K_2$ there is a coordinate neighborhood U of x which does not meet K_1 In U choose a compact disk D containing x in its interior. Then finitely many such disks, say $D_1, \ldots, D_k$, cover X. Set

$$Y \coloneqq K_2 \setminus (D_1 \cup \cdots \cup D_k).$$

Then Y is open and $K_1 \subset Y \subset K_2$. Let C_j, $j \in J$ be the connected components of $X \setminus K_2$. By assumption they are not relatively compact. Every D_k is connected and meets at least one C_j. Hence no connected component of $X \setminus Y$ is relatively compact, i.e., $Y = h(Y)$. Finally, it's easy to verify that all the boundary points of Y are regular by definition.

■

Theorem 1.3.4. *Suppose X is a non-compact Riemann surface and $Y \Subset X$ a relatively compact Runge domain. Then there exists a sequence $Y_0 \coloneqq Y \Subset Y_1 \Subset \ldots$ of relatively compact Runge domains with $\bigcup Y_j = X$ and so that every Y_j has a regular boundary.*

Proof

Since X is second countable and locally Hausdorff, there exists a compact exhaustion $K_1 \subset K_2 \subset \ldots$ of X. Fix a exhaustion, then we can construct Y_j inductively. Let $K \supset Y_{j-1}$ be a compact subset of X, choose a compact subset $K' \subset X$ such that $K \cup K_j \subset \operatorname{int}(K')$. Set

$$A = K \cup K_j, \quad B = h(K').$$

Then A and B are compact sets satisfying $A \subset \operatorname{int}(B)$ and $B = h(B)$. By (1.3.3) we can take Y_j to be a Runge domain with $A \subset Y_j \subset B$ and so that its boundary is regular. One can easily check $Y_{j-1} \Subset Y_j$ and $K_j \subset Y_j$. Consequently, $\bigcup Y_j \supset \bigcup K_j = X$, and thus the sequence satisfies our conditions.

■

Definition 1.3.5. Suppose X is a Riemann surface and $Y \subset X$ is an open subset. We can choose a countable family of compact sets $K_j \subset Y$, $j \in J$, with $\bigcup \operatorname{int}(K_j) = Y$ and such that K_j is contained in some coordinate neighborhood (U_j, z_j). For every j and $\alpha = (\alpha_1, \alpha_2)$ define a semi-norm

$$p_{j\alpha}(f) \coloneqq \sup_{z \in K_j} |D_j^\alpha f(z)|, \quad \text{where } D_j^\alpha = \left(\frac{\partial}{\partial x_j}\right)^{\alpha_1} \left(\frac{\partial}{\partial y_j}\right)^{\alpha_2}$$

is the appropriate differential operator relative to the coordinate z_j. These countably many semi-norms define a topology on $C^\infty(Y)$. A neighborhood basis of zero is given by finite intersections of sets of the form

$$\mathscr{U}(p_{j\alpha}, \epsilon) \coloneqq \{f \in C^\infty(Y) : p_{j\alpha}(f) < \epsilon\}, \quad \epsilon > 0.$$

Then the convergence with respect to this topology means uniform convergence in any order of derivatives on every K_j. This topology is called the *Fréchet structure* on $C^\infty(Y)$.

One can easily check that this topology is independent of the choice of K_j and (U_j, z_j). Let $C^\infty(Y)'$ have the topology given by another choice of countable compact family and local coordinates, then the identity map is sequentially continuous, because they define the same convergence. Since they are first countable, sequential continuity implies continuity. Thus, the identity map is continuous, and hence homeomorphic by the symmetricity. On the vector subspace $\mathcal{O}(Y) \subset C^\infty(Y)$ the induced topology coincides with the topology of uniform convergence on compact subsets. Analogously one can introduce the Fréchet structure on the vector space $\bigwedge^{0,1}(Y)$ of $(0, 1)$-forms on Y with smooth coefficients. An element $\omega \in \bigwedge^{0,1}(Y)$ may be written $\omega = f_j d\bar{z}_j$ on U_j. Set

$$p_{j\alpha}(\omega) = p_{j\alpha}(f_j).$$

Then the topology is obtained as above from the semi-norms $p_{j\alpha}$.

Lemma 1.3.6. *Suppose Y is an open subset of a Riemann surface X. Then every continuous linear map $T : C^\infty(Y) \to \mathbb{C}$ has compact support, i.e., there exists a compact subset $K \subset Y$ such that*

$$T[f] = 0 \textit{ for every } f \in C^\infty(Y) \textit{ with } \operatorname{Supp}(f) \subset Y \setminus K.$$

An analogous result is also true for $\bigwedge^{0,1}(Y)$.

Proof

Since T is continuous, there exists a neighborhood U of zero in $C^\infty(Y)$ such that $|T[f]| < 1$ for every $f \in U$. By the definition of the topology on $C^\infty(Y)$ there exist elements $j_1, \ldots, j_m$, $\alpha_1, \ldots, \alpha_m$ and $\epsilon > 0$, such that

$$\mathscr{U}(p_{j_1\alpha_1}, \epsilon) \cap \cdots \cap \mathscr{U}(p_{j_m\alpha_m}, \epsilon) \subset U.$$

Let $K \coloneqq K_{j_1} \cup \cdots \cup K_{j_m}$. If $f \in C^\infty(Y)$ with $\operatorname{Supp}(f) \subset Y \setminus K$, then for arbitrary $\lambda > 0$,

$$p_{j_1\alpha_1}(\lambda f) = \cdots = p_{j_m\alpha_m}(\lambda f) = 0.$$

Thus $\lambda f \in U$ and $|T[f]| = |T[\lambda f]|/\lambda < 1/\lambda$. This is possible only if $T[f] = 0$.

■

Lemma 1.3.7. *Suppose Z is an open subset of a Riemann surface X and $S : \bigwedge^{0,1}(X) \to \mathbb{C}$ is a continuous linear mapping with $S[\overline{\partial} g] = 0$ for every $g \in C^\infty(Y)$ with $\operatorname{Supp}(g) \Subset Z$. Then there exists a holomorphic 1-form $\sigma \in \Omega(X)$ such that*

$$S[\omega] = \iint_Z \sigma \wedge \omega$$

for every $\omega \in \bigwedge^{0,1}(X)$ with $\operatorname{Supp}(\omega) \Subset Z$.

Proof

Suppose $z : U \to V \subset \mathbb{C}$ is a chart on X which lies in Z. Identify U with V. For $\varphi \in C_0^\infty(U)$ denote by $\tilde{\varphi}$ the 1-form which equals $\varphi \, d\overline{z}$ on U and zero on $X \setminus U$. Then the mapping

$$S_U : C_0^\infty(U) \to \mathbb{C}, \quad \varphi \mapsto S[\tilde{\varphi}]$$

is a distribution on U with $\partial S_U / \partial \overline{z} = 0$. Hence by corollary of Weyl's Lemma (1.1.8) there exists a unique holomorphic function $h_U \in \mathcal{O}(U)$ with

$$S[\tilde{\varphi}] = \iint_U h_U(z)\varphi(z) \, dz \wedge d\overline{z} \quad \text{for every } \varphi \in C_0^\infty(U).$$

Setting $\sigma_U := h_U \, dz$, we get

$$S[\omega] = \iint_U \sigma_U \wedge \omega$$

for every $\omega \in \bigwedge^{0,1}(U)$ with $\operatorname{Supp}(\omega) \Subset U$. Now carry out the same construction with respect to another chart U', then

$$\iint_U \sigma_U \wedge \omega = \iint_{U'} \sigma_{U'} \wedge \omega$$

for every $\omega \in \bigwedge^{0,1}(X)$ with $\operatorname{Supp}(\omega) \Subset U \cap U'$. This implies $\sigma_U = \sigma_{U'}$ on $U \cap U'$. Thus the σ_U piece together to give a 1-form $\sigma \in \Omega(Z)$. If $\omega \in \bigwedge^{0,1}(X)$ is an arbitrary 1-form with $\operatorname{Supp}(\omega) \Subset Z$, then using a partition of unitity one can write $\omega = \omega_1 + \cdots + \omega_m$, $\operatorname{Supp}(\omega_j) \Subset U_j$ with each U_j is in a local chart. Thus

$$S[\omega] = \sum_{j=1}^n S[\omega_j] = \sum_{j=1}^n \iint_{U_j} \sigma_{U_j} \wedge \omega_j = \iint_Z \sigma \wedge \omega.$$

■

Lemma 1.3.8. *Suppose Y is a relatively compact open Runge subset of a non-compact Riemann surface X. Then for every open subset Y' with $Y \Subset Y' \subset X$ the image of the restriction map $\mathcal{O}(Y') \to \mathcal{O}(Y)$ is dense, where the topology is uniform convergence on compact subsets.*

Proof

Denote by $\beta : C^\infty(Y') \to C^\infty(Y)$ the restriction map. By the Hahn-Banach theorem, if there is some $f \in \mathcal{O}(Y)$ not in the closure of $\beta(\mathcal{O}(Y'))$, then we can construct a continuous linear functional T so that $T|_{\beta(\mathcal{O}(Y'))} = 0$ and $T(f) = 1$. Thus, it suffices to show that every continuous functional on $C^\infty(Y)$ which vanishes on $\beta(\mathcal{O}(Y'))$ also vanishes on $\mathcal{O}(Y)$.

To prove this, define a linear mapping

$$S : \bigwedge^{0,1}(X) \to \mathbb{C}$$

in the following way. By (1.2.3) given $\omega \in \bigwedge^{0,1}(X)$ there exists a function $f \in C^\infty(Y')$ with $\overline{\partial} f = \omega|_{Y'}$. Then set

$$S[\omega] := T[f|_Y].$$

This definition is independent of choice of the function f. For, if $\overline{\partial} g = \omega|_{Y'}$ then $f - g \in \mathcal{O}(Y')$ and thus by assumption $T[f|_Y] = T[g|_Y]$, We will now show that S is continuous. Consider the vector space

$$V := \{(\omega,\ f) \in \bigwedge^{0,1}(X) \times C^\infty(Y') : \overline{\partial} f = \omega|_{Y'}\}.$$

Since $\overline{\partial} : C^\infty(Y') \to \bigwedge^{0,1}(Y')$ is continuous, by the closed graph theorem V is a closed vector subspace of $\bigwedge^{0,1}(X) \times C^\infty(Y')$ and thus a Fréchet space. Now the projection π_1 is surjective and thus is open by the Theorem of Banach. Also the mapping $\beta \circ \pi_2$ is continuous, for every open subset $U \in \mathbb{C}$, one can see that

$$S^{-1}(U) = \pi_1((T \circ \beta \circ \pi_2)^{-1}(U)) \text{ is open.}$$

Therefore S is a continuous map.

By (1.3.6) there exists a compact subset $K \subset Y$ and a compact subset $L \subset X$ with

(a) $T[f] = 0$ for every $f \in C^\infty(Y)$ with $\mathrm{Supp}(f) \subset Y \setminus K$

(b) $S[\omega] = 0$ for every $\omega \in \bigwedge^{0,1}(Y)$ with $\mathrm{Supp}(\omega) \subset X \setminus L$

If $g \in C^\infty(X)$ is a function with $\mathrm{Supp}(g) \Subset X \setminus K$, then $S[\overline{\partial} g] = T[g|_Y] = 0$. Thus by (1.3.7) there exists a holomorphic 1-form $\sigma \in \Omega(X \setminus K)$ such that

$$S[\omega] = \iint\limits_{X \setminus K} \sigma \wedge \omega$$

for every $\omega \in \bigwedge^{0,1}(X)$ with $\mathrm{Supp}(\omega) \Subset X \setminus K$. Because of (b) it must be the case that $\sigma|_{X \setminus L} = 0$. Every connected component of $X \setminus h(K)$ is not relatively compact by definition, in particular, is not contained in L and hence meets $X \setminus L$. Thus by the identity theorem $\sigma|_{X \setminus h(K)} = 0$, i.e.

$$S[\omega] = 0 \text{ for every } \omega \in \bigwedge^{0,1}(X) \text{ with } \mathrm{Supp}(\omega) \Subset X \setminus h(K).$$

Now suppose $f \in \mathcal{O}(Y)$. We have to show $T[f] = 0$. Since Y is Runge, $h(K) \subset Y$. Hence there is a function $g \in C^\infty(X)$ with $f = g$in a neighborhood of $h(K)$ abd $\mathrm{Supp}(g) \Subset Y$. Then $T[f] = T[g|_Y] = S[\overline{\partial} g]$ by (a), and $\mathrm{Supp}(\overline{\partial} g) \Subset X \setminus h(K)$ since g is holomorphic on a neighborhood of $h(K)$. Thus $T[f] = S[\overline{\partial} g] = 0$ for every $f \in \mathcal{O}(Y)$.

■

Theorem 1.3.9 (The Runge Approximation Theorem)**.** *Suppose X is a non-compact Riemann surface and $Y \subset X$ is a Runge domain. Then every holomorphic function on Y can be approximated uniformly on every compact subset of Y by holomorphic functions on X.*

Proof

It suffices to consider the case when Y is relatively compact in X. Suppose $f \in \mathcal{O}(Y)$, a compact subset $K \subset Y$ and $\epsilon > 0$ are given. By (1.3.4) there exists an exhaustion $Y_1 \Subset Y_2 \Subset \cdots$ of X by Runge domains with $Y_0 := Y \Subset Y_1$. By (1.3.8) there is a holomorphic function $f_1 \in \mathcal{O}(Y_1)$ with

$$||f_1 - f||_K < 2^{-1}\epsilon,$$

where $||\cdot||_K$ denotes the supremum norm on K.

Now using (1.3.8) and induction one gets a sequence of functions $f_n \in \mathcal{O}(Y_n)$ with

$$||f_n - f_{n-1}||_{\overline{Y}_{n-2}} < 2^{-n}\epsilon \quad \text{for every } n \geq 2.$$

For every $n \geq 0$ the sequence $(f_j)_{j>n}$ converges uniformly on Y_n. Hence there exists a function $F \in \mathcal{O}(X)$, holomorphic on all of X, which on each Y_n is the limit of the sequence $(f_j)_{j>n}$. Thus, by construction, $||F - f||_K < \epsilon$.

■

2 Applications

2.1 Finiteness Theorem

Theorem 2.1.1 (Montel)**.** *Suppose U is an open subset of $\mathbb{C}$, a family of holomorphic functions $\mathcal{F} \subset \mathcal{O}(U)$ is normal if and only if it is locally uniformly bounded.*

Note that under the Fréchet structure, normality is equivalent to compactness.

Theorem 2.1.2 (Schwartz)**.** *Suppose E and F are Fréchet spaces and φ, $\psi : E \to F$ are continuous linear mappings such that φ is surjective and ψ is compact. Then the image of the mapping $\varphi - \psi$ has finite codimension in F.*

Theorem 2.1.3. *Suppose Y is a relatively compact open subset of a Riemann surface X and E is a holomorphic vector bundle on X. Then $H^1(Y, \mathcal{O}_E)$ is finite dimensional.*

Proof

There is an open set Y' with $Y \Subset Y' \Subset X$ and open sets $V_j \Subset U_j$, $j = 1, \ldots, r$, in X with the following properties:

i. $\bigcup_{j=1}^r V_j = Y$, $\bigcup_{j=1}^r U_j = Y'$.

ii. Every U_j is biholomorphic to an open subset of C.

iii. On every U_j there is a holomorphic linear chart $h_j : E_{U_j} \to U_j \times \mathbb{C}^n$.

Now $\mathfrak{U} := \{U_j\}$ and $\mathfrak{V} := \{V_j\}$ are Leray coverings of Y' resp. Y for the sheaf $\mathcal{O}_E$. We claim that the restriction mapping $H^1(\mathfrak{U}, \mathcal{O}_E) \to H^1(\mathfrak{V}, \mathcal{O}_E)$ is surjective. To show this, set

$$Y_k := Y \cup \bigcup_{j=1}^{k} U_j.$$

Clearly it suffices to show that the mappings

$$H^1(Y_k,\, \mathcal{O}_E) \to H^1(Y_{k-1},\, \mathcal{O}_E)$$

for $k = 1, \ldots, r$ are surjective. Fix k and let

$$\begin{aligned} W_j &:= U_j \cap Y_{k-1} && \text{for } j = 1,\, \ldots,\, r, \\ W_j' &:= W_j && \text{for } j \neq k \quad \text{and } W_k' := U_k. \end{aligned}$$

Then $\mathfrak{W} = \{W_j\}$ and $\mathfrak{W}' = \{W_j'\}$ are Leray coverings of Y_{k-1} resp. Y_k. Since $W_j \cap W_j = W_j' \cap W_l'$ for every $j \neq l$, one has $Z^1(\mathfrak{W},\, \mathcal{O}_E) = Z^1(\mathfrak{W}',\, \mathcal{O}_E)$. Thus $H^1(\mathfrak{W}',\, \mathcal{O}_E) \to H^1(\mathfrak{W},\, \mathcal{O}_E)$ is surjective, and our claim is true. This implies that the mapping

$$\varphi : C^0(\mathfrak{B},\, \mathcal{O}_E) \times Z^1(\mathfrak{U},\, \mathcal{O}_E) \to (\mathfrak{B},\, \mathcal{O}_E)$$

$$(\eta,\, \xi) \mapsto \delta(\eta) + \beta(\xi)$$

is surjective, where β is the restriction map. One can make the space $Z^1(\mathfrak{U},\, \mathcal{O}_E)$ into Fréchet space in the following way. First $\mathcal{O}_{\mathcal{E}}(U_j \cap U_k) \simeq \mathcal{O}(U_j \cap U_k)^n$ with topology of uniformly convergence on compact subsets is a Fréchet space. Thus so is $C^1(\mathfrak{U},\, \mathcal{O}_E) = \prod_{j,\,k} \mathcal{O}(U_j \cap U_k)$ with the product topology. This point of view can also be applied to $C^k(\mathfrak{U},\, \mathcal{O}_E)$. It's easy to see that δ is continuous and $Z^1(\mathfrak{U},\, \mathcal{O}_E)$ is a closed subset of $C^1(\mathfrak{U},\, \mathcal{O}_E)$. Similarly, we can define the Fréchet structures on $Z^1(\mathfrak{B},\, \mathcal{O}_E)$ and $C^0(\mathfrak{U},\, \mathcal{O}_E)$. With respect to these topologies the mappings $\delta : C^0(\mathfrak{B},\, \mathcal{O}_E) \to Z^1(\mathfrak{B},\, \mathcal{O}_E)$ and β are continuous. Then (2.1.1) implies that β is a compact operator. Hence

$$\psi : C^0(\mathfrak{B},\, \mathcal{O}_E) \times Z^1(\mathfrak{U},\, \mathcal{O}_E) \to Z^1(\mathfrak{B},\, \mathcal{O}_E)$$

$$(\eta,\, \xi) \mapsto \beta(\xi)$$

is also compact. By (2.1.2) the mapping $\varphi - \psi$ has finite codimensional image. However, one can see that its image is precisely $B^1(\mathfrak{B},\, \mathcal{O}_E)$. Thus $H^1(Y,\, \mathcal{O}_E) \simeq H^1(\mathfrak{B},\, \mathcal{O}_E)$ is finite dimensional.

■

2.2 Triviality of Holomorphic Vector Bundles

Theorem 2.2.1. *Suppose X is a non-compact Riemann surface. Then given a 1-form $\omega \in \bigwedge^{0,1}(X)$ there exists a function $f \in C^\infty(X)$ with $\overline{\partial} f = \omega$.*

Proof

Suppose $Y_0 \Subset Y_1 \Subset \cdots$ is an exhaustion of X by Runge domains (1.3.4). By induction on n we will construct functions $f_n \in C^\infty(Y_n)$ such that

i. $\overline{\partial} f_n = \omega|_{Y_n}$,

ii. $||f_{n+1} - f_n||_{Y_n} \leq 2^{-n}$.

To begin choose any function $f_0 \in C^\infty(Y_0)$ which is a solution of the differential equation $\overline{\partial} f_0 = \omega|_{Y_0}$ (1.2.3). Now suppose f_n has been constructed. There exists $g_{n+1} \in C^\infty(Y_{n+1})$ with $\overline{\partial} g_{n+1} = \omega|_{Y_{n+!}}$. On Y_n one has $\overline{\partial} g_{n+1} = \overline{\partial} f_n$ and thus $g_{n+1} - f_n$ is holomorphic on Y_n. By the Runge Approximation Theorem (1.3.9) there exists $h \in \mathcal{O}(Y_{n+1})$ such that

$$||(g_{n+1} - f_n) - h||_{Y_n} \leq 2^{-n}.$$

Set $f_{n+1} = g_{n+1} - h$, it satisfies the desired properties. Now follows that the functions f_n converge to a solution $f \in C^\infty(X)$ of the differential equation $\overline{\partial} f = \omega$.

■

Corollary 2.2.2. *Suppose X is a non-compact Riemann surface. Then*

$$H^1(X, \mathcal{O}) = 0.$$

Proof

By the Doubeault Theorem one has $H^1(X, \mathcal{O}) \simeq \bigwedge^{0,1}(X)/\mathrm{Im}\,\overline{\partial}$. But by (2.2.1) $\overline{\partial}$ is surjective.

■

Definition 2.2.3. Suppose D is a divisor of a Riemann surface X, let

$$X_D \coloneqq \{x \in X : D(x) \geq 0\}.$$

By a *weak solution* of D we mean a function $f \in C^\infty(X_D)$ with the following property. For every point $x \in X$ there exists a coordinate neighborhood (U, z) with $z(x) = 0$ and a function $\psi \in C^\infty(U)$ with $\psi(x) \neq 0$, such that

$$f = \psi z^k \quad \text{on } U \cap X_D, \quad \text{where } k = D(x).$$

Lemma 2.2.4. *Suppose X is a Riemann surface, $c : [0, 1] \to X$ is a curve and U is a relatively compact open neighborhood of $c([0, 1])$. Then there exists a weak solution φ of the divisor ∂c with $f|_{X \setminus U} = 1$, such that for every closed 1-form ω one has*

$$\int_c \omega = \frac{1}{2\pi i} \iint_X \frac{d\varphi}{\varphi} \wedge \omega.$$

Lemma 2.2.5. *Every divisor D on a non-compact Riemann surface X has a weak solution.*

Proof

Choose a sequence $K_1, K_2, \ldots$ of compact subsets of X, using a similar method to (1.3.4) we can assume them to have the following properties:

i. $K_j = h(K_j)$ for every $j \geq 1$,

ii. $K_j \subset \mathrm{int}(K_{j+1})$ for every $j \geq 1$,

iii. $\bigcup K_j = X$.

We claim that given $a_0 \in X \setminus K_j$, there exists a weak solution φ of the divisor (a_0) with $\varphi|_{K_j} = 1$.

In order to prove the claim, note that since $K_j = h(K_j)$, the point lies in a connected component U of $X \setminus K_j$ which is not relatively compact. Hence there exists a point $a_1 \in U \setminus K_{j+1}$ and a curve c_0 in U with $\partial c_0 = (a_0) - (a_1)$. By (2.2.4) there is a weal solution φ_0 of ∂c_0 with $\varphi_0|_{K_j} = 1$. Repeating the construction gives a sequence of points $a_k \in X \setminus K_{j+k}$, curves $c_k$in $X \setminus K_{j+k}$ with $\partial c_k = (a_k) - (a_{k+1})$ and weak solutions φ_k of ∂c_k with $\varphi|_{K_{j+k}} = 1$. The infinite product

$$\varphi := \prod_{k=0}^{\infty} \varphi_k$$

converges, since on every K_j there are only finitely many factors which are not identically 1. Now φ is the desired weak solution of the divisor (a_0).

Suppose D is an arbitrary divisor on X. For each j set

$$D_j(x) := \begin{cases} D(x), & \text{if } x \in K_j \setminus K_{j-1}, \\ 0, & \text{otherwise,} \end{cases}$$

where $K_0 := \varnothing$. Then $D = \sum D_k$. Since D_k is non-zero at a finite number of points, there is a weak solution ψ_k of the divisor D_k with $\psi_k|_{K_k} = 1$. The product

$$\psi := \prod_{k-1}^{\infty} \psi_k$$

is thus a weak solution of D.

■

Theorem 2.2.6 (Weierstrass)**.** *On a non-compact Riemann surface X every divisor D is a divisor of a meromorphic function $f \in \mathscr{M}^*(X)$.*

Proof

Since the problem has a solution locally, there exists an open covering $\mathfrak{U} = (U_i)_{i \in I}$ of X and meromorphic functions $f_i \in \mathscr{M}^*(U_i)$ such that the divisor of f_i coincides with $D|_{U_i}$. We may assume that all the U_i are simply connected. On the intersection $U_i \cap U_j$ the functions f_i and f_j have the same zeros and poles, i.e.,

$$\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j) \quad \text{for every } i,\, j \in I.$$

Now suppose ψ is a weak solution of D (2.2.5). Then $\psi = \psi_i f_i$ on U_i, where the function $\psi_i \in C^\infty(U_i)$ has no zeros. Since U_i is simply connected, there exists a function $\varphi_i \in C^\infty(U_i)$ with $\psi_i = e^{\varphi_i}$. Then on $U_i \cap U_j$ one has

$$e^{\varphi_j - \varphi_i} = \frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$$

and thus $\varphi_{ij} := \varphi_j - \varphi_i \in \mathcal{O}(U_i \cap U_j)$. One can see that the family (φ_{ij}) is a cocycle in $Z^1(\mathfrak{U}, \mathcal{O})$. Because $H^1(X, \mathcal{O}) = 0$ (2.2.2), this cocycle splits. Thus there exists holomorphic functions $g_i \in \mathcal{O}(U_i)$ with

$$\varphi_{ij} = \varphi_j - \varphi_i = g_j - g_i, \quad e^{g_j} f_j = e^{g_i} f_i \quad \text{on } U_i \cap U_j$$

for every $i, j \in I$. Hence there exists a global meromorphic function f with $f = e^{g_i} f_i$ on U_i for every $i \in I$. One has $(f) = D$.

■

Lemma 2.2.7. *Suppose E is a holomorphic vector bundle on a Riemann surface X and Y is a relatively compact open subset of X. Then given any $a \in Y$ there exists a meromorphic section of E over Y which has a pole at a and is holomorphic on $Y \setminus \{a\}$.*

PROOF

Suppose (U_1, z_0) is a coordinate neighborhood of a with $z_0(a) = 0$ and local trivialization $h : E|_{U_1} \simeq U_1 \times \mathbb{C}^n$. Set $U_2 := X \setminus \{a\}$. Then $\mathfrak{U} = \{U_1, U_2\}$ a open covering of X. The functions $z^{-\overline{\alpha}} := (z_1^{-\alpha_1}, \dots, z_n^{-\alpha_1})$, where $h = (z_0, z)$, are holomorphic on $U_1 \cap U_2 = U_1 \setminus \{a\}$ and represent cocycles $\zeta_{\overline{\alpha}} \in Z^1(\mathfrak{U}, \mathcal{O}_E)$. Let $k := \dim H^1(Y, \mathcal{O}_E) < \infty$ (2.1.3), take $\overline{\alpha}_1, \dots, \overline{\alpha}_{k+1} \in \mathbb{N}^n$ different, the cocycles

$$\zeta_{\overline{\alpha}_j}|_Y \in Z^1(\mathfrak{U} \cap Y, \mathcal{O}_E), \quad 1 \le j \le k+1,$$

are linearly dependent modulo the coboundaries. Thus there exist complex numbers $c_1, \dots, c_{k+1}$, not all zero, and a cochain $\eta = (f_1, f_2) \in C^0(\mathfrak{U} \cap Y, \mathcal{O}_E)$ such that

$$c_1 \zeta_{\overline{\alpha}_1} + \cdots + c_{k+1} \zeta_{\overline{\alpha}_{k+1}} = \delta\eta, \quad \text{with respect to } \mathfrak{U} \cap Y,$$

i.e.,

$$\sum_{j=1}^{k+1} c_j z^{-\overline{\alpha}_j} = f_2 - f_1 \quad \text{on } U_2 \cap U_2 \cap Y.$$

Hence there is a function $f \in \mathscr{M}(Y)$, which coincides with

$$f_1 + \sum_{j=1}^{k+1} c_j z^{-\overline{\alpha}_j}$$

on $U_1 \cap Y$ and which is equal to f_2 on $U_2 \cap Y = Y \setminus \{a\}$. This is the desired function.

■

Proposition 2.2.8. *Suppose E is a holomorphic vector bundle of rank n on a Riemann surface X. Let $\mathfrak{U} = \{U_j\}$ be an open covering of X, $h_j : E_{U_j} \to U_J \times \mathbb{C}^n$, $j \in J$, be a holomorphic atlas for E and $(g_{jk}) \in Z^1(\mathfrak{U})$, $\mathrm{GL}_n(\mathcal{O})$ be the corresponding cocycle of transition functions. Then TFAE:*

i. E is holomorphically trivial.

ii. There exist n global holomorphic sections $F_1, \dots, F_n$ of E such that for each point $x \in X$ the vectors $F_1(x), \dots, F_n(x) \in E_x$ are linearly independent.

iii. The cocycle (g_{jk}) splits, i.e., there exists a cochain $(g_l) \in C^0(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$ with

$$g_{jk} = g_j g_k^{-1} \quad \text{on } U_i \cap U_j \quad \text{for every } j,\, k \in J.$$

Lemma 2.2.9. *Suppose X is a non-compact Riemann surface and E is a holomorphic vector bundle on X. If E has a non-trivial global meromorphic section, then E also has a global holomorphic section which has no zeros.*

PROOF

Suppose f is a non-trivial meromorphic section of E over X and $A \subset X$ is a discrete subset consisting of ite zeros and poles. Suppose $a \in A$ and $h : E_U \to U \times \mathbb{C}^n$ is a holomorphic linear chart of E on an open neighborhood U of a. Relative to the chart h we may represent f as $(f_1, \ldots, f_n) \in \mathscr{M}(U)^n$. Let $k(a)$ be the minimum of the orders of the functions f_j at the point a. By (2.2.6) there exists a meromorphic function $\varphi \in \mathscr{M}(X)$ which at each point $a \in A$ has order $-k(a)$ and is holomorphic and non-zero on $X \setminus A$. Then $f \coloneqq \varphi f$ is a holomorphic section of E which has no zeros.

■

Theorem 2.2.10. *Every holomorphic line bundle E on a non-compact Riemann surface X is holomorphically trivial.*

PROOF

Suppose $\varnothing \neq Y_0 \Subset Y_1 \Subset \cdots$ is a sequence of relatively compact Runge domains in X with $\bigcup Y_j = X$. By (2.2.7) over every Y_j there is a meromorphic section. Thus by (2.2.9) there is also a holomorphic section which does not vanish. Hence E is trivial over each Y_j by (2.2.8). It then follows from the Runge Approximation Theorem (1.3.9) that every holomorphic section of E over Y_j can be approximated uniformly on compact subsets by holomorphic sections of E over Y_{j+1}. Let $f_0 \in \mathcal{O}_E(Y_0)$ be a section which is not zero at some point $a \in Y_0$. One can now construct a sequence $f_j \in \mathcal{O}_E(Y_j)$, $j \geq 1$, such that $\lim_{j\to\infty} f_j(a) \neq 0$ and such that for each $j \in \mathbb{N}$ the sequence $(f_k|_{Y_j})_{k>j}$ converges in $\mathcal{O}_E(Y_j)$, Then the limit of the sequence (f_j) is a section $f \in \mathcal{O}_E(X)$ which does not vanish identically. As above this implies that E is trivial over X.

■

Theorem 2.2.11. *Every holomorphic vector bundle E on a non-compact Riemann surface X is holomorphically trivial.*

PROOF

The theorem will be proved by induction on n, the rank of E. The case $n = 1$ is given by (2.2.10). Now assume the result has been proved for all bundles of rank $n-1$ and suppose E is a bundle of rank n.

First we assume that there exists a section $F_n \in \mathcal{O}_E(X)$ which does not vanish anywhere. Since E is locally trivial, there exists an open covering $\mathfrak{U} = \{U_j\}_{j\in J}$ of X with the property that for every $j \in J$ ther are sections $F_1^j, \ldots, F_{n-1}^j \in \mathcal{O}_E(U_j)$ such that $F_1^j(x), \ldots, F_{n-1}^j(x), F_n(x)$ are linearly independent for every $x \in U_j$. On any intersection $U_j \cap U_k$ these systems are related to each other in the following way:

$$\begin{pmatrix} F^j \\ F_n \end{pmatrix} = \begin{pmatrix} G^{jk} & a^{jk} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F^k \\ F_n \end{pmatrix}, \tag{2}$$

where F^j denotes the column vector with entries $F_1^j, \ldots, F_{n-1}^j$, the matrix G^{jk} is an element of $\mathrm{GL}_{n-1}(\mathcal{O}(U_j \cap U_k))$ and a^{jk} is a column vector with $n-1$ rows having coefficients in $\mathcal{O}(U_j \cap U_k)$. Then $G^{jk}G^{kl} = G^{jl}$ on $U_j \cap U_k \cap U_l$. Hence by the induction hypothesis there exist matrices $G^j \in \mathrm{GL}_{n-1}(\mathcal{O}(U_j))$ with

$$G^{jk} = G^j (G^k)^{-1} \quad \text{on } U_j \cap U_k.$$

Setting $\widetilde{F}^i := (G^j)^{-1} F^j$ and using (2) gives

$$\begin{pmatrix} \widetilde{F}^j \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & b^{jk} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{F}^k \\ F_n \end{pmatrix}, \tag{3}$$

for some $b^{jk} \in \mathcal{O}(U_j \cap U_k)^{n-1}$. On $U_j \cap U_k \cap U_l$ one has the relation $b^{jk} + b^{kl} = b^{jl}$. Since $H^1(\mathfrak{U}, \mathcal{O}) = 0$, one can thus find holomorphic column vectors $b^j \in \mathcal{O}(U_j)^{n-1}$ having $(n-1)$ rows with

$$b^{jk} = b^j - b^k \quad \text{on } U_j \cap U_k.$$

Set $\widehat{F}^j = \widetilde{F}^j - b^j F_n$. Then it follows from (3) that

$$\begin{pmatrix} \widehat{F}^j \\ F_n \end{pmatrix} = \begin{pmatrix} \widehat{F}^k \\ F_n \end{pmatrix} \quad \text{on } U_j \cap U_k.$$

Hence the $\widehat{F}^j$ piece together to form a global $(n-1)$-tuple $(F_1, \ldots, F_{n-1}) \in \mathcal{O}_E(X)^{n-1}$. By construction $F_1(x), \ldots, F_n(x)$ are linearly independent for every $x \in X$. Thus E is holomorphically trivial.

The only remaining thing to do is to show that E has a holomorphic section which does not vanish. By (2.2.7) and (2.2.9) this is the case over any relatively compact domain $Y \Subset X$. Thus one has that E is trivial over Y. As in the proof of (2.2.10) one can now construct with the help of the Runge Approximation Theorem (1.3.9) a non-trivial holomorphic section of E over X. By (2.2.9) then E also has a nowhere vanishing holomorphic section. This completes the proof of the theorem.

■

Corollary 2.2.12. *Suppose X is a non-compact Riemann surface. Then*

$$H^1(X, \mathrm{GL}_n(\mathcal{O})) = 0.$$

In particular, $H^1(X, \mathcal{O}^*) = 0$.

Reference

[1] Otto Forster. *Lectures on Riemann Surfaces.* Springer-verlag, 1981.