Final report: Bogomolov-Tian-Todorov theorem and its extension to dGBV categories

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1 Introduction

Let $X = (M, I)$ be a complex manifold. A small deformation of the complex structure gives a splitting $T^*M_{\mathbf{C}} = T_t^{1,0} \oplus T_t^{0,1}$, which is equivalently determined by a map

$$\varphi(t) : T^{0,1} \to T^{1,0}$$

characterized by $v + \varphi(t)v \in T_t^{0,1}$ for small t. Conversely, if given a family of tensors $\varphi(t) = \varphi_\mu^\nu(t) d\overline{z}^\mu \otimes \dfrac{\partial}{\partial z^\nu} \in \mathcal{A}^{0,1}(\mathcal{T}_X)$, with $\varphi(0) = 0$,

1. the almost complex structure $I(t)$ for small t is given, with respect to the basis $\{\partial/\partial z^\mu\} \cup \{\partial/\partial \overline{z}^\mu\}$ by

$$I(t) = P \begin{pmatrix} i & \\ & -i \end{pmatrix} p^{-1}, \ P = \begin{pmatrix} \delta_\mu^\nu & \varphi_\mu^\nu(t) \\ \overline{\varphi}_\mu^\nu(t) & \delta_\mu^\nu \end{pmatrix}.$$

2. $\overline{\partial}_\varphi := \overline{\partial}(t) = \overline{\partial} + \varphi(t)$

The almost complex structure is integrable (i.e. $\overline{\partial}_\varphi^2 = 0$, i.e. $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$) if and only if the Maurer-Cartan equation

$$\overline{\partial}\varphi(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0 \tag{1}$$

holds in $\mathcal{A}^{0,2}(\mathcal{T}_X)$.

Remark 1.1. Instead of (1), the equation given in [Huy05] is $\overline{\partial}\varphi(t) + [\varphi(t), \varphi(t)] = 0$, which I believe is a mistake since the last 2 equations in p258 uses inconsistent conventions: $d\overline{z}^\mu \wedge d\overline{z}^\gamma = d\overline{z}^\mu \otimes d\overline{z}^\gamma - d\overline{z}^\gamma \wedge d\overline{z}^\mu$ and $d\overline{z}^\mu \wedge d\overline{z}^\gamma = (d\overline{z}^\mu \otimes d\overline{z}^\gamma - d\overline{z}^\gamma \wedge d\overline{z}^\mu)/2$.

Remark 1.2. [MK71] takes another approach in defining the $\mathcal{T}_X$-valued (0,1)-form φ, associated to a deformation. It turns out that they differ by a minus sign.

If we consider the power series expansion $\varphi = \sum_{i\geq 1} \varphi_i t^i$, (1) becomes a recursive system of equations :

$$0 = \overline{\partial}\varphi_1 \tag{2}$$

$$0 = \overline{\partial}\varphi_2 + \frac{1}{2}[\varphi_1, \varphi_1] \tag{3}$$

$$\vdots \tag{4}$$

$$0 = \overline{\partial}\varphi_k + \frac{1}{2}\sum_{0<i<k}[\varphi_i, \varphi_{k-i}]. \tag{5}$$

Hence $\varphi(t)$ is determined by the tangential part φ_1, the deformation theory asks whether there exists a convergent solution prescribing a $\overline{\partial}$ closed form φ_1, i.e. deform the complex structure in direction $[\varphi_1] \in H^1_{\overline{\partial}}(X, T_X) \simeq H^1(X, T_X)$.

In this final report, we prove that for a compact Kähler Calabi-Yau manifold, the Bogomolov-Tian-Todorov lemma and solve the Maurer-Cartan equation (1) by "dualize" the $\mathcal{T}_X$-valued differential forms to the honest differential forms where we have the $\partial\overline{\partial}$-lemma.

The existence theorem of Kodaira-Spencer-Nirenberg tells us that subject to the condition

$$\overline{\partial}^*\varphi = 0, \tag{6}$$

the recursively defined KSN ansatz

$$\varphi(t) = \varphi_1 t - \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)] \tag{7}$$

is a solution if and only if the harmonic part of $[\varphi, \varphi]$ is 0. We show that by choosing the Ricci-flat metric on X, the KSN-ansatz can be recovered, and the convergence follows.

In other words, the Kuranishi space of X is an open ball of $H^1(X, \mathcal{T}_X)$.

2 Bogomolov-Tian-Todorov lemma and solution to Maurer-Cartan equation

Definition 2.1. *A compact Kähler manifold X is Calabi-Yau (CY) if $K_X \simeq \mathscr{O}_X$.*

Let $\Omega \in H^0(X, K_X)$ be a holomorphic volume form, it induces a nondegenerate pairing

$$\wedge^p \mathcal{T}_X \otimes \wedge^{n-p}\mathcal{T}_X \to \mathscr{O}_X, \ \alpha \wedge \beta \mapsto \Omega(\alpha \wedge \beta)$$

inducing a natural isomorphism

$$\eta : \wedge^p \mathcal{T}_X \simeq \Omega_X^{n-p} \tag{8}$$

characterized by

$$\eta(\alpha)(\beta) = (-1)^{p(p-1)/2}\Omega(\alpha \wedge \beta).$$

In terms of local coordinate, $\eta(v_1 \wedge \cdots \wedge v_p) = \iota_{v_1} \cdots \iota_{v_p}\Omega$.

Moreover,

$$\eta : \mathcal{A}^{0,q}(\wedge^p \mathcal{T}_X) \simeq \mathcal{A}^{n-p,q}(X), \tag{9}$$

distinguished features such as ∂ and $\partial\overline{\partial}$-lemma carried over.

Definition-Lemma. *We define an operator* $\Delta = \eta^{-1}\partial\eta : \mathcal{A}^{0,q}(\wedge^p \mathcal{T}_X) \to \mathcal{A}^{0,q}(\wedge^{p-1}\mathcal{T}_X)$. *Then*

1. $\Delta^2 = 0$,
2. $\overline{\partial} \circ \eta = \eta \circ \overline{\partial}$,
3. $\Delta \circ \overline{\partial} = -\overline{\partial} \circ \Delta$.

Note that φ is Δ-exact/closed if and only if $\eta(\varphi)$ is ∂-exact/closed.

Proof. 2. holds trivially since Ω is holomorphic, $\overline{\partial}$ acts on $(0, q)$-form part while η contracts p-vector part. 1. and 2. follows from 2. $\square$

$\mathcal{A}^{0,*}(\wedge^* \mathcal{T}_X)$ admits $\mathbf{Z}_2$-graded product: $(\alpha \otimes v) \wedge (\beta \otimes w) = (-1)^{\overline{\beta}\overline{v}}\alpha \wedge \beta \otimes v \wedge w$

Lemma 2.2 (Bogomolov-Tian-Todorov)**.** *If* $\alpha \in \mathcal{A}^{0,p}(\mathcal{T}_X),\ \beta \in \mathcal{A}^{0,q}(\mathcal{T}_X)$, *then*

$$(-1)^p[\alpha, \beta] = \Delta(\alpha \wedge \beta) - \Delta\alpha \wedge \beta - (-1)^{p+1}\alpha \wedge \Delta\beta$$

Proof. Let $G(\alpha, \beta) = \Delta(\alpha \wedge \beta) - \Delta\alpha \wedge \beta - (-1)^{p+1}\alpha \wedge \Delta\beta$. We check the identity locally. Suppose $\alpha = d\overline{z}^I \otimes a\partial_i,\ \beta = d\overline{z}^J \otimes b\partial_j$ where $\partial_i := \partial/\partial z^i$.

$$\begin{aligned}
\Delta(\alpha \wedge \beta) &= (-1)^q \eta^{-1}(d\overline{z}^I \wedge d\overline{z}^J \otimes a\partial_i \wedge b\partial_j) && (10)\\
&= (-1)^q \eta^{-1}\partial(d\overline{z}^I \wedge d\overline{z}^J \wedge \eta(a\partial_i \wedge b\partial_j) && (11)\\
&= (-1)^p d\overline{z}^I \wedge d\overline{z}^J \otimes \Delta(a\partial_i \wedge b\partial_j) && (12)\\
\Delta\alpha \wedge \beta &= (-1)^p d\overline{z}^I \otimes \Delta(a\partial_i) \wedge (d\overline{z}^J \otimes b\partial_j) && (13)\\
&= (-1)^p d\overline{z}^I \wedge d\overline{z}^J \otimes (\Delta(a\partial_i) \wedge b\partial_j) && (14)\\
\alpha \wedge \Delta\beta &= (d\overline{z}^I \otimes a\partial_i) \wedge (-1)^q(d\overline{z}^J \otimes \Delta(b\partial_j)) && (15)\\
&= d\overline{z}^I \wedge d\overline{z}^J \otimes (a\partial_i \wedge \Delta(b\partial_j)). && (16)
\end{aligned}$$

Thus

$$G(\alpha,\beta) = (-1)^p d\bar{z}^I \wedge d\bar{z}^J \otimes G(a\partial_i, b\partial_j). \tag{17}$$

We claim that $G(a\partial_i, b\partial_j) = [a\partial_i, b\partial_j]$, since both G and $[\ ,\ ]$ are skew-symmetric, it suffices to prove for the case $i > j$.

We assume that the holomorphic volume is $\Omega = f dz^1 \wedge \cdots \wedge dz^n$, then

$$\Delta(a\partial_i) = (-1)^{i-1}\eta^{-1}\partial(af \cdot dz^{[n]-i}) \tag{18}$$
$$= (-1)^{i-1}\eta^{-1}(\partial_i(af) \cdot dz^i \wedge dz^{[n]-i}) \tag{19}$$
$$= \eta^{-1}(\partial_i(af) \cdot dz^{[n]} \tag{20}$$
$$= \frac{1}{f}\frac{\partial af}{\partial z^i}. \tag{21}$$
$$\Delta(a\partial_i \wedge b\partial_j) = (-1)^{i+j}\eta^{-1}\partial(abf \cdot dz^{[n]-i-j)} \tag{22}$$
$$= \eta^{-1}((-1)^{j-1}\partial_i(abf)dz^{[n]-j} + (-1)^i\partial_j(abf) \cdot dz^{[n]-i}) \tag{23}$$
$$= \frac{1}{f}\left(\frac{\partial abf}{\partial z^i}\frac{\partial}{\partial z^j} - \frac{\partial abf}{\partial z^j}\frac{\partial}{\partial z^i}\right) \tag{24}$$

Applying the Leibniz rule and putting it altogether,

$$G(a\partial_i, b\partial_j) = \frac{1}{f}\left(af\frac{\partial b}{\partial z^i}\frac{\partial}{\partial z^j} - bf\frac{\partial a}{\partial z^j}\frac{\partial}{\partial z^i}\right) \tag{25}$$
$$= [a\partial_i, b\partial_j]. \tag{26}$$

□

2.1 Existence

Proposition 2.3. *Let X be a Calabi-Yau manifold and let $v \in H^1(X, \mathcal{T}_X)$. The Maurer-Cartan equation (1) is solvable such that*

1. $\varphi_1 \in v$
2. $\eta(\varphi_i) \in \mathcal{A}^{n-1,n}(X)$ *is ∂-exact for $i > 1$.*

Proof. Pick $\varphi_1 \in v$ such that $\eta(\varphi_1)$ is a harmonic $(n-1,1)$-form, in any case φ_1 is $\bar{\partial}$-closed. We solve for $\varphi_{i>1}$ inductively and to make the picture clearer, we first try to solve for φ_2. By (2.2),

$$[\varphi_1, \varphi_1] = -\Delta(\varphi_1 \wedge \varphi_1), \tag{27}$$

in particular, $\eta[\varphi_1, \varphi_1]$ is ∂-exact and $\bar{\partial}$-closed $(n-1,1)$-form. By $\partial\bar{\partial}$-lemma, there exists $\alpha \in \mathcal{A}^{n-2,0}$ such that

$$\eta[\varphi_1, \varphi_1] = -\bar{\partial}\partial\beta. \tag{28}$$

Take $\varphi_2 = \frac{1}{2}\eta^{-1}\partial\beta$. For $i > 1$, suppose $\varphi_1, \ldots, \varphi_{k-1}$ are picked such that

$$\eta(\varphi_i) \text{ is } \partial\text{-exact and } \overline{\partial}\varphi_l + \frac{1}{2}\sum_{0<i<l}[\varphi_{l-i}, \varphi_i] = 0 \ \forall 2 \leq l \leq k-1. \tag{29}$$

Then by (2.2),

$$-\eta \sum_{0<i<k}[\varphi_{k-i}, \varphi_i] = \partial\eta \sum_{0<i<k} \varphi_i \wedge \varphi_{k-i} \tag{30}$$

We claim that it is $\overline{\partial}$-closed, then by $\partial\overline{\partial}$-lemma, we obtain a Δ-exact φ_{k+1}.

$$\overline{\partial}\sum_{0<i<k}[\varphi_i, \varphi_{k-i}] = -2\sum_{0<i<k}[\varphi_{k-i}, \overline{\partial}\varphi_i] \tag{31}$$

$$= -2\sum_{0<i<k}\sum_{0<j<i}[\varphi_{k-i}, [\varphi_j, \varphi_{i-j}]] \tag{32}$$

$$= -2\sum_{0<i<k}\sum_{0<j<i}[\varphi_{k-i}, [\varphi_j, \varphi_{i-j}]] \tag{33}$$

$$= -2\sum_{\substack{a,b,c>0 \\ a+b+c=k}}[\varphi_a, [\varphi_b, \varphi_c]] \tag{34}$$

Since the summation is symmetric, by Jacobi identity, it is 0. □

2.2 Convergence

Each step in (2.3) does not produce unique φ_i's, for example, $\varphi_2' = \varphi_2 + \partial\overline{\partial}\beta$ is another desired solution. To kill the ambiguity, we make a modification so that $\eta(\varphi_k) \in \mathrm{Im}\overline{\partial}^*$. By (2.2), $\gamma := \eta \sum_{0<i<k}[\varphi_i, \varphi_{k-i}]$ is ∂-exact, so

$$\gamma = \Delta_{\overline{\partial}}G\gamma = \overline{\partial}(\overline{\partial}^* G\gamma), \tag{35}$$

the $\overline{\partial}^*\overline{\partial}$ part vanishes since γ is also $\overline{\partial}$-closed as shown in (2.3). Hence we take

$$\varphi_k = -\frac{1}{2}\eta^{-1}\overline{\partial}^* G\eta \sum_{0<i<k}[\varphi_i, \varphi_{k-i}], \tag{36}$$

which is already close to the KSN-ansatz (7). So far the choice of Kähler metric is irrelevant, in fact, one good choice of Kähler metric makes η commutes with $\overline{\partial}^*$ and G.

Lemma 2.4. *Let $\wedge^p \mathcal{T}_X$ and Ω^{n-p} be endowed with the canonical hermitian metric. Then the isomorphism $\eta : \mathcal{T} \to \Omega^{n-1}$ is an isometry (up to a constant) if and only if*

$$\omega^n = c \cdot \Omega \wedge \overline{\Omega} \tag{37}$$

for some constant c.

Proof.

$$\langle \eta(\partial/\partial z^i), \eta(\partial/\partial z^j)) \rangle = (-1)^{i+j} |f|^2 \langle \cdots \widehat{dz^i} \cdots, \cdots \widehat{dz^j} \cdots \rangle \tag{38}$$

$$= |f|^2 C^{i\bar{j}} \tag{39}$$

$$= |f|^2 \det(g^{i\bar{j}}) \overline{g}_{j\bar{i}} \tag{40}$$

$$= |f|^2 \det(g^{i\bar{j}}) g_{i\bar{j}} \tag{41}$$

where $C^{i\bar{j}}$ is the (i,j)-cofactor of $(\langle dz^i, dz^j \rangle) = (\overline{g}^{i\bar{j}})$. Hence η is isometry if and only if $|f|^2 = \det(g_{i\bar{j}})$. On the otherhand,

$$\frac{\omega^n}{\Omega \wedge \overline{\Omega}} = c \frac{\det(g)}{|f|^2} \tag{42}$$

for some constant c. $\square$

Suppose ω is the desired metric. In this case, η commutes with $\overline{\partial}^*$ (since it commutes with $\overline{\partial}$), and thus with G. Therefore, the arguments in the existence theorem of [MK71] apply.

Initially, we have $\omega_0^n = e^f \Omega \wedge \overline{\Omega}$ for some global smooth function f. By the $\partial\overline{\partial}$-lemma, any other Kähler form in $[\omega_0]$ is of the form $\omega = \omega_0 + \partial\overline{\partial}\varphi$. Requiring $\omega^n = \Omega \wedge \overline{\Omega}$ is then equivalent to solving the complex Monge-Amperé equation:

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{-f} \det(g_{i\bar{j}}). \tag{43}$$

The existence of a smooth solution φ to this equation was established by Yau in his proof of the Calabi conjecture [Yau78].

3 dGBV algebra structure

Let X be a general complex manifold. The object of study in this section is the complex

$$L = \mathcal{A}_X = \bigoplus_{p,q} \mathcal{A}^{0,q}(X, \wedge^p \mathcal{T}_X) \tag{44}$$

and the general Maurer-Cartan equation on L. $(L, d, [,])$ has a structure of differential Lie $\mathbf{Z}_2$-graded algebra:

1. (Grading) $L^k := \bigoplus_{p+q-1=k} L^{q,p},\ L^{q,p} := \mathcal{A}^{0,q}(X, \wedge^p \mathcal{T}_X)$

2. (multiplication) If $\theta \otimes v \in L^{q,p},\ \delta \otimes w \in L^{q',p'}$,

$$(\theta \otimes v) \wedge (\delta \otimes w) := (-1)^{pq'}(\theta \wedge \delta) \otimes (v \wedge w)$$

3. (Bracket) If $d\overline{z}^I \otimes v \in L^{q,p},\ d\overline{z}^J \otimes w \in L^{q',p'}$,

$$[d\overline{z}^I \otimes v, d\overline{z}^J \otimes w] := (-1)^{q'(p+1)}(d\overline{z}^I \wedge d\overline{z}^J) \otimes [v, w]_{SN}$$

where the *Schouten-Nijenhuis* bracket

$$[\ ,\ ]_{SN} : \wedge^* \mathcal{T}_X \times \wedge^{*'} \mathcal{T}_X \to \wedge^{*+*'-1} \mathcal{T}_X$$

is the generalization of Lie bracket characterized as follows: If $v \in \wedge^p,\ w \in \wedge^{p'},\ u \in \wedge^{p''}$,

(a) (Graded skew-symmetry) $[v, w] = -(-1)^{(p+1)(p'+1)}[w, v]$

(b) If $v \in \mathcal{T}_X$, then $[v,\ -\ ]$ is the unique extension of $\mathcal{L}_v$ such that

$$[v, w \wedge u] = [v, w] \wedge u + w \wedge [v, u]$$

(c) If $f \in \mathcal{A}^0$,

$$[f, w \wedge u] = [f, w] \wedge u + (-1)^q w \wedge [f, u]$$

In general, we have

$$[v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_{p'}] := \sum_{i,j} (-1)^{i+j} [v_i, w_j] v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_{p'} \tag{45}$$

and

$$[v, w \wedge u] = [v, w] \wedge u + (-1)^{(p+1)p'} w \wedge [v, u]. \tag{46}$$

Remark 3.1. The degree is shifted by 1 to make $(\mathcal{A}_X, \overline{\partial}, [\ ,\ ])$ a $\mathbf{Z}_2$-graded Lie algebra.

Proposition 3.2. *If $\alpha \in L^k,\ \overline{\alpha} := k$. Then*

1. $L^k \wedge L^\ell \subset L^{k+l-1}$ *and* $\alpha \wedge \beta = (-1)^{(\overline{\alpha}+1)(\overline{\beta}+1)} \beta \wedge \alpha$,

2. $[L^k, L^\ell] \subset L^{k+\ell}$ *and* $[\alpha, \beta] = -(-1)^{\overline{\alpha}\overline{\beta}}[\beta, \alpha]$,

3. $\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{\overline{\alpha}+1} \alpha \wedge \overline{\partial}\beta$,

4. $\overline{\partial}[\alpha,\beta] = [\overline{\partial}\alpha,\beta] + (-1)^{\overline{\alpha}}[\alpha,\overline{\partial}\beta]$,

5. $[\alpha,\beta\wedge\gamma] = [\alpha,\beta]\wedge\gamma + (-1)^{\overline{\alpha}(\beta+1)}\beta\wedge[\alpha,\gamma]$,

6. *(Jacobi-identity)* $(-1)^{\overline{\alpha\gamma}}[\alpha,[\beta,\gamma]] + (-1)^{\overline{\gamma\beta}}[\gamma,[\alpha,\beta]] + (-1)^{\overline{\beta\alpha}}[\beta,[\gamma,\alpha]] = 0$.

Remark 3.3. Property $1,2,5,6$ make $L[1]$ a *Gerstenhaber* algebra.

When X is a Calabi-Yau manifold, then $\mathcal{A}_X$ carries another operator

$$\Delta : \mathcal{A}_X \to \mathcal{A}_X, \gamma \mapsto \iota_\gamma\Omega.$$

From now on, we consider a general Maurer-Cartan equation in $\mathcal{A}_X$,

$$\overline{\partial}\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \tag{47}$$

which occurs in the moduli problem of complex structure of a complex manifold, in that case we restrict ourselves to the Lie subalgebra $(\mathcal{A}^{0,*}(\mathcal{T}_X), [,], \overline{\partial})$.

We will see that when X is a Calabi-Yau manifold, the Maurer-Cartan equation is solvable in $\mathcal{A}_X$. Moreover, the multiplicative structure on $\mathcal{A}_X$ which we don't have in $\mathcal{A}^{0,*}(\mathcal{T}_X)$ defines a Frobenius structure [BK98].

The starting point is the generalization of (2.2) in [BK98]

Lemma 3.4 (Generelized Bogomolov-Tian-Todorov lemma)**.** *For* $\alpha \in L^k,\ \beta \in L^\ell$,

$$(-1)^{\overline{\alpha}}[\alpha,\beta] = \Delta(\alpha\wedge\beta) - \Delta\alpha\wedge\beta + (-1)^{\overline{\alpha}}\alpha\wedge\Delta\beta. \tag{48}$$

Proof. The idea of the proof of (2.2) applied here, except that we have to take care of the sign convention. □

As a corollary, the Maurer-Cartan equation is solvable. There is a solution parametrized by $H(\mathcal{A}_X,\overline{\partial})$. Let $H = H(\mathcal{A}_X,\overline{\partial})$. We adjoin the $\mathbf{Z}_2$-graded variables corresponding to elements in H^*, and denote the set of the formal series in t by $\mathcal{A}_X[[t]]$. The grading is given as follows:

Let $\Delta_0 = 1 \in H^0(X,\wedge^0 T_X)$ and $\{\Delta_a\}$ be a homogeneous basis of $H(\mathcal{A}_X,\overline{\partial}) = \oplus_{p,q}H^q(X,\wedge^p\mathcal{T}_X)$. We let $\deg\Delta_a = p+q-2$ for $\Delta_a \in H^q(X,\wedge^p\mathcal{T}_X)$, while the corresponding dual coordinate t^a has $\deg t^a = -\deg\Delta_a$.

Under the convention, $\sum_a \varphi_a t^a$ has odd degree in $H[[t]]$, where $\varphi_a \in \mathcal{A}_X^{0,q}(\wedge^p\mathcal{T}_X)$

Since there might be element of even degree, i.e. when $p+q$ is odd, the product is only $\mathbf{Z}_2$-commutative, so the space cohomology space $H(\mathcal{A}_X[[t]],\overline{\partial})$ is no longer a formal scheme.

The operation Δ, $\wedge$, $[\ ,\ ]$ extended canonically to $\mathcal{A}_X[[t]]$ so that (3.2) holds.

Proposition 3.5 (Generalized Bogomolov-Tian-Todorov Theorem)**.** *There exists a solution to the Maurer-Cartan equation*

$$\overline{\partial}\hat{\varphi}(t) + \frac{1}{2}[\hat{\varphi}(t), \hat{\varphi}(t)] = 0 \tag{49}$$

in $\mathcal{A}_X[[t]]$

$$\hat{\varphi}(t) = \sum_a \varphi_a t^a + \frac{1}{2!}\sum_{a_1,a_2} \varphi_{a_1a_2} t^{a_1} t^{a_2} + \cdots \in (\mathcal{A}_X[[t]])^{odd}. \tag{50}$$

such that

1. *The cohomology classes* $\{[\varphi_a]\}_a$ *form a basis of* $H(\mathcal{A}_X[[t]], \overline{\partial})$.
2. $\varphi_a \in \ker\Delta$ *and* $\varphi_{a_1\cdots a_k} \in \operatorname{Im}\Delta$ *for* $k \geq 2$.
3. $\partial_0\hat{\varphi}(t) = 1_H$.

Proof. The exact same proof of the classical case applied with the use of (3.4) and $\partial\overline{\partial}$-lemma. □

4 Deformation of algebra $(H, \wedge)$

So far we haven't seen advantages of considering the big complex $\mathcal{A}_X$, instead of the classical one $\mathcal{A}^{0,*}(\mathcal{T}_X)$, in this section we will see that a solution to Maurer-Cartan equation deforms the algebra structure of $H^{0,*}(X \wedge^* \mathcal{T}_X)$.

Proposition 4.1. *Let* $\varphi \in \mathcal{A}_X[[t]]^{odd}$, *then* $\overline{\partial}_\varphi = \overline{\partial} + [\varphi, -]$ *is a differential of degree 1, and* $\overline{\partial}_\varphi^2 = 0$ *if and only if the (so-called master equation) holds*

$$[\overline{\partial}\varphi + \frac{1}{2}[\varphi,\varphi], -] = 0 \tag{51}$$

Proof. When φ is odd, by (3.2), $[\varphi, -]$ is a derivation of type same as $\overline{\partial}$. On the other hand, by graded version of Jacobi identity,

$$\overline{\partial}_\varphi^2 B = [\overline{\varphi} + \frac{1}{2}[\varphi,\varphi], B]$$

□

Suppose γ is a solution to (51), we define the deformed differential $\overline{\partial}_\gamma = \overline{\partial} + [\gamma,\ -]$, and consider

$$T_\gamma = H(\mathcal{A}_X[[t]], \overline{\partial}_\gamma).$$

If we restrict ourselves to the classscial case $H^1(X, \mathcal{T}_X)$, T_γ is nothing but the tangent space to the kuranishi space $S \subset H^1(X, \mathcal{T}_X)$ at γ: If $\gamma(t) \in S$ is a curve with $\gamma(0) = \gamma$, then by differentiating the Maurer-Cartan equation, we get

$$\overline{\partial}_\gamma \gamma'(0) = \overline{\partial}\gamma'(0) + [\gamma, \gamma'(0)] = 0.$$

Now we make use of the solution $\hat{\varphi}$ in (3.5), then by diffentiating the Mauerer-Cartan equation with respect to the direction $\partial_a := \partial/\partial t^a$, we get that

1. the cohomology classes of $\partial_a \hat{\varphi} = \varphi_a + O(t)$ form a $\mathbf{C}[[t]]$-basis of $T_{\hat{\varphi}}$.

And by (3.2),

2. $T_{\hat{\varphi}}$ is closed under wedge product.

Hence we have

Proposition-Definition. *There exists formal (super)-power series $A_{ab}^c(t) \in \mathbf{C}[[t]]$ satisfying*

$$\partial_a \hat{\varphi} \wedge \partial \hat{\varphi}_b = \sum_c A_{ab}^c(t) \partial_c \hat{\varphi} \ (mod\ \overline{\partial}_{\hat{\varphi}(t)}). \tag{52}$$

The series $A_{ab}^c(t)$ defines structure constant of a $\mathbf{Z}_2$-commutative associative $\mathbf{C}[[t]]$-algebra structure on $H[[t]]$

4.1 Non-degenerate pairing

Introduce a linear functional on $\mathcal{A}_X$

$$\int \gamma = \int_X \eta(\gamma) \wedge \Omega = \int_X \iota_\gamma \Omega \wedge \Omega \tag{53}$$

which is supported on $\mathcal{A}^{0,n}(\wedge^n \mathcal{T}_X)$. The pairing $\langle \cdot, \cdot \rangle$ is defined to be

$$\langle \gamma_1, \gamma_2 \rangle = \int \gamma_1 \wedge \gamma_2, \tag{54}$$

which is graded symemetric and non-degenerate since Ω is nowhere vanishing. The operators behave well under the pairing:

$$\langle \overline{\partial}\gamma_1, \gamma_2 \rangle = (-1)^{\overline{\gamma_1}} \langle \gamma_1, \overline{\partial}\gamma_2 \rangle \tag{55}$$

$$\langle \Delta\gamma_1, \gamma_2 \rangle = (-1)^{\overline{\gamma_1}+1} \langle \gamma_1, \Delta\gamma_2 \rangle \tag{56}$$

By associativity of $\wedge$, the multiplication is compatible with the algebra structure.

The pairing is not positive definite (even hermitian), however, the deformed basis $\{\partial_a \hat{\varphi}(t)\}$ is flat in the following sense:

Proposition 4.2. *The pairing $\langle \partial_a \hat{\gamma}(t), \partial_b \hat{\varphi}(t) \rangle$ is independent of t.*

Proof. We have chosen φ so that the conditions in (3.5) are satisfied, but $\ker \Delta$ and $\operatorname{Im} \Delta$ are perpendicular with respect to the pairing by (55). □

4.2 Flat connection

$(H[[t]], \wedge_t, \langle \cdot, \cdot \rangle)$. We give a geometric interpretation of the algebra structure. Define a connection $\mathcal{T}_H^*$ as follows: Let $\{p^a\}_a$ be framing dual to $\{\partial_a\}$, we define

$$\nabla_\partial p^c = A^c_{ab} p^b \tag{57}$$

Proposition 4.3. *The connection is flat, i.e. $\nabla^2 = 0$. In fact, the connection matrix $A_{abc} = A^e_{ab} g_{ec} = \partial_a \partial_b \partial_c \Phi$ where*

$$\Phi = \int -\frac{1}{2} \overline{\partial} \alpha \wedge \Delta \alpha + \frac{1}{6} \varphi^3, \tag{58}$$

if $\varphi = \varphi_1 + \Delta \alpha$.

Remark 4.4. Explicit flat sections was constructed in [BK98] using the deformed holomorphic volume form $\Omega(t) = e^{\iota_{\hat{\varphi}(t)}} \Omega$. In the classical case, $\Omega(t) \in H^0(X_t, K_{X_t})$.

Proof. For simplicity, we restrict ourselves to coordinate with even degree H^{odd} i.e. $p+q$ is even. Let $\delta = \sum \tau^a \partial_a$, we claim that

$$\delta^3 \Phi = (\delta \hat{\varphi})^3.$$

We omit $\wedge$ for brevity and denote $\varphi := \hat{\varphi}$. Note that φ is odd, it commutes with everything since $\varphi \psi = (-1)^{(\overline{\varphi}+1)(\overline{\psi}+1)}$. The main facts we use here is

1. $\langle \overline{\partial} \gamma_1, \gamma_2 \rangle = (-1)^{\overline{\gamma_1}} \langle \gamma_1, \overline{\partial} \gamma_2 \rangle$
2. $\langle \Delta \gamma_1, \gamma_2 \rangle = (-1)^{\overline{\gamma_1}+1} \langle \gamma_1, \Delta \gamma_2 \rangle$
3. $\overline{\partial} \varphi = \frac{1}{2} \Delta \varphi^2$ (by (1), (3.4) and $\Delta \varphi = 0$).

$$\delta^3\frac{1}{6}\int\varphi^3=\int(\delta\varphi)^3+3\varphi\delta\varphi\delta^2\varphi+\frac{1}{2}\varphi^2\delta^3\varphi \tag{59}$$

$$\frac{1}{2}\int\overline{\partial}\alpha\Delta\alpha=-\frac{1}{2}\int\alpha\overline{\partial}\Delta\alpha=-\frac{1}{2}\int\alpha\overline{\partial}\varphi=\frac{1}{2}\int\alpha\Delta\varphi^2=\frac{1}{4}\int\Delta\alpha\cdot\varphi^2 \tag{60}$$

$$\frac{1}{4}\delta^4\int\Delta\alpha\cdot\varphi^2=\frac{1}{4}\int\underbrace{\delta^3(\Delta\alpha)\cdot\varphi^2}_{=\delta^3\varphi\cdot\varphi^2}+\underbrace{3\delta^2(\Delta\alpha)\delta\varphi^2}_{4\delta^2\varphi\cdot\delta(\varphi^2)}+3\delta(\Delta\alpha)\delta^2(\varphi^2)+\Delta\alpha\delta^3(\varphi^2) \tag{61}$$

$$\int\delta(\Delta\alpha)\delta^2(\varphi^2)=\int\delta(\varphi^2)\delta^2(\varphi) \tag{62}$$

$$\int\Delta\alpha\delta^3(\varphi^2)=\int\varphi^2\delta^3\varphi. \tag{63}$$

We then get the result by putting all together. □

Hence $(H[[t]],\wedge_t,\langle\cdot,\cdot\rangle,\nabla)$ defines a *Frobenius manifold:*

Definition 4.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be a finite-dimensional $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space equipped with a nondegenerate graded-symmetric pairing. With respect to to a basis $\{\partial_a\}$, a formal power series $A^c_{ab}\in\mathbf{C}[[H^*]]$ defining structure constant of an algebra structure on $H[[H^*]]$:*

$$\partial_a\circ\partial_b=A^c_{ab}\partial_c$$

such that

1. *$\circ$ is $\mathbf{Z}_2$-commutative, associative,*
2. *$\langle\alpha\circ\beta,\gamma\rangle=\langle\alpha,\beta\circ\gamma\rangle$*
3. *$\forall a,b,c,d,\ \partial_dA^c_{ab}=(-1)^{\overline{a}\overline{d}}\partial_aA^c_{db}$*

The construction of Frobenius manifold was generalized to arbitrary dGBV algebra on which (3.5) is valid [Man99].

Definition 4.6. *A differential Lie $\mathbf{Z}_2$-graded algebra $(L,d,[,])$ is called a differential Gerstenhaber-Batalin-Vilkovisky algebra (dGBV) if it is endowed with an odd $\mathbf{C}$-linear map $\Delta:L\to L$ such that*

1. *$\Delta^2=0$*
2. *for any $\alpha\in L$, the map*

$$\delta_a:L\to L,\ \beta\mapsto(-1)^{\overline{\alpha}}(\Delta(\alpha\wedge\beta)-\Delta\alpha\wedge\beta+(-1)^{\overline{\alpha}}\alpha\wedge\Delta\beta).$$

is a derivation of parity $\overline{\alpha}+1$.

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