Hormander's L^2 estimate and extension theorem

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Contents

The main reference is [Dem12], especially chapter VIII, and the reference looks like VII, 4.2 means the (4.2) in chapter VII in [Dem12].

1 Preliminaries about functional ananysis

Let $\mathscr{H}_1, \mathscr{H}_2$ be complex Hilbert spaces. A linear operator T defined on a subspace $\operatorname{Dom} T \subset \mathscr{H}_1$ into $\mathscr{H}_2$. Then T is said to be *densely defined* if $\operatorname{Dom} T$ is dense in $\mathscr{H}_1$, and *closed* if its graph

$$\operatorname{Gr} T = \{(x, Tx) | x \in \operatorname{Dom} T\}$$

is closed in $\mathscr{H}_1 \times \mathscr{H}_2$. (closed imply continuous if $\operatorname{Dom} T = \mathscr{H}_1$.)

Assume T is closed and densely defined.

Definition 1.1. The adjoint T^* of T in Von Neumann's sense is constructed as follows: $\operatorname{Dom} T^*$ is the set of $y \in \mathscr{H}_2$ such that the linear functional define on $x \in \operatorname{Dom} T$ by

$$f_y(x) := \langle Tx, y \rangle_2$$

is bounded in $\mathscr{H}_1$-norm. Since $\operatorname{Dom} T$ is dense, we can extend the bounded operator from $\operatorname{Dom} T$ uniquely to a bounded linear functional on the whole $\mathscr{H}_1$, then by Riesz representation theorem we get a unique element $T^* y \in \mathscr{H}_1$ such that $\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1$ for all $x \in \operatorname{Dom} T$.

undone

Theorem 1.1 (Von Neumann 1929). *If $T : \mathscr{H}_1 \to \mathscr{H}_2$ is closed and densely defined. then its adjoint T^* is also closed and densely defined and $(T^*)^* = T$. We also have $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ and its dual $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Im} T^*}$.*

Now for two closed and densely defined operators T, S:

$$\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2 \xrightarrow{S} \mathscr{H}_3$$

such that $S \circ T = 0$ (i.e. $\operatorname{Im} T = T(\operatorname{Dom} T) \subset \operatorname{Ker} S \subset \operatorname{Dom} S$), we have:

Theorem 1.2.

$$\mathscr{H}_2 = (\operatorname{Ker} S \cap \operatorname{Ker} T^*) \oplus \overline{\operatorname{Im} T} \oplus \overline{\operatorname{Im} S^*},$$
$$\operatorname{Ker} S = (\operatorname{Ker} S \cap \operatorname{Ker} T^*) \oplus \overline{\operatorname{Im} T}.$$

And in order that $\operatorname{Im} T = \operatorname{Ker} S$ (i.e. exact), it suffices that

$$||T^* x||_1^2 + ||Sx||_3^2 \geq C ||x||_2^2, \quad \forall x \in \operatorname{Dom} S \cap \operatorname{Dom} T^* \tag{1}$$

for some constant $C > 0$. *In this case,* $\forall v \in \mathscr{H}_2$ *with* $Sv = 0$, *there exists* $u \in \mathscr{H}_1$ *such that* $Tu = v$ *and* $\|u\|_1^2 \leq \frac{1}{C}\|v\|_2^2$. *In particular,*

$$\overline{\operatorname{Im} T} = \operatorname{Im} T = \operatorname{Ker} S, \quad \overline{\operatorname{Im} S^*} = \operatorname{Im} S^* = \operatorname{Ker} T^*.$$

Proof. □

2 Preliminaries on complete Riemannian manifolds

3 L^2 Hodge theory on complete Riemannian manifolds

4 General estimates for $\overline{\partial}$ on complete Hermitian manifolds

Let (X, ω) be a *complete* hermitian manifold and E a holomorphic hermitian vector bundle of rank r over X. Consider now the operator

$$A_{E,\omega} = [i\Theta(E), \Lambda] + T_\omega$$

on $\Lambda^{p,q}T^*X \otimes E$ (cf. Chapter VII in [Dem12]). This operator comes from the difference of two Laplace operators:

Theorem 4.1 (VII-Thm 1.4). *The operator $\Delta'_\tau = [D' + \tau, \delta' + \tau^*]$ is a positive and formally self-adjoint operator. Moreover*

$$\Delta'' = \Delta'_\tau + [i\Theta(E), \Lambda] + T_\omega$$

where T_ω is an operator of order 0 depending only on the torsion of ω.

Proof. see [Dem12]. □

Thus the operator is formally self-adjoint (pointwise?) acting pointwise on $\Lambda^{p,q}T^*X \otimes E$. As a corollary, we have the following important inequality

Lemma 4.1 (Bochner-Kodaira-Nakano inequality, VII-2.1). *For $u \in \mathscr{C}^\infty_{p,q}(X, E)$ (compactly supported E-valued (p, q)-forms), we have*

$$||D''u||^2 + ||\delta''u||^2 \geq \int_X \langle A_{E,\omega}u, u\rangle dV \tag{2}$$

Proof.

$$\langle\langle \Delta''u, u\rangle\rangle = \int_X \langle \Delta''u, u\rangle dV = ||D''u||^2 + ||\delta''u||^2$$
$$\langle\langle \Delta'_\tau u, u\rangle\rangle = ||D'u + \tau u||^2 + ||\delta'u + \tau^* u||^2 \geq 0.$$

□

Assume now that $A_{E,\omega}$ is semi-positive on $\Lambda^{p,q}T^*X \otimes E$ (i.e. $\langle A_{E,\omega}u, u\rangle \geq 0$ pointwise). Then by density of $\mathscr{C}^\infty_{p,q}(X, E)$ in $\operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$, we can find $u_j \in \mathscr{C}^\infty_{p,q}(X, E)$ such that $u_j \to u$, $D''u_j \to D''u, \delta''u_j \to \delta''u$ in L^2 norm. Then by taking a subsequence, we have $u_j \to u$ pointwise almost everywhere. And since $A_{E,\omega}$ act pointwise on fiber, we have $\langle A_{E,\omega}u_j, u_j\rangle \to \langle A_{E,\omega}u, u\rangle$ pointwise almost everywhere. Then

$$\int_X |\langle A_{E,\omega}u_j, u_j\rangle| dV = \int_X \langle A_{E,\omega}u_j, u_j\rangle dV \leq ||D''u_j||^2 + ||\delta''u_j||^2 \to ||D''u||^2 + ||\delta''u||^2$$

and thus by Fatou's lemma

$$\int_X \langle A_{E,\omega} u, u\rangle dV = \int_X \liminf \langle A_{E,\omega} u_j, u_j\rangle dV \leq \liminf \int_X \langle A_{E,\omega} u_j, u_j\rangle dV \leq \|D'' u\|^2 + \|\delta'' u\|^2.$$

(i.e. Equation 2 holds for all $u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$.)

Now given $g \in L^2_{p,q}(X, E)$ such that

$$D'' g = 0. \tag{3}$$

In addition to this, we also assume that g satisfies condition (⋆):

1. for almost every $x \in X$, there exists $\alpha(x) \in [0, \infty)$ such that

$$|\langle g(x), u\rangle|^2 \leq \alpha \langle A_{E,\omega} u, u\rangle$$

   for every $u \in (\Lambda^{p,q} T^* X \otimes E)_x$.

   Remark. If the operator $A_{E,\omega}$ is invertible (i.e. positive), the minimal such α is $\langle A^{-1}_{E,\omega} g(x), g(x)\rangle$, so we denoted α this way even when $A_{E,\omega}$ is not invertible.

   Proof. By the Cauchy-Schwarz inequality on $\langle u, v\rangle_A := \langle Au, v\rangle$, we have

$$|\langle g, u\rangle|^2 = |\langle A^{-1} g, u\rangle_A|^2 \leq \langle A^{-1} g, A^{-1} g\rangle_A \cdot \langle u, u\rangle_A = \langle A^{-1} g, g\rangle \cdot \langle Au, u\rangle.$$

   □

2. Additionally, we assume the global constraint

$$\int_X \langle A^{-1}_{E,\omega} g, g\rangle \, dV < +\infty.$$

Then the basic result of L^2 theory can be stated as follows.

Theorem 4.2 (VIII-Thm 4.5). *If (X, ω) is complete and $A_{E,\omega} \geq 0$ in bidegree (p, q), then for any $g \in L^2_{p,q}(X, E)$ satisfying (⋆) and $D'' g = 0$, there exists $f \in L^2_{p,q-1}(X, E)$ such that $D'' f = g$ and*

$$\|f\|^2 \leq \int_X \langle A^{-1}_{E,\omega} g, g\rangle dV.$$

Proof. For every $u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$ we have

$$\begin{aligned} |\langle\langle u, g\rangle\rangle|^2 &= \left|\int_X \langle u, g\rangle dV\right|^2 \leq \left(\int_X |\langle u, g\rangle| dV\right)^2 \\ &\leq \left(\int_X \langle A_{E,\omega} u, u\rangle^{\frac{1}{2}} \langle A^{-1}_{E,\omega} g, g\rangle^{\frac{1}{2}} dV\right)^2 \\ &\leq \int_X \langle A^{-1}_{E,\omega} g, g\rangle dV \cdot \int_X \langle A_{E,\omega} u, u\rangle dV \end{aligned}$$

by the definition of $\langle A_{E,\omega}^{-1} g, g\rangle$ and the Cauchy-Schwarz inequality. Then Equation 2 implies

$$|\langle\!\langle u, g\rangle\!\rangle|^2 \le C(\|D'' u\|^2 + \|\delta'' u\|^2), \quad \forall u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$$

where C is the integral $\int_X \langle A_{E,\omega}^{-1} g, g\rangle\, dV$. We now repeat the proof of Thm 1.2: For any $u \in \operatorname{Dom} \delta''$[1], let us write

$$u = u_1 + u_2, \quad u_1 \in \operatorname{Ker} D'', \quad u_2 \in (\operatorname{Ker} D'')^{\perp} = \overline{\operatorname{Im} \delta''}.$$

Then $D'' u_1 = 0$ and $\delta'' u_2 = 0$ (Since $\delta'' u_2 = 0 \iff \langle\!\langle u_2, D'' h\rangle\!\rangle = 0, \forall h \in \mathscr{C}^{\infty}$). Since $g \in \operatorname{Ker} D''$, we get

$$|\langle\!\langle u, g\rangle\!\rangle|^2 = |\langle\!\langle u_1, g\rangle\!\rangle|^2 \le C\|\delta'' u_1\|^2 = C\|\delta'' u\|^2.$$

The Hahn-Banach theorem shows that the bounded linear functional defined on $\operatorname{Im} \delta''$

$$\delta'' u \mapsto \langle\!\langle u, g\rangle\!\rangle$$

can be extended to a linear functional $v \mapsto \langle\!\langle v, f\rangle\!\rangle$, $f \in L^2_{p,q-1}$, of norm $\|f\| \le C^{\frac{1}{2}}$. This means that

$$\langle\!\langle u, g\rangle\!\rangle = \langle\!\langle \delta'' u, f\rangle\!\rangle, \quad \forall u \in \operatorname{Dom} \delta'',$$

i.e. that $D'' f = g$ (as we define D'' in distribution sense, or by $\delta''^{*} = D''$). The theorem is proved.

□

Remark. We can always find a solution $f \in (\operatorname{Ker} D'')^{\perp}$ by taking the orthogonal projection to $(\operatorname{Ker} D'')^{\perp}$. Then this solution is clearly unique and is precisely the one with minimal L^2 norm of equation $D'' f = g$. We thus have

$$\begin{aligned}\langle\!\langle \Delta'' f, h\rangle\!\rangle &= \langle\!\langle f, \delta'' D'' h\rangle\!\rangle + \langle\!\langle f, D'' \delta'' h\rangle\!\rangle = \langle\!\langle f, \delta'' D'' h\rangle\!\rangle \\ &= \langle\!\langle D'' f, D'' h\rangle\!\rangle = \langle\!\langle g, D'' h\rangle\!\rangle = \langle\!\langle \delta'' g, h\rangle\!\rangle\end{aligned}$$

for all $h \in \mathscr{C}^{\infty}_{p,q-1}(X, E)$ and consequently $\Delta'' f = \delta'' g$ in distribution sense. Hence if $g \in C^{\infty}_{p,q}(X, E)$, the ellipticity of Δ'' shows that $f \in C^{\infty}_{p,q-1}(X, E)$.

5 Estimates on weakly pseudoconvex manifolds

We now introduce a large class of complex manifolds such that the L^2 estimates will still be easily tractable.

[1] we don't need $u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$ since we only aply the estimate on $u_1 \in \operatorname{Ker} D'' \cap \operatorname{Dom} \delta''$

Definition 5.1. A complex manifolds X is said to be *weakly pseudoconvex* if there exists a plurisubharmonic exhaustion function $\psi \in C^\infty(X, \mathbb{R})$. That is $i\partial\bar{\partial}\psi \geq 0$ on X, and $\forall c \in \mathbb{R}$, the subset $X_c = \{x \in X; \psi(x) < c\}$ is relatively compact in X.

Examples

Theorem 5.1 (VIII-Thm 5.2). *Every weakly pseudoconvex Kähler manifold (X, ω) carries a complete Kähler metric $\hat{\omega}$.*

Proof. Let $\psi \in C^\infty(X, \mathbb{R})$ be the plurisubharmonic exhaustive function on X. As ψ is exhaustive, we have $\{\psi < 0\}$ is relative compact. Thus $\inf \psi > -\infty$ and we may assume $\psi \geq 0$ by adding a constant to it. Then $\hat{\omega} = \omega + i\partial\bar{\partial}(\psi^2)$ is Kähler and

$$\hat{\omega} = \omega + 2i\psi\partial\bar{\partial}\psi + 2i\partial\psi \wedge \bar{\partial}\psi \geq \omega + 2i\partial\psi \wedge \bar{\partial}\psi > 0.$$

We have $|d\psi|_{\hat{\omega}} = |\partial\psi + \bar{\partial}\psi|_{\hat{\omega}} \leq 2|\partial\psi|_{\hat{\omega}} \leq \sqrt{2}$. Then by Lem 2.4 we have $\hat{\omega}$ is complete ($|d\psi|_{\hat{\omega}}$ bounded is enough). □

More generally, we can set $\hat{\omega} = \omega + i\partial\bar{\partial}(\chi \circ \psi)$ where χ is a convex increasing function (so $\chi' > 0$, $\chi'' > 0$). Then

$$\begin{aligned}\hat{\omega} &= \omega + i(\chi' \circ \psi)\partial\bar{\partial}\psi + i(\chi'' \circ \psi)\partial\psi \wedge \bar{\partial}\psi \\ &\geq \omega + i\sqrt{\chi''(\psi)}\partial\psi \wedge \sqrt{\chi''(\psi)}\bar{\partial}\psi = \omega + i\partial(\rho \circ \psi) \wedge \bar{\partial}(\rho \circ \psi) := \omega',\end{aligned}$$

where $\rho(t) = \int_0^t \sqrt{\chi''(u)}\, du$. Then we have $|\partial(\rho \circ \psi)|_{\hat{\omega}} \leq |\partial(\rho \circ \psi)|_{\omega'} \leq 1$ since we can choose coorinates such that $\omega_{i\bar{j}} = \delta_{ij}$ at x_0, then for real function f,

$$\begin{aligned}|\partial f|^2_{\delta_{ij} + f_i f_j} &= \bar{f}_i(\delta_{ij} + f_i\bar{f}_j)^{-1} f_j = \bar{f}_i(\delta_{ij} - \frac{f_i\bar{f}_j}{1 + \sum_i |f_i|^2}) f_j \\ &= \sum_i |f_i|^2 - \frac{(\sum_i |f_i|^2)^2}{1 + \sum_i |f_i|^2} = \frac{\sum_i |f_i|^2}{1 + \sum_i |f_i|^2} < 1.\end{aligned}$$

we can regard this as adding the direction of $d\psi$ to the metric to control $|d\psi|$.

And for $\rho \circ \psi$ to remain exhaustive we need

$$\lim_{t \to \infty} \int_0^t \sqrt{\chi''(u)} du = +\infty$$

so that there does not exist $c < \infty$ such that $\{\rho < c\} = \{\psi < \infty\} = X$ may not be relatively compact. We can take for example $\chi(t) = t^2$ or $\chi(t) = t - \log t$ for $t \geq 1$.

Then many vanish theorems can now be generalized to weakly pseudoconvex domain and reduced to finding suitable Kähler metric on X and hermitian metric on E such that the conditions for Theorem 4.2 are satisfied.

see [Dem12] VIII-5

6 Hormander's L^2 estimates for non complete Kähler metrics

In this section, we generalize the previous results to estimates in non complete Kähler metric, for example the standard metric on a bounded domain $\Omega \subset \mathbb{C}^n$. The idea is to approximate the given metric by complete Kähler metrics.

Theorem 6.1 (VIII-6.1). *Let $(X, \hat{\omega})$ be a complete Kähler manifold, ω another Kähler metric, and $E \to X$ a m-positive* [2] *bundle. Let $g \in L^2_{n,q}(X, E)$ with $D''g = 0$ and*

$$\int_X \langle A_q^{-1} g, g\rangle dV < +\infty \tag{$\star$}$$

with respect to ω, with $A_q = [i\Theta(E), \Lambda]$ in bidegree (n, q) and $q \geq 1$, $m \geq \min\{n-q+1, r\}$ (We have A_q is positive definite by VII-Lem 7.2). Then there exist $f \in L^2_{n,q-1}(X, E)$ such that $D''f = g$ and

$$\|f\|^2 \leq \int_X \langle A_q^{-1} g, g\rangle dV.$$

Proof. First note that by Hopf-Rinow, we have a metric g is complete iff every closed geodesic ball $\overline{B}_g(r)$ is compact. Then for every $\varepsilon > 0$, the Kähler metric $\omega_\varepsilon = \omega + \varepsilon\hat{\omega}$ is complete. (Since $\overline{B}_{\varepsilon\hat{\omega}}(r) = \overline{B}_{\hat{\omega}}(\varepsilon r)$, and for $g \geq h$ we have $d_g(x, y) \geq d_h(x, y)$ and $\overline{B}_g(r) \subset \overline{B}_h(r)$ which is compact if h is complete.)

Now let us put an index ε on objects depending on ω_ε. It follows from Lemma 6.1 below that

$$|u|^2_\varepsilon dV_\varepsilon \leq |u|^2 dV, \quad \langle A^{-1}_{q,\varepsilon} u, u\rangle_\varepsilon\, dV_\varepsilon \leq \langle A_q^{-1} u, u\rangle dV. \tag{4}$$

Then Theorem 4.2 applies to ω_ε (as $g \in L^2_{n,q}(X, E)_\varepsilon$ and ($\star$) holds in ω_ε) yields a solution $f_\varepsilon \in L^2_{n,q-1}(X, E)_\varepsilon$ such that $D''f_\varepsilon = g$ in distribution sense with respect to ω_ε and

$$\int_X |f_\varepsilon|^2_\varepsilon dV_\varepsilon \leq \int_X \langle A^{-1}_{q,\varepsilon} g, g\rangle_\varepsilon dV_\varepsilon \leq \int_X \langle A_q^{-1} g, g\rangle dV.$$

This means that the family (f_ε) is bounded in L^2 norm (in ω) on every compact subset of X (as ω_ε are quasi isometric to ω for ε small on a fixed compact set[3]). Now fixed a compact exhaustion X_i, then by the Banach–Alaoglu theorem, there is a weakly convergent subsequence of (f_ε) in $L^2_{n,q-1}(X_i, E)$. By the diagonal method, we get a subsequence weakly converges to $f \in L^2_{loc,\omega}$ on every X_i. Now for every $h \in \mathscr{C}^\infty_{n,q}(X, E)$, said $\operatorname{Supp} h \subset X_i$, we want

$$\langle\!\langle f, \delta''h\rangle\!\rangle \overset{\text{I}}{\leftarrow} \langle\!\langle f_\varepsilon, \delta''h\rangle\!\rangle \overset{\text{II}}{\longleftarrow} \langle\!\langle f_\varepsilon, \delta''_\varepsilon h\rangle\!\rangle \overset{\text{III}}{\longleftarrow} \langle\!\langle f_\varepsilon, \delta''_\varepsilon h\rangle\!\rangle_\varepsilon = \langle\!\langle g, h\rangle\!\rangle_\varepsilon \overset{\text{IV}}{\longrightarrow} \langle\!\langle g, h\rangle\!\rangle.$$

[2]It's m-semi-positive in [Dem12], I don't know whether this is enough for A_q to be positive definite. Compare VII-Lem 7.2

[3]by controlling the biggest eigenvalues of $\hat{\omega}$ with respect to ω on compact set.

First, we have **I** is due to the weak convergence of $f_\varepsilon \to f$ in $L^2(X_i)$. Then for **II**, we have

$$\begin{aligned}\langle\!\langle f_\varepsilon, \delta'' h - \delta''_\varepsilon h\rangle\!\rangle &\le \|f_\varepsilon\| \cdot \|\delta'' h - \delta''_\varepsilon h\| \le C\|f_\varepsilon\|_\varepsilon \cdot \|\delta'' h - \delta''_\varepsilon h\| \\ &\le C\left(\int_X \langle A_q^{-1} g, g\rangle dV\right)^{\frac{1}{2}} \cdot \|\delta'' h - \delta''_\varepsilon h\|,\end{aligned}$$

and $\delta''_\varepsilon h$ are uniformly bounded on X_i and converge to $\delta'' h$ pointwise. Hence by the Dominated convergence theorem, we get **II**.

For **III**,

$$\int_{X_i} \langle f_\varepsilon, \delta''_\varepsilon h\rangle dV \int_{X_i} \langle f_\varepsilon, \delta''_\varepsilon h\rangle_\varepsilon dV_\varepsilon$$

undone

For **IV**, we have

$$\langle g, h\rangle_\varepsilon dV_\varepsilon \to \langle g, h\rangle dV$$

pointwise and

$$\int_X \langle g, h\rangle_\varepsilon dV_\varepsilon = \int_{X_i} \langle g, h\rangle_\varepsilon dV_\varepsilon \le \int_{X_i} |g|_\varepsilon |h|_\varepsilon dV_\varepsilon \le C\int_{X_i} |g|\cdot|h| dV \le C' \int_{X_i} dV \cdot \int_{X_i} |g|^2 dV.$$

For the last inequality, we use that $|h|$ is bounded on X_i and Cauchy-Schwarz. Therefore by DCT again, we have **IV**.

For the norm of f, on every compact set X_i, we have by Cauchy-Schwarz inequality

$$\langle\!\langle f, f\rangle\!\rangle_{X_i} = \lim_{\varepsilon\to 0} \langle\!\langle f_\varepsilon, f\rangle\!\rangle_{X_i} \le \liminf \|f_\varepsilon\|_{X_i} \|f\|_{X_i},$$

and thus

$$\begin{aligned}\|f\|_{X_i} &\le \liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{X_i} \le \liminf_{\varepsilon\to 0} C_{\varepsilon, X_i} \|f_\varepsilon\|_{\varepsilon, X_i} \\ &\le \liminf_{\varepsilon\to 0} C_{\varepsilon, X_i} \cdot \left(\int_X \langle A_q^{-1} g, g\rangle dV\right)^{\frac{1}{2}} = \left(\int_X \langle A_q^{-1} g, g\rangle dV\right)^{\frac{1}{2}}.\end{aligned}$$

Where C_{ε, X_i} depend on largest eigenvalues of ω_ε with respect to ω on X_i, and hence $C_{\varepsilon, X_i} \to 1$ as $\varepsilon \to 0$. Finally, let X_i increase to X [4] and we get

$$\|f\|^2 \le \int_X \langle A_q^{-1} g, g\rangle dV.$$

□

[4] Let $Y_i = X_i - X_{i-1}$, then $X = \bigsqcup Y_i$ and so $\int_X |f|^2 dV = \sum \int_{Y_i} |f|^2 dV$, with $\sum_{i=1}^n \int_{Y_i} |f|^2 dV \le M$ bounded and increasing, thus $\|f\|^2 \le M$.

Lemma 6.1 (VIII-Lem 6.3). *Let ω, γ be hermitian metrics on X such that $\gamma \geq \omega$. Then for every $u \in \Lambda^{n,q}T^*X \otimes E$, $q \geq 1$, we have*

$$|u|_\gamma^2 dV_\gamma \leq |u|^2 dV, \quad \langle A_{q,\gamma}^{-1}u, u\rangle_\gamma \, dV_\gamma \leq \langle A_q^{-1}u, u\rangle dV$$

where an index γ means the term is computed in terms of γ instead of ω.

Proof. Locally at $x_0 \in X$, there exists a coordinates [5] $(z^1, \dots, z^n)$ such that

$$\omega = i \sum_j dz^j \wedge d\bar{z}^j, \quad \gamma = i \sum_j \gamma_j \, dz^j \wedge d\bar{z}^j \quad \text{at } x_0,$$

where $\gamma_1 \leq \cdots \leq \gamma_n$ are the eigen values with respect to ω. Then $\gamma \geq \omega$ implies $\gamma_1 \geq 1$. We have $|dz^j|_\gamma^2 = \gamma_j^{-1}$ and $|dz^K|_\gamma^2 = \gamma_K^{-1}$ where $\gamma_K = \prod_{k \in K} \gamma_k$. Now for any (n,q) form $u = \sum u_{K,\lambda} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^K \otimes e_\lambda$, $|K| = q$, $\{e_\lambda\}_{\lambda=1}^r$ is a orthonormal frame of E. Then

$$|u|_\gamma^2 = \sum_{K,\lambda} (\gamma_1 \cdots \gamma_n)^{-1} \gamma_K^{-1} |u_{K,\lambda}|^2, \quad dV_\gamma = \gamma_1 \cdots \gamma_n dV,$$

$$|u|_\gamma^2 dV_\gamma = \sum_{K,\lambda} \gamma_K^{-1} |u_{K,\lambda}|^2 dV \leq \sum |u_{K,\lambda}|^2 dV = |u|^2 dV,$$

$$\Lambda_\gamma u = \sum_{|I|=q-1} \sum_{j,\lambda} i(-1)^{n+j-1} \gamma_j^{-1} u_{jI,\lambda} (\widehat{dz^j}) \wedge d\bar{z}^I \otimes e_\lambda,$$

where $(\widehat{dz^j}) = dz^1 \wedge \cdots \wedge \widehat{dz^j} \wedge \cdots \wedge dz^n$. And thus for $i\Theta(E) = i \sum c_{j\bar{k}\lambda}^\mu dz^j \wedge d\bar{z}^k \otimes e_\lambda^* \otimes e_\mu$, since u is (n,q)-form,

$$\begin{aligned}
A_{q,\gamma} u &= [i\Theta(E), \Lambda_\gamma] u = i\Theta(E) \wedge (\Lambda_\gamma u) \\
&= \sum_{|I|=q-1} i^2 \sum_{j,\lambda} c_{j\bar{k}\lambda}^\mu dz^j \wedge d\bar{z}^k \wedge \left((-1)^{n+j-1} \gamma_j^{-1} u_{jI,\lambda} \widehat{dz^j} \wedge d\bar{z}^I \right) \otimes e_\mu \\
&= \sum_{|I|=q-1} \sum_{j,\lambda} \gamma_j^{-1} c_{j\bar{k}\lambda}^\mu u_{jI,\lambda} (-1)^{n+j} \widehat{dz^j} \wedge dz^j \wedge d\bar{z}^k \wedge d\bar{z}^I \otimes e_\mu \\
&= \sum_{|I|=q-1} \sum_{j,\lambda} \gamma_j^{-1} c_{j\bar{k}\lambda}^\mu u_{jI,\lambda} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{kI} \otimes e_\mu, \\
\langle A_{q,\gamma} u, u \rangle_\gamma &= (\gamma_1 \cdots \gamma_n)^{-1} \sum_{|I|=q-1} \gamma_I^{-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{j\bar{k}\lambda}^\mu u_{jI,\lambda} \bar{u}_{kI,\mu} \\
&\geq (\gamma_1 \cdots \gamma_n)^{-1} \sum_{|I|=q-1} \gamma_I^{-2} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{j\bar{k}\lambda}^\mu u_{jI,\lambda} \bar{u}_{kI,\mu} \\
&= \gamma_1 \cdots \gamma_n \langle A_q S_\gamma u, S_\gamma u \rangle
\end{aligned}$$

[5] We can first use linear coordinate change to let $\omega_{ij} = \delta_{ij}$, then use unitary diagonalization to diagonalize γ and preserve $\omega = \delta_{ij}$.

where

$$S_\gamma u = \sum_K (\gamma_1 \cdots \gamma_n)^{-1} \gamma_K^{-1} u_{K,\lambda} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^K \otimes e_\lambda.$$

Therefore we get

$$\begin{aligned}
|\langle u, v\rangle_\gamma|^2 = |\langle u, S_\gamma v\rangle|^2 = |\langle A_q^{-1} u, S_\gamma v\rangle_{A_q}|^2 &\le \langle A_q^{-1} u, A_q^{-1} u\rangle_{A_q} \cdot \langle S_\gamma v, S_\gamma v\rangle_{A_q} \\
&= \langle A_q^{-1} u, u\rangle \langle A_q S_\gamma v, S_\gamma v\rangle \\
&\le (\gamma_1 \cdots \gamma_n)^{-1} \langle A_q^{-1} u, u\rangle \langle A_{q,\gamma} v, v\rangle_\gamma,
\end{aligned}$$

and let $v = A_{q,\gamma}^{-1} u$ we get

$$\langle A_{q,\gamma}^{-1} u, u\rangle_\gamma \le (\gamma_1 \cdots \gamma_n)^{-1} \langle A_q^{-1} u, u\rangle, \quad \langle A_{q,\gamma}^{-1} u, u\rangle_\gamma dV_\gamma \le \langle A_q^{-1} u, u\rangle dV.$$

□

We are now interested in the case where E is a line bundle, then $i\Theta(E)$ is a closed real valued $(1,1)$-form. In general, for a real $(1,1)$-form $\gamma \in \Lambda^{1,1} T^* X$. There exist ω-orthogonal basis $(\zeta_1, \dots, \zeta_n)$ in $T^{1,0} X$ which diagonalizes both ω and γ:

$$\omega = i \sum_{j=1}^n \zeta_j^* \wedge \overline{\zeta}_j^*, \quad \gamma = i \sum_{j=1}^n \gamma_j \zeta_j^* \wedge \overline{\zeta}_j^*, \ \gamma_j \in \mathbb{R}.$$

Proposition 6.1 (VI-Porp 5.8). *For every form $u = \sum u_{J,K} \zeta_J^* \wedge \overline{\zeta}_K^*$ [6], one has*

$$[\gamma, \Lambda] u = \sum_{J,K} \Big(\sum_{j \in J} \gamma_j + \sum_{k \in K} \gamma_k - \sum_{j=1}^n \gamma_j\Big) u_{J,K} \zeta_J^* \wedge \overline{\zeta}_K^*.$$

Proof. For (p,q)-form u, we have

$$\Lambda u = i(-1)^p \sum_{J,K,l} u_{J,K} (\zeta_l \lrcorner \zeta_J^*) \wedge (\bar{\zeta}_l \lrcorner \bar{\zeta}_K^*).$$

$$\gamma \wedge u = i(-1)^p \sum_{J,K,m} \gamma_m u_{J,K}\, \zeta_m^* \wedge \zeta_J^* \wedge \bar{\zeta}_m^* \wedge \bar{\zeta}_K^*,$$

$$\begin{aligned}
[\gamma, \Lambda] u &= \sum_{J,K,l,m} \gamma_m u_{J,K} \left(\zeta_l^* \wedge (\zeta_m \lrcorner \zeta_J^*) \wedge \bar{\zeta}_l^* \wedge (\bar{\zeta}_m \lrcorner \bar{\zeta}_K^*) - (\zeta_m \lrcorner (\zeta_l^* \wedge \zeta_J) \wedge \bar{\zeta}_m \lrcorner (\bar{\zeta}_l^* \wedge \bar{\zeta}_K)) \right) \\
&= \sum_{J,K} \Big(\sum_{j \in J} \gamma_j + \sum_{k \in K} \gamma_k - \sum_{j=1}^n \gamma_j\Big) u_{J,K} \zeta_J^* \wedge \overline{\zeta}_K^*.
\end{aligned}$$

□

[6] $J = (j_1, \dots, j_p)$ is a multi-index with $j_1 < \cdots < j_p$.

With this we can apply Theorem 6.1 to an important special case of semi-positive line bundle E. If we let $0 \le \lambda_1(x) \le \cdots \le \lambda_n(x)$ be the eigenvalues of $i\Theta(E)_x$ with respect to ω_x for all $x \in X$, then Proposition 6.1 implies for (n,q)-form u

$$\langle A_q u, u\rangle \ge (\lambda_1 + \cdots + \lambda_q)|u|^2,$$

and thus

$$\langle g, u\rangle \le |g|^2 \cdot |u|^2 \le \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2\langle A_q u, u\rangle.$$

By previous remark 1, we have

$$\langle A_q^{-1} g, g\rangle \le \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2 \implies \int_X \langle A_q^{-1} g, g\rangle dV \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2 dV.$$

For example, we can apply this to the case when E is the trivial line bundle $X \times \mathbb{C}$ with metric given by a weight $e^{-\varphi}$. One can assume that φ is plurisubharmonic and $i\partial\bar\partial\varphi$ has at least $n - q + 1$ positive eigenvalues at every point, i.e. $\lambda_q > 0$ on X. This leads to the L^2 estimates originally given by [Hör65]. We state here a slightly more general result.

Theorem 6.2 (VIII-Thm 6.5). *Let (X, ω) be a weakly pseudoconvex Kähler manifold, E a hermitian line bundle on X, $\varphi \in C^\infty(X, \mathbb{R})$ a weight function such that the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ of $i\Theta(E) + i\partial\bar\partial\varphi$ are ≥ 0. Then for every form g, of type (n, q), $q \ge 1$, with L^2_{loc} (resp. C^∞) coefficients such that $D''g = 0$ and*

$$\int_X \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2 e^{-\varphi} dV < +\infty,$$

we can find a L^2_{loc} (resp. C^∞) form of type $(n, q-1)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi} dV \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2 e^{-\varphi} dV.$$

Proof. We apply the general result on E_φ (E with metric twisted by $e^{-\varphi}$), then $i\Theta(E_\varphi) = -i\partial\bar\partial\log(e^{-\varphi}h) = i\Theta(E) + i\partial\bar\partial\varphi$. We can exhaust X by relatively compact weakly pseudoconvex domains

$$X_c = \{x \in X | \psi(x) < c\}$$

where $\psi \in C^\infty(X, \mathbb{R})$ is a plurisubharmonic exhaustion function. Then $-\log(c - \psi)$ is a psh. exhaustion function on X_c, and since

$$\int_X \langle A_{q,E_\varphi} g, g\rangle_\varphi dV \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2_\varphi dV = \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q}|g|^2_E e^{-\varphi} dV < +\infty.$$

By Theorem 6.1, we get solution f_c on X_c with

$$\int_{X_c} |f_c|^2 e^{-\varphi} dV \leq \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi} dV.$$

As before, by the Banach-Alaoglu theorem and diagonal method, we get a subsequence that weakly converges to $f \in L^2_{loc}$ on every X_c. Then clearly $D'' f = g$ in distribution sense, and again we have

$$\int_K |f|^2 e^{-\varphi} dV \leq \liminf \int_K |f_c|^2 e^{-\varphi} dV \leq \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi} dV,$$

for K a compact subset of X. Let K increase to X, and we get the estimates we want. □

If we need estimates for (p, q)-forms instead of (n, q)-forms, we can use the isomorphism $\Lambda^p T_X^{*1,0} \simeq \Lambda^{n-p} T_X^{1,0} \otimes \Lambda^n T_X^{*1,0}$ obtained by contraction of n-forms with (n-p)-vectors to get

$$\Lambda^{p,q} T^* X \otimes E \simeq \Lambda^{n,q} T^* X \otimes (\Lambda^{n-p} T_X^{1,0} \otimes E).$$

In case of $p = 0$, we have

Definition 6.1. $\operatorname{Ric} \omega = i\Theta(\Lambda^n T_X^{1,0}) = i \operatorname{Tr} \Theta(T_X^{1,0})$.

For any local coordinates $(z^1, \dots, z^n)$, the holomorphic n-form $dz^1 \wedge \cdots \wedge dz^n$ is a local section of $\Lambda^n T^{*1,0} X$, hence we have

$$\operatorname{Ric} \omega = i\Theta(\Lambda^n TX) = i\partial\bar{\partial} \log |dz^1 \wedge \cdots \wedge dz^n|^2_\omega = -i\partial\bar{\partial} \log \det \omega_{j\bar{k}}$$

Then Theorem 6.2 can be apply to $(0, q)$-form g, with condition on eigenvalues of

$$i\Theta(E) + \operatorname{Ric} \omega + i\partial\bar{\partial}\varphi$$

in the place of $i\Theta(E_\varphi)$.

7 Extension of holomorphic functions from subvarieties

With the capability of solving $\bar{\partial}$-equation, we can now try to extend holomorphic section defined on (a neighborhood of) subvariety. Suppose f is a section of line bundle L defined on a neighborhood U of subvariety Y of X, the idea is to first multiply by a bump function to get a global section ψf on X, then consider $g = \bar{\partial}(\psi f)$ satisfying $\bar{\partial} g = 0$. If we can find u such that $\bar{\partial} u = g = \bar{\partial}(\psi f)$ and $u|_Y = 0$, then $F = \psi f - u$ satisfy $\bar{\partial} F = 0$ and $F|_Y = f|_Y$ is the holomorphic extension we want.

Now the difficulty lies in how to ensure $u|_Y = 0$ and to find a suitable weight function φ such that we can apply the L^2 estimates on $i\Theta(L_\varphi)$ (for example we need $i\Theta(L_\varphi) \geq 0$ and some control on the L^2 norm of $|g|^2 e^{-\varphi}$, see Theorem 6.2). A method is to use a non integrable weight on Y like $e^{-\varphi} = |d(x,Y)|^{-2p}$, where $d(x,Y)$ is the distance to Y and p is the codimension of Y. Then the estimates from Theorem 6.2 will gives

$$\int_X |u|^2 e^{-\varphi} dV < \infty,$$

which will make sure $u|_Y = 0$.

Suppose now $Y = \sigma^{-1}(0)$, where σ is a holomorphic section of a hermitian vector bundle E, we may replace $d(x,Y)$ by $\sigma(x)$ in above discussion. That is, consider the weight $\varphi = p\log|\sigma|^2$, which will contribute $ip\partial\bar{\partial}\log|\sigma|^2$ in the curvature. To calculate it, we define for $s = \sigma_\lambda \otimes e_\lambda \in \Lambda^p T^*X \otimes E$, $t = \tau_\mu \otimes e_\mu \in \Lambda^q T^*X \otimes E$,

$$\{s,t\} := \sigma_\lambda \wedge \overline{\tau_\mu} \otimes \langle e_\lambda, e_\mu \rangle \in \Lambda^{p+q} \otimes E$$

and we have $d\{s,t\} = \{Ds,t\} + (-1)^p\{s,Dt\}$ since Chern connection is compatible with metric. (see V-7.2 in [Dem12]). Then

$$\begin{aligned} \partial|\sigma|^2 &= \pi^{1,0}(d|\sigma|^2) = \pi^{1,0}(\{D\sigma,\sigma\} + \{\sigma, D\sigma\}) \\ &= \{D^{1,0}\sigma,\sigma\} + \{\sigma, D^{0,1}\sigma\} = \{D^{1,0}\sigma,\sigma\} \end{aligned}$$

as $D^{0,1} = \bar{\partial}$ and σ is holomorphic. Therefore $\partial\log|\sigma|^2 = \dfrac{\partial|\sigma|^2}{|\sigma|^2} = \dfrac{\{D^{1,0}\sigma,\sigma\}}{|\sigma|^2}$ and also $D^{0,1}D^{1,0}\sigma = D^2\sigma = \Theta(E)\sigma$. Then

$$\begin{aligned} i\partial\bar{\partial}\log|\sigma|^2 &= -i\bar{\partial}\partial\log|\sigma|^2 = -i\bar{\partial}\left(\frac{\{D^{1,0}\sigma,\sigma\}}{|\sigma|^2}\right) \\ &= -i\frac{-\{\sigma, D^{1,0}\sigma\} \wedge \{D^{1,0}\sigma,\sigma\}}{|\sigma|^4} - i\frac{\{D^{0,1}D^{1,0}\sigma,\sigma\}}{|\sigma|^4} + i\frac{\{D^{1,0}\sigma, D^{1,0}\sigma\}}{|\sigma|^2} \\ &= i\frac{\{D^{1,0}\sigma, D^{1,0}\sigma\}}{|\sigma|^2} - i\frac{\{D^{1,0}\sigma,\sigma\} \wedge \{\sigma, D^{1,0}\sigma\}}{|\sigma|^4} - \frac{\{i\Theta(E)\sigma,\sigma\}}{|\sigma|^4} \end{aligned} \tag{5}$$

And we have

$$i\frac{\{D^{1,0}\sigma, D^{1,0}\sigma\}}{|\sigma|^2} - i\frac{\{D^{1,0}\sigma,\sigma\} \wedge \{\sigma, D^{1,0}\sigma\}}{|\sigma|^4} \geq 0, \tag{6}$$

as

$$|\sigma|^2|\xi \lrcorner D^{1,0}\sigma|^2 - |\langle \xi \lrcorner D^{1,0}\sigma, \sigma\rangle_E|^2 \geq 0,\ \forall \xi \in T_X^{1,0}$$

by the Cauchy-Schwarz inequality.

Similarly,

$$
\begin{aligned}
i\partial\bar{\partial}\log(1+|\sigma|^2) &= \frac{i(1+|\sigma|^2)\{D^{1,0}\sigma, D^{1,0}\sigma\} - i\{D^{1,0}\sigma,\sigma\}\wedge\{\sigma, D^{1,0}\sigma\}}{(1+|\sigma|^2)^2} - \frac{\{i\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2} \\
&\geq \frac{i\{D^{1,0}\sigma, D^{1,0}\sigma\}}{(1+|\sigma|^2)^2} - \frac{\{i\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2}.
\end{aligned}
\tag{7}
$$

This turns out will be the what we use to control the contribution of bump function in curvature. Now since the weight is singular along Y, we actually want to apply the theorem to $X\backslash Y$, then we need to know whether $X\backslash Y$ has a complete metric.

Lemma 7.1 (VIII-Lem 7.2). *Let (X,ω) be a Kähler manifold, and $Y = \sigma^{-1}(0)$ an analytic subset defined by a section of a hermitian vector bundle E. If X is weakly pseudoconvex and exhausted by $X_c = \{\psi < c\}$, then $X_c\backslash Y$ has a complete Kähler metric for all $c \in \mathbb{R}$.*

Proof. We need to take care of two parts, when we approach Y and when we near ∂X_c. undone □

We can now prove the following,

Theorem 7.1 (VIII-Thm 7.1). *Let (X,ω) be a weakly pseudoconvex Kähler manifold, L a hermitian line bundle and E a hermitian vector bundle over X. Let $Y = \sigma^{-1}(0)$ for some section σ of E, and p the maximal codimension of the irreducible components of Y. Let f be a holomorphic section of $K_X \otimes E$ defined in the open set $Y \subset U = \{|\sigma| < 1\}$. If $\int_U |f|^2 dV < +\infty$ and if the curvature form of L satisfies*

$$
i\Theta(L) \geq \left(\frac{p}{|\sigma|^2} + \frac{\varepsilon}{1+|\sigma|^2}\right)\{i\Theta(E)\sigma,\sigma\}
$$

for some $\varepsilon > 0$. Then there is a section $F \in H^0(X, K_X \otimes L)$ such that $F_Y = f|_Y$ and

$$
\int_X \frac{|F|^2}{(1+|\sigma|^2)^{p+\varepsilon}} dV \leq \left(1 + \frac{p+1}{\varepsilon}\right)\int_U |f|^2 dV.
$$

Proof. Let h be the continuous section of L defined by $h = (1-|\sigma|^{p+1})f$ on U and $h = 0$ on $X\backslash U$. [7] We have $h|_Y = f|_Y$ and since f is holomorphic, the nontrivial term in $\bar{\partial}h$ only comes from the bump function. Therefore

$$
\bar{\partial}h = -\frac{p+1}{2}|\sigma|^{p-1}\{\sigma, D^{1,0}\sigma\} \otimes f \text{ on } U, \quad \bar{\partial}h = 0 \text{ on } X\backslash U.
$$

[7] We may replace σ by $(1+\eta)\sigma$ to assume f is defined in a neighborhood of $\overline{U}$, then let $\eta \to 0$. So that f is bounded and $(1-|\sigma|^{p+1})f$ will tend to 0 when approaching ∂U. undone

We consider $g = \bar{\partial} h$ as a $(n,1)$-form with values in L. And we twist the metric by weight $e^{-\varphi}$ given by

$$\varphi = p \log|\sigma|^2 + \varepsilon \log(1+|\sigma|^2).$$

Note that φ is singular along Y. The above calculation and the condition on curvature of L imply that

$$\begin{aligned} i\Theta(L_\varphi) &= i\Theta(L) + pi\partial\bar{\partial}\log|\sigma|^2 + \varepsilon i\partial\bar{\partial}\log(1+|\sigma|^2) \\ &\geq i\Theta(L) - \left(\frac{p}{|\sigma|^2} + \frac{\varepsilon}{1+|\sigma|^2}\right)\{i\Theta(E)\sigma,\sigma\} + \varepsilon\frac{i\{D^{1,0}\sigma, D^{1,0}\sigma\}}{(1+|\sigma|^2)} \\ &\geq \varepsilon\frac{i\{D^{1,0}\sigma, D^{1,0}\sigma\}}{(1+|\sigma|^2)} \geq i\varepsilon\frac{\{D^{1,0}\sigma,\sigma\}\wedge\{\sigma, D^{1,0}\sigma\}}{|\sigma|^2}(1+|\sigma|^2)^2 \geq 0. \end{aligned} \tag{8}$$

Set $\overline{\bar{\partial}(1-|\sigma|^{p+1})} = \xi = -\frac{p+1}{2}|\sigma|^{p-1}\{D^{1,0}\sigma,\sigma\} = \sum \xi_j dz^j$ [8] in a $\omega-$orthonomal basis $\frac{\partial}{\partial z^j}$ at x_0, and let $\hat{\xi} = \sum \xi_j \frac{\partial}{\partial \bar{z}^j}$ br the dual $(0,1)-$vector field (same coefficients since $\frac{\partial}{\partial z^j}$ orthonormal). Then for every L-valued $(n,1)$-form v, we find (on U)

$$|\langle\bar{\partial}h, v\rangle| = |\langle\bar{\xi}\wedge f, v\rangle| = |\langle f, \hat{\xi} \lrcorner v\rangle| \leq |f|\cdot|\hat{\xi}\lrcorner v|.$$

Now for $\hat{\xi}\lrcorner v$, we can write

$$\hat{\xi}\lrcorner v = \sum -i\xi_j dz^j \wedge \Lambda v = -i\xi\wedge\Lambda v,$$

since v is of type $(n,1)$. Then

$$\begin{aligned} |\langle\bar{\partial}h, v\rangle|^2 &\leq |f|^2|\hat{\xi}\lrcorner v|^2 = |f|^2\langle -i\xi\wedge\Lambda v, \hat{\xi}\lrcorner v\rangle \\ &= |f|^2\langle -i\bar{\xi}\wedge\xi\wedge\Lambda v, v\rangle = |f|^2\langle[i\xi\wedge\bar{\xi},\Lambda]v, v\rangle \\ &\leq \frac{(p+1)^2}{4\varepsilon}|\sigma|^{2p}(1+|\sigma|^2)^2|f|^2\langle[i\Theta(L_\varphi),\Lambda]v,v\rangle, \end{aligned}$$

since we have by Equation 6,

$$i\xi\wedge\bar{\xi} = \frac{(p+1)^2}{4}|\sigma|^{2p-2}\{D^{1,0}\sigma,\sigma\}\wedge\{\sigma, D^{1,0}\sigma\} \leq \frac{(p+1)^2}{4\varepsilon}|\sigma|^{2p}(1+|\sigma|^2)^2|f|^2 i\Theta(L_\varphi).$$

And for $\gamma \geq 0 \in \Lambda^{1,1}T^*X$, we get $\langle[\gamma,\Lambda]v,v\rangle \geq 0$ like in Theorem 6.2. Thus in the notation of previous section (see 4), the form $g = \bar{\partial}h$ satisfies

$$\langle A_{L_\varphi}^{-1}g, g\rangle \leq \frac{(p+1)^2}{4\varepsilon}|\sigma|^{2p}(1+|\sigma|^2)^2|f|^2 \leq \frac{(p+1)^2}{\varepsilon}|f|^2|\sigma|^{2p} \leq \frac{(p+1)^2}{\varepsilon}|f|^2e^{\varphi},$$

[8] There's a difference of sign compare to [Dem12].

where we use $(1+|\sigma|^2)^2 \le 4$ on $U = \{|\sigma| < 1\} \supset \operatorname{Supp} g$. Hence we have

$$\int_X \langle A^{-1}g, g\rangle_\varphi dV = \int_U \langle A^{-1}g, g\rangle e^{-\varphi} dV \le \frac{(p+1)^2}{\varepsilon} \int_U |f|^2 dV < \infty.$$

Then Lemma 7.1 shows that Theorem 6.1 can be applied on each set $X_c \backslash Y$. Let c tend to infinity and taking the weak limit like before, we then get a L-valued $(n,0)$-form u such that $\bar\partial u = g$ on $X \backslash Y$ and

$$\int_{X\backslash Y} \frac{|u|^2}{|\sigma|^{2p}(1+|\sigma|^2)^\varepsilon} dV = \int_{X\backslash Y} |u|^2 e^{-\varphi} dV \le \frac{(p+1)^2}{\varepsilon} \int_U |f|^2 dV$$

In particular, we have $\dfrac{|u|^2}{|\sigma|^{2p}}$ is locally L^1 near Y. Now as g is continuous almost everywhere, Lemma 7.2 below shows that the equality $\bar\partial u = g = \bar\partial h$ extends to X, thus $F = h - u$ is holomorphic everywhere. Thus $u = h - F$ is continuous on X, and as $\sigma(x) \le C d(x, Y)$ in a neighborhood of every point of Y, we see that $|\sigma|^{-2p}$ is non integrable at every point $x_0 \in Y_{\text{reg}}$ since $\operatorname{codim} Y \le p$. It follows that $u = 0$ on Y, so

$$F|_Y = h|_Y = f|_Y.$$

Finally, we have

$$|F|^2 = |h-u|^2 \le (1+|\sigma|^{-2p})|u|^2 + (1+|\sigma|^{2p})|f|^2 \textit{undone}$$

which implies

$$\frac{|F|^2}{(1+|\sigma|^2)^p} \le \frac{|u|^2}{|\sigma|^{2p}} + |f|^2$$

since

$$1 + |\sigma|^{2p} \le (1+|\sigma|^2)^p.$$

So

$$\int_X \frac{|F|^2}{(1+|\sigma|^2)^{p+\varepsilon}} dV \le \int_X \frac{|u|^2}{|\sigma|^{2p}(1+|\sigma|^2)^\varepsilon} + \frac{|f|^2}{(1+|\sigma|^2)^\varepsilon} dV \le \left(1 + \frac{p+1}{\varepsilon}\right) \int_U |f|^2 dV.$$

□

Lemma 7.2 (VIII-Lem 7.3). *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p, q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $\bar\partial v = w$ on $\Omega\backslash Y$ (in the sense of distribution theory). Then $\bar\partial v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it suffices to prove the result in a neighborhood of a regular point $a \in Y$. By using local isomorphism, we reduced to the case where Y is contained in the hyperplane $z^1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function with $\lambda(t) = 0$ for $t \leq \frac{1}{2}$ and $\lambda(t) = 1$ for $t \geq 1$. We must show that

$$\int_\Omega w \wedge \alpha = (-1)^{p+q} \int_\Omega v \wedge \bar{\partial}\alpha \tag{9}$$

for all $\alpha \in \mathscr{C}^\infty_{n-p,n-q}(\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(\frac{|z^1|}{\varepsilon})$ and replace α in the integral by $\lambda_\varepsilon \alpha$. Then $\lambda_\varepsilon \alpha \in \mathscr{C}^\infty_{n-p,n-q}(\Omega \backslash Y)$ and we have

$$\int_\Omega w \wedge \lambda_\varepsilon \alpha = (-1)^{p+q} \int_\Omega v \wedge \bar{\partial}(\lambda_\varepsilon \alpha) = (-1)^{p+q} \int_\Omega v \wedge (\bar{\partial}\lambda_\varepsilon \alpha + \lambda_\varepsilon \bar{\partial}\alpha).$$

As w, v has L^1_{loc} coefficients on Ω,

$$\int_\Omega w \wedge \lambda_\varepsilon \alpha \to \int_\Omega w \wedge \alpha, \quad \int_\Omega v \wedge \lambda_\varepsilon \bar{\partial}\alpha \to \int_\Omega v \wedge \bar{\partial}\alpha \quad \text{as } \varepsilon \to 0.$$

The remaining term can be estimated by Cauchy-Schwarz inequality:

$$\left| \int_\Omega v \wedge \bar{\partial}\lambda_\varepsilon \wedge \alpha \right|^2 \leq \int_{|z^1| \leq \varepsilon} |v \wedge \alpha|^2 dV \cdot \int_{\operatorname{Supp} \alpha} |\bar{\partial}\lambda_\varepsilon|^2 dV;$$

as $v \in L^2_{loc}(\Omega)$, then

$$\int_{|z^1| \leq \varepsilon} |v \wedge \alpha|^2 dV \to 0$$

as $\varepsilon \to 0$, whereas

$$\int_{\operatorname{Supp} \alpha} |\bar{\partial}\lambda_\varepsilon|^2 dV \leq \frac{C}{\varepsilon^2} \operatorname{Vol}(\operatorname{Supp} \alpha \cap \{|z^1| \leq \varepsilon\}) \leq C''.$$

Hence Equation 9 follows when ε tends to 0. □

Corollary 7.1 (VIII-Cor. 7.5). *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and let φ, ψ be plurisubharmonic functions on Ω, where ψ is finite and continuous. Let $\sigma = (\sigma_1, \dots, \sigma_r)$ be a family of holomorphic functions on Ω, let $Y = \sigma^{-1}(0)$, p be the maximal codimension of Y and set*

1. *$U = \{z \in \Omega; |\sigma(z)|^2 < e^{-\psi(z)}\}$,*
2. *$U' = \{z \in \Omega; |\sigma(z)|^2 < e^{\psi(z)}\}$.*

Then for every $\varepsilon > 0$ and every holomorphic function f on U (resp. U'), there exists a holomorphic function F on Ω such that $F|_Y = f|_Y$ and

1. $$\int_\Omega \frac{|F|^2 e^{-\varphi + p\psi}}{(1+|\sigma|^2 e^{\psi})^{p+\varepsilon}} dV \leq \left(1 + \frac{(p+1)^2}{\varepsilon}\right) \int_U |f|^2 e^{-\varphi+p\psi} dV,$$
2. $$\int_\Omega \frac{|F|^2 e^{-\varphi}}{(e^{\psi}+|\sigma|^2)^{p+\varepsilon}} dV \leq \left(1 + \frac{(p+1)^2}{\varepsilon}\right) \int_{U'} |f|^2 e^{-\varphi-+(p+\varepsilon)\psi} dV.$$

Proof. Assume φ, ψ smooth[9]. Either case will follows when we apply Theorem 7.1 to

1. $E = \Omega \times \mathbb{C}^r$ with the weight e^{ψ}, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi+p\psi}$, and $U = \{|\sigma|^2 e^{\psi} < 1\}$. Then
$$i\Theta(E) = -i\partial\bar{\partial}\psi \otimes \mathrm{Id}_E \leq 0, \quad i\Theta(L) = i\partial\bar{\partial}\varphi - pi\partial\bar{\partial}\psi \geq pi\Theta(E).$$
2. $E = \Omega \times \mathbb{C}^r$ with the weight $e^{-\psi}$, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi-(p+\varepsilon)\psi}$, and $U' = \{|\sigma|^2 e^{-\psi} < 1\}$. Then
$$i\Theta(E) = -i\partial\bar{\partial}\psi \otimes \mathrm{Id}_E \geq 0, \quad i\Theta(L) = i\partial\bar{\partial}\varphi + (p+\varepsilon)i\partial\bar{\partial}\psi \geq (p+\varepsilon)i\Theta(E).$$

Then the curvature condition is satisfied and K_X is trivial. □

Theorem 7.2 (Hörmander-Bombieri-Skoda theorem, VIII-Thm 7.6). *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and φ a plurisubharmonic function on Ω. For every $\varepsilon > 0$ and every point $z_0 \in \Omega$ auch that $e^{-\varphi}$ is integrable in a neighborhood of z_0, there exists a holomorphic function F on Ω such that $F(z_0) = 1$ and*

$$\int_\Omega \frac{|F(z)|^2 e^{-\varphi(z)}}{(1+|z|^2)^{n+\varepsilon}} dV < \infty.$$

Proof. Apply Corollary 7.1 to $f \equiv 1$, $\sigma(z) = z - z_0$, $p = n$ and $\psi = \log r^2$ where $U = B(z_0, r)$ is a ball such that $\int_U e^{-\varphi} dV < \infty$. □

Corollary 7.2. *Let φ be a plurisubharmonic function on a complex manifold X. Let A be the set of points $z \in X$ such that $e^{-\varphi}$ is not locally integrable in a neighborhood of z. Then A is an analytic subset of X.*

Proof. undone □

[9] By taking convolution with smooth kernels on the pseudoconvex domain $\Omega_c \subset \Omega$.

References

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