Unobstructedness of deformations of smooth Calabi-Yau varieties

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Contents

In this report, our final goal is to present the unobstructedness of deformations of smooth Calabi–Yau varieties via Ran-Kawamata T^1-lifting theorem (Theorem 2.7). Before doing so, we require the foundational terminology and framework established by Schlessinger in [Sch68]. The preliminary of deformation theory, particularly the following topics:

- $\mathcal{T}^i$ functor (for the smooth case, only $\mathcal{T}^0$ relevant, as $\mathcal{T}^1$ and $\mathcal{T}^2$ vanish),
- the deformations of smooth schemes,
- obstruction for extensions of deformations, the exact sequences of deformations, and automorphisms of deformation (cf. Theorem B.5 and Remark B.6)

will be included in the Appendix. Throughout this report, we assume everything [Har77].

1 Functors of artin rings and Schlessinger's criterion

In this section, we adopt notations in [Sch68].

1.1 The category $\mathcal{C}_\Lambda$

Let Λ be a complete noetherian local ring, μ be its maximal ideal, and $k = \Lambda/\mu$ be the residue field. Let $\mathcal{C} = \mathcal{C}_\Lambda$ be the category of artinian local Λ-algebras having residue field k induces from structure map $\Lambda \to A$. Morphisms in $\mathcal{C}$ are local homomorphisms of Λ-algebras. Let $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_\Lambda$ be the category of complete noetherian local Λ-algebras A for which $A/\mathfrak{m}^n \in \mathcal{C}$ for all n. Notice that $\mathcal{C}$ is a full subcategory of $\widehat{\mathcal{C}}$.

If $p : A \to B$, $q : C \to B$ are morphisms in $\mathcal{C}$, let

$$A \times_B C \coloneqq \{(a, c) \in A \times C \mid p(a) = q(b)\} \in \mathcal{C}$$

For any $A \in \mathcal{C}$, let

$$t_A^* \coloneqq {}^{\mathfrak{m}}\!/_{\mathfrak{m}^2 + \mu A}$$

be the **Zariski cotangent space** of A over Λ, or abbreviate as t_A^*. It is clear that the dual vector space t_A is isomorphic to $\mathrm{Der}_\Lambda(A, k)$.

Lemma 1.1

A morphism $B \to A$ in $\widehat{\mathcal{C}}$ is surjective if and only if the induced map $t_B^* \to t_A^*$ is surjective.

Proof. Notice that A as the Λ-module is generated by $\mathfrak{m}_A$ and the image of Λ in A, since $\Lambda/\mu \simeq A/\mathfrak{m}_A$. Thus, the induced map $\mu/\mu^2 \to \mu A/(\mathfrak{m}_A^2 \cap \mu A)$ is surjective. If $B \to A$ is a morphism in $\widehat{\mathcal{C}}$, then we get commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu A/(\mu A + \mathfrak{m}_A^2) & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & t_A^* & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mu B/(\mu B + \mathfrak{m}_B^2) & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & t_B^* & \longrightarrow & 0.
\end{array}$$

If $t_B^* \to t_A^*$ is surjective, then $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. By induction, we have $B/\mathfrak{m}_B^k \to A/\mathfrak{m}_A^k$ is surjective for all k, and thus

$$B \simeq \varprojlim B/\mathfrak{m}_B^k \twoheadrightarrow \varprojlim A/\mathfrak{m}_A^k \simeq A.$$

Conversely, if $B \to A$ is surjective, then it is clear that $t_B^* \to t_A^*$ is surjective. □

Definition 1.2. Let $p : B \to A$ be a surjection in $\mathcal{C}$.

- p is a **small extension** if $\ker p = (t) \neq 0$ such that $\mathfrak{m}_B t = (0)$, namely, $\ker p$ is a k-vector space of dimension 1.
- p is **essential** if for any morphism $q : C \to B$ in $\mathcal{C}$ such that pq is surjective will implies q is surjective.

Lemma 1.3

Let $p : B \to A$ be a surjection in $\mathcal{C}$. Then

(1) p is essential if and only the induced map $p_* : t_B^* \to t_A^*$ is an isomorphism.

(2) If p is a small extension, then p is not essential if and only if p has a section $s : A \to B$ with $ps = 1_A$.

Proof.

(1) If p_* is an isomorphism, then p is essential by Lemma 1.1. Conversely, let $\{\widetilde{t}_1, \ldots, \widetilde{t}_r\}$ be a basis of t_A^* and lift the $\widetilde{t}_i$ back to elements t_i in B. Set

$$C = \Lambda[t_1, \ldots, t_r] \subseteq B.$$

Then p induces a surjection from C to A, and thus $C = B$ by p is essential. This forces

$$\dim_k t_B^* \leq r = \dim_k t_A^* \implies t_B^* \simeq t_A^*.$$

(2) If p has a section s, then s is not surjective implies p is not essential. If p is not essential, then the subring C constructed above is a proper subring of B. Since p is a small extension, we have $\text{length}(B) = \text{length}(A) + 1$, and thus $C \simeq A$ yields the section.

□

1.2 Functors on $\mathcal{C}$

We shall consider only covariant functors F from $\mathcal{C}$ to Sets such that $|F(k)| = 1$. A **couple** for F means a pair (A, ξ), where $A \in \mathcal{C}$ and $\xi \in F(A)$. A morphism of couples $u : (A, \xi) \to (A', \xi')$ is a morphism $u : A \to A'$ in $\mathcal{C}$ such that $F(u)(\xi) = \xi'$. If we extend F to $\widehat{\mathcal{C}}$ by

$$\widehat{F}(A) = \varprojlim F(A/\mathfrak{m}^n),$$

then we may speak analogously of **pro-couples** and morphisms of pro-couples.

For any ring R in $\widehat{\mathcal{C}}$, we define a functor $h_R = \operatorname{Hom}(R, -)$ on $\mathcal{C}$. For any functor F on $\mathcal{C}$, we have a canonical bijection

$$\widehat{F}(R) \xrightarrow{\sim} \operatorname{Hom}(h_R, F).$$

Indeed, notice that every morphism $u : R \to A$ factors through $u_n : R/\mathfrak{m}_R^n \to A$ for some n, since A is an artinian local ring implies $\sqrt{\ker u} = m_R$. For any $\xi = \varprojlim \xi_n$ in $\widehat{F}(R)$, we may assign $u \in h_R(A)$ to the element $F(u_n)(\xi_n) \in F(A)$, which is well-defined since ξ_n form a inverse system. Conversely, let $\pi_n : R \to R/\mathfrak{m}_R^n$ be the canonical projection. For any $\eta \in \operatorname{Hom}(h_R, F)$, $\xi_n = \eta(R/\mathfrak{m}_R^n)(\pi_n)$ form a inverse system and obtains $\xi = \varprojlim \xi_n \in \widehat{F}(R)$. It is clear that this two map give a canonical bijection.

Definition 1.4. A pro-couple (R, ξ) for F is **pro-represents** F if the morphism $h_R \to F$ induced by ξ is an isomorphism.

Example 1.5

Let G be a contravariant functor on the category of scheme over $\operatorname{Spec} \Lambda$ and fixed $e \in G(\operatorname{Spec} k)$. For $A \in \mathcal{C}$, define

$$F(A) = \{\xi \in G(\operatorname{Spec} A) \mid G(i)(\xi) = e\},$$

where $i : \operatorname{Spec} k \to \operatorname{Spec} A$ induces by residue field. If G is represented by a scheme X, then e determines a k-rational point $x \in X$, and it is clear that $F(a) = \operatorname{Hom}_\Lambda(\mathcal{O}_{X,x}, A)$. Thus the completion of $\mathcal{O}_{X,x}$ is pro-represents F.

Unfortunately, many interesting functors are not pro-representable. However, one can still look for a "universal object" in some sense.

Definition 1.6. A morphism $F \to G$ of functors is **smooth** if for any surjection $B \to A$ in $\mathcal{C}$, the morphism

$$F(B) \to F(A) \times_{G(A)} G(B) \tag{1}$$

Remark 1.7.

(1) It is enough to check surjectivity in (1) for small extensions $B \to A$.

(2) If $F \to G$ is smooth, then $\widehat{F} \to \widehat{G}$ is surjective, in the sense that $\widehat{F}(A) \to \widehat{G}(A)$ is surjective for all A in $\widehat{\mathcal{C}}$.

Proposition 1.8

(1) Let $R \to S$ be a morphism in $\widehat{\mathcal{C}}$. Then $h_S \to h_R$ is smooth if and only if S is a power series ring over R.

(2) If $F \to G$ and $G \to H$ are smooth morphism of functors, then the composition

$F \to H$ is smooth.

(3) If $u : F \to G$ and $v \in G \to H$ are morphisms of functors such that u is surjective and vu is smooth, then v is smooth.

(4) If $F \to G$ and $H \to G$ are morphisms of functors such that $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.

Proof.

(1) Pick $x_1, \dots, x_n$ in S which induce a basis of $t^*_{S/R} = \mathfrak{m}_S/(\mathfrak{m}_S^2 + \mathfrak{m}_R S)$, then

$$\mathfrak{m}_S = \mathfrak{m}_S^2 + \mathfrak{m}_R S + \sum S x_i \implies \mathfrak{m}_S = \mathfrak{m}_R S + \sum S x_i$$

by Nakayama's lemma. Set $T = R[[X_1, \dots, X_n]]$ and define a morphism of local R-algebra $u_1 : S \to T/(\mathfrak{m}_T^2 + \mathfrak{m}_R T)$ by $x_i \mapsto \overline{X}_i$. We check that u_1 is well-defined. Notice that

$$S = R \cdot 1_S + \mathfrak{m}_S = R \cdot 1_S + \mathfrak{m}_R(R \cdot 1_S + \mathfrak{m}_R S) + \sum (R \cdot 1_S + \mathfrak{m}_R S) x_i,$$

and $\mathfrak{m}_R^2 S + \sum \mathfrak{m}_R S x_i$ maps to $\mathfrak{m}_T^2 + \mathfrak{m}_R T$ under $x_i \mapsto X_i$. Suppose that

$$0 = r \cdot 1_S + \sum r_i x_i + \lambda$$

for some r, $r_i \in R$ and $\lambda \in (\mathfrak{m}_R^2 S + \sum \mathfrak{m}_R x_i)$. Then $r \cdot 1_S \in \mathfrak{m}_S$, and thus $r \in \mathfrak{m}_R$. Then

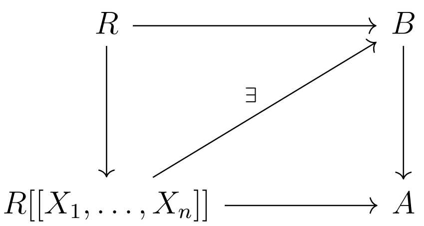
$$\sum r_i x_i \equiv 0 \pmod{\mathfrak{m}_S^2 + \mathfrak{m}_R S} \implies r_i \in \mathfrak{m}_R,$$

and thus

$$u_1(r \cdot 1_S + \sum r_i x_i + \lambda) = \sum r_i \overline{X}_i = 0.$$

By smoothness, we may lift u_1 to $u_2 : S \to T/\mathfrak{m}_T^2$. By induction, we may lift u_1 to $u_k : S \to T/\mathfrak{m}_T^k$ for all $k \in \mathbb{N}$, and thus we get $u : S \to T = \varprojlim T/\mathfrak{m}_T^k$ which induces an isomorphism of $t^*_{S/R}$ with $t^*_{T/R}$ by the choice of u_1. By Lemma 1.1, u is surjective. Choose $y_i \in S$ such that $u(y_i) = X_i$. Then $X_i \mapsto y_i$ define a morphism of local R-algebra $v : T \to S$ such that $uv = \mathrm{id}_T$. In particular, v is injective. Since v induces a bijection on the cotangent spaces, it follows that v is surjective by Lemma 1.1. Hence, v induces an isomorphism of $T = R[[X_1, \dots, X_n]]$ with S.

Conversely, if S is a power series ring over R, then it is obvious that $h_S \to h_R$ is smooth.



(2) For any surjection $B \to A$ in $\mathcal{C}$, we have the following diagram

$$\begin{array}{ccccc} F(B) & & & & \\ & \searrow\!\!\!\!\searrow & & & \\ & & F(A) \times_{G(A)} G(B) & \longrightarrow & G(B) \\ & & \downarrow & \lrcorner & \downarrow\!\!\!\downarrow \\ & & F(A) \times_{H(A)} H(B) & \longrightarrow & G(A) \times_{H(A)} H(B), \end{array}$$

and thus $F(B) \to F(A) \times_{H(A)} H(B)$ is surjective.

(3) For any surjection $B \to A$ in $\mathcal{C}$. Since $F(A) \to G(A)$ is surjective, we have

$$F(A) \times_{H(A)} H(B) \twoheadrightarrow G(A) \times_{H(A)} H(B)$$

Since vu is smooth, we have $F(B) \twoheadrightarrow F(A) \times_{H(A)} H(B)$. Hence,

$$F(B) \twoheadrightarrow G(A) \times_{H(A)} H(B).$$

(4) For any surjection $B \to A$ in $\mathcal{C}$, we have the following diagram

$$\begin{array}{ccccc} F(B) \times_{G(B)} H(B) & \longrightarrow & F(A) \times_{G(A)} H(B) & \longrightarrow & H(B) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ F(B) & \twoheadrightarrow & F(A) \times_{G(A)} G(B) & \longrightarrow & G(B) \\ & & \downarrow & \lrcorner & \downarrow \\ & & F(A) & \longrightarrow & G(A) \end{array}$$

Hence,

$$F(B) \times_{G(B)} H(B) \twoheadrightarrow F(A) \times_{G(A)} H(B) \simeq (F(A) \times_{G(A)} H(A)) \times_{H(A)} H(B)$$

implies $F \times_G H \to H$ is smooth.

□

The **ring of dual numbers** over k is denoted by $k[\varepsilon]$, where $\varepsilon^2 = 0$. For any functor F, the set $F(k[\varepsilon])$ is called the **tangent space** to F, and is denoted by t_F. If $F = h_R$, then there is a canonical isomorphism $t_F \simeq t_R$ given by

$$\begin{array}{ccc} \mathrm{Hom}_\Lambda(R, k[\varepsilon]) & \longrightarrow & \mathrm{Der}_\Lambda(R, k) \simeq t_R \\ f & \longmapsto & \pi \circ f, \end{array}$$

where $\pi : k[\varepsilon] \to k$ defined by $\pi(a + b\varepsilon) = b$.

Definition 1.9. A pro-couple (R, ξ) for a functor F is called a **pro-representable hull** of F (or abbreviated as **hull** of F), if the induced map $h_R \to F$ is smooth and $t_R \to t_F$ of tangent spaces is a bijection.

If (R, ξ) pro-represents F, then (R, ξ) is a hull of F. In this case, (R, ξ) is unique up to canonical isomorphism. In general, we only have non-canonical isomorphism in the following proposition.

Proposition 1.10

Let (R, ξ) and (R', ξ') be hulls of F. Then there exists an isomorphism $u : R \to R'$ such that $F(u)(\xi) = \xi'$.

Proof. Since $h_R \to F$ is smooth, by Remark 1.7 we have the surjection

$$\begin{array}{rcl} \mathrm{Hom}(R, R') \simeq \widehat{h}_R(R') & \twoheadrightarrow & \widehat{F}(R') \\ u & \longrightarrow & \varprojlim F(\overline{u}_n)(\xi_n), \end{array}$$

where $\overline{u}_n : R/\mathfrak{m}_R^n \to R'/\mathfrak{m}_{R'}^n$ induces from $u : R \to R'$. So there exists $u \in \mathrm{Hom}(R, R')$ such that $\xi' = \varprojlim F(\overline{u}_n)(\xi_n)$, that is, $u : (R, \xi) \to (R', \xi')$ define a morphism of pro-couple. Then u induces the morphism on tangent space

$$\begin{array}{ccccc} \mathrm{Hom}_\Lambda(R', k[\varepsilon]) & \longrightarrow & \mathrm{Hom}_\Lambda(R, k[\varepsilon]) & \longleftrightarrow & F(k[\varepsilon]) \\ & & g & \longrightarrow & F(g_2)(\xi_2) \\ f & \longrightarrow & f \circ u & \longrightarrow & F(f_2 \circ u_2)(\xi_2) = F(f_2)(\xi_2') \end{array}$$

and the bijection is given by (R, ξ) is a hull of F. Since (R', ξ') is a hull of F, the map $f \mapsto F(f_2)(\xi_2')$ is bijection, and thus u induces isomorphism between tangent space. By Lemma 1.1, u is surjective. Similarly, we have a surjective morphism $u' : (R', \xi') \to (R, \xi)$. Then $u'u : R \to R$ is a surjective endomorphism.

Claim. The surjective endomorphism of a Noetherian ring A is an isomorphism.

subproof. Let $f \in \mathrm{End}(A)$ and $I = \ker f$. Then

$$0 \to I \otimes_A \kappa(\mathfrak{m}) \to \kappa(\mathfrak{p}) \xrightarrow{f} \kappa(\mathfrak{p}) \to 0$$

as $\kappa(\mathfrak{p})$-module for all $\mathfrak{p} \in \mathrm{Spec}\, A$. By Nakayama's lemma, $0 = I \otimes_A \kappa(\mathfrak{p}) = I_\mathfrak{p} \otimes A_\mathfrak{p}/\mathfrak{p}$ implies $I_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\, A$. Hence, $I = 0$. □

By Claim, we conclude that $u'u$ and uu' are isomorphisms, and thus $u : R \to R'$ as required. □

Remark 1.11. Let (R, ξ) be a hull of F. Then R is a power series ring over Λ if and only if F transforms surjections $B \to A$ in C into surjections $F(B) \to F(A)$. Indeed, the stated condition on F is equivalent to the smoothness of the natural transform $F \to h_\Lambda$. Applying

Proposition 1.8 (2) and (3) to the diagram

$$\begin{array}{ccc} h_R & \longrightarrow & h_\Lambda \\ & {\scriptstyle \text{sm.}}\searrow \quad \nearrow & \\ & F & \end{array}$$

we conclude that $h_R \to h_\Lambda$ is smooth if and only if $F \to h_\Lambda$ is (notice that $h_R \to F$ is surjective by Remark 1.7).

Lemma 1.12

Suppose F is a functor such that the canonical map

$$F(k[V] \times_k k[W]) \longrightarrow F(k[V]) \times F(k[W])$$

is bijective for every k-vector spaces V and W, where $k[V]$ denotes the ring $k \oplus V$ such that V is a square zero ideal. Then $F(k[V])$ has a canonical vector space structure, such that $F(k[V]) \simeq t_F \otimes V$ as k-vector space.

Proof. Define the addition map $k[V] \times_k k[V] \to k[V]$ by $(x, y) \mapsto x + y$ for $x, y \in V$, and the scalar multiplication $x \mapsto ax$ $(a \in k)$ on V. Since F is commutes with the necessary products, $F(k[V])$ inherent a k-vector space structure. Under the identification,

$$\begin{aligned} \operatorname{Hom}(k[\varepsilon], k[V]) &\longrightarrow V \\ f &\longrightarrow f(\varepsilon) \end{aligned}$$

we get a map

$$\begin{aligned} t_F \otimes V &\longrightarrow F(k[V]) \\ \xi \otimes f &\longrightarrow F(f)(\xi) \end{aligned}$$

which is an k-vector space isomorphism, since $k[V]$ is the product of $\dim V$ copies of $k[\varepsilon]$. □

Remark 1.13. Suppose F is a functor in Lemma 1.12. For any $R \in \widehat{\mathcal{C}}$ and $\eta \in \operatorname{Hom}(h_R, F)$, by functoriality, the map $\eta(k[V])$ is a k-vector space homomorphism.

Theorem 1.14

Let $F : \mathcal{C} \to \mathsf{Sets}$ be a functor such that $F(k) = \{e\}$ consists of one element. Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the canonical map

$$F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A''). \tag{2}$$

Then

(1) F has a hull if and only if F has properties (H_1), (H_2), (H_3) below:

(H$_1$) (2) is a surjection whenever $A'' \to A$ is a small extension.

(H$_2$) (2) is a bijection when $A = k$, $A'' = k[\varepsilon]$.

(H$_3$) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property (H$_4$):

$$F(A' \times_A A') \xrightarrow{\sim} F(A') \times_{F(A)} F(A') \tag{3}$$

for any small extension $A' \to A$.

As preparation for the proof, we investigate what these conditions entail.

(a) If F is isomorphic to some h_R, then (2) is an isomorphism for any morphism $A' \to A$, $A'' \to A$ by the universal property of fiber product.

(b) By (H$_2$) and induction, it is clear that the hypothesis of Lemma 1.12 holds, and thus t_F is a vector space.

(c) By induction on $\text{length}_{A''} A$, it follows from (H$_1$) that (2) is surjective for any surjection $A'' \to A$.

(d) Let I be the kernel of small extension $A' \to A$, then we have an isomorphism

$$\begin{aligned} A' \times_A A' &\xrightarrow{\sim} A \times_k k[I] \\ (x, y) &\longrightarrow (x, \overline{x} + y - x), \end{aligned}$$

where $\overline{x} \in k$ and $y - x \in I$. Then (H$_2$) induces

$$\begin{array}{ccc} F(A' \times_k k[I]) & \xrightarrow{\sim} & F(A' \times_A A') \\ \downarrow \wr & & \downarrow \\ F(A') \times (t_F \otimes I) & \overset{\exists}{\dashrightarrow} & F(A') \times_{F(A)} F(A') \end{array}$$

a $t_F \otimes I$ action on $F(A')$. For any $\eta \in F(A)$, $t_F \otimes I$ acts on the subset $F(p)^{-1}(\eta) \subseteq F(A')$. Then (H$_1$) implies this action is transitive, while (H$_4$) is precisely the condition that $t_F \otimes I$ action make $F(p)^{-1}(\eta)$ be a formally principal homogeneous space.

Proof. Suppose that F satisfies (H$_1$), (H$_2$), (H$_3$). Let $r = \dim t_F$ and set $S = \Lambda[[T_1, \ldots, T_r]]$. We will construct R as the inverse limit of successive quotients R_n of S. To begin,

$$R_2 = S/(\mathfrak{m}_S^2 + \mu S) \simeq k[\varepsilon] \times_k \cdots \times_k k[\varepsilon] \quad (r \text{ times}).$$

By (H$_2$), we have

$$\begin{array}{ccccccc} \operatorname{Hom}(h_{R_2}, F) & \xrightarrow{\sim} & F(R_2) & \xrightarrow{\sim} & \prod F(k[\varepsilon]) & \xrightarrow{\sim} & \prod \operatorname{Hom}(h_{k[\varepsilon]}, F) \\ \eta & & \longrightarrow & & \big((\eta \circ \pi_i^*)(k[\varepsilon])(\text{id}_{k[\varepsilon]})\big)_i & \longrightarrow & (\eta \circ \pi_i^*)_i \end{array}$$

where $\pi_i : R_2 \to k[\varepsilon]$ is projection to i-th component. Since $\eta(k[\varepsilon])$ is a k-vector space homomorphism between two k-vector space of dimension r, it follows that $\eta(k[\varepsilon])$ is bijective if and only if $\left\{(\eta \circ \pi_i^*)(k[\varepsilon])(\mathrm{id}_{k[\varepsilon]}) \middle| 1 \le i \le r\right\}$ form a basis of t_F. Hence, there exists $\xi_2 \in F(R_2)$ which induces a bijection between t_{R_2} and t_F.

Suppose we have found (R_q, ξ_q), where $R_q = S/J_q$. Let $\mathscr{S}$ be the set of ideals J in S such that $\mathfrak{m}_S J_q \subseteq J \subseteq J_q$ and ξ_q can be lifted to S/J. We claim that $\mathscr{S}$ has the minimal element. Notice that $J_q \in \mathscr{S}$ and $J \in \mathscr{S}$ can be regard as a subset of finite dimensional k-vector space $J_q/\mathfrak{m}_S J_q$, so it suffices to show that $\mathscr{S}$ is stable under pairwise intersection. Suppose that J and J' are in $\mathscr{S}$, enlarge J without changing the intersection $J \cap J'$ if necessary, we may assume $J + J' = J_q$. Apply (H_1) on

$$S/J \times_{S/J_q} S/J' \simeq S/(J \cap J'),$$

we conclude that $J \cap J' \in \mathscr{J} \in \mathscr{S}$. Let J_{q+1} be the minimal element in $\mathscr{S}$ and pick any lifting $\xi_{q+1} \in F(R_{q+1})$ of $\xi_q \in F(R_q)$.

Let $J = \bigcap_{n \ge 2} J_n$ and $R = S/J$. Since $\mathfrak{m}_S^q \subseteq J_q$, it follows that $\{J_n/J\}_{n \ge 2}$ form a base for topology in R. Hence, $R = \varprojlim S/J_q$, and $\xi := \varprojlim \xi_n \in \widehat{F}(R)$ is defined. By our choice of R_2, we have $t_F \simeq t_R$.

Claim. $h_R \to F$ is smooth.

subproof. Let $p : (A', \eta') \to (A, \eta)$ be a morphism of couples of F, where p is a small extension and let $A = A'/I$. To lift the given morphism $u : (R, \xi) \to (A, \eta)$ to $(R, \xi) \to (A', \eta')$, it suffices to find a $u' : R \to A'$ such that $pu' = u$. Indeed, we have a transitive action of $t_F \otimes I$ on $F(p)^{-1}(\eta)$. Given such a u', there exists $\sigma \in t_F \otimes I$ such that $(F(u')(\xi))^\sigma = \eta'$. Since the action is functorial and $t_R \simeq t_F$, the following diagram

$$\begin{array}{ccc} h_R(A') \otimes (t_R \otimes I) & \longrightarrow & h_R(A') \times_{h_R(A)} h_R(A') \\ \downarrow & & \downarrow \\ F(A') \otimes (t_F \otimes I) & \longrightarrow & F(A') \times_{F(A)} F(A') \end{array}$$

implies $v' := (u')^\sigma$ will satisfy $F(v')(\xi) = \eta'$ and $pv' = u$.

Now, u factors as $(R, \xi) \to (R_n, \xi_n) \to (A, \eta)$ for some n. It suffices to complete the diagram

$$\begin{array}{ccc} R_{n+1} & \overset{\exists}{\dashrightarrow} & A' \\ \downarrow & & \downarrow p \\ R_n & \longrightarrow & A \end{array}$$

or equivalently, the diagram

$$\begin{array}{ccc} \Lambda[[T_1, \ldots, T_r]] = S & \overset{w}{\longrightarrow} & R_q \times_A A' \\ \downarrow & \overset{v}{\nearrow} & \downarrow \mathrm{pr}_1 \\ R_{q+1} & \longrightarrow & R_q \end{array}$$

where w has been chosen to make the square commute. If the small extension pr_1 has a section, then v obviously exists. Otherwise, by Lemma 1.3, pr_1 is essential, and thus w is surjective. Apply (H_1) on $R_n \times_A A'$, $(\xi_n, \eta') \in F(R_n) \times_{F(A)} F(A')$ lifts to $R_n \times_A A'$. By our choice of J_{n+1}, $\ker w \supseteq J_{n+1}$, and thus w factors through $S/J_{n+1} = R_{n+1}$. This completes the proof that (R, ξ) is a hull of F. □

Conversely, suppose that (R, ξ) is a hull of F. To verify (H_1), let $p' : (A', \eta') \to (A, \eta)$ and $p'' : (A'', \eta'') \to (A, \eta)$ be morphisms of couples, where p'' is surjective. Since $h_R \to F$ is surjective, there exists a $u' : (R, \xi) \to (A', \eta')$. By smoothness, there exists $u'' : (R, \xi) \to (A'', \eta'')$. Consider $u' \times u'' : R \to A' \times_A A''$, then $\zeta := F(u' \times u'')(\xi)$ projects onto η' and η''. Hence, (H_1) is satisfied.

For the case of $(A, \eta) = (k, e)$ and $A'' = k[\varepsilon]$. If ζ_1, $\zeta_2 \in F(A' \times_k k[\varepsilon])$ have same projections η' and η'' on $F(A')$ and $F(k[\varepsilon])$, respectively. Let $u' : (R, \xi) \to (A', \eta')$ as above. Apply smoothness to the projection $A' \times_k k[\varepsilon] \to A'$, we get morphisms

$$u' \times u_i : (R, \xi) \longrightarrow (A' \times_k k[\varepsilon], \zeta_i)$$

for $i = 1$, 2, that is, $u_i \in h_R(k[\varepsilon])$ send to η'' via ξ under the isomorphism

$$\xi \in \widehat{F}(R) \simeq \mathrm{Hom}(h_R, F).$$

Since (R, ξ) is a hull of F, ξ induces an isomorphism on $t_R \xrightarrow{\sim} t_F$, and thus $u_1 = u_2$. Therefore, $\zeta_1 = \zeta_2$, which proves (H_2). From $t_R \simeq t_F$, we also have (H_3).

For part (2), suppose that F satisfies $(\mathsf{H}_1), \ldots, (\mathsf{H}_4)$. By part (1), there exists a hull (R, ξ) of F. We prove that $h_R(A) \simeq F(A)$ by induction on $\mathrm{length}_\Lambda(A)$. Consider a small extension $p : A' \to A = A'/I$, and assume that $h_R(A) \xrightarrow{\sim} F(A)$. For each $\eta \in F(A)$, both $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are formally principal homogeneous spaces under $t_F \otimes I$. Since $h_R(A')$ surjects to $F(A')$, we conclude that $h_R(A') \xrightarrow{\sim} F(A')$. □

1.3 Picard functor

If X is scheme, we define $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. Recall that the group of automorphisms of an invertible sheaf is canonically isomorphic to $H^0(X, \mathcal{O}_X^*)$.

Suppose that X is a scheme over $\mathrm{Spec}\,\Lambda$. Abbreviate $X \times_{\mathrm{Spec}\,\Lambda} \mathrm{Spec}\,A$ as X_A for any A in $\mathcal{C}$, and set $X_0 = X_k$. Notice that $|X_A| = |X_0|$ as the topological space, since $\mathfrak{m}_A \mathcal{O}_X$ contained in the nilradical ideal of $\mathcal{O}_X$. If $\eta \in \mathrm{Pic}(X_A)$, let $\eta \otimes_A B \in \mathrm{Pic}(X_B)$ denoted the pull-back of η. Fix $\xi_0 \in \mathrm{Pic}(X_0)$ in this discussion, and let

$$\mathsf{P}(A) := \{\eta \in \mathrm{Pic}(X_A) \mid \eta \otimes_A k = \xi_0\}.$$

Proposition 1.15

Assume that

(i) X is flat over Λ,

(ii) $A \xrightarrow{\sim} H^0(X_A, \mathcal{O}_{X_A})$ for each $A \in \mathcal{C}$,

(iii) $\dim_k H^1(X_0, \mathcal{O}_{X_0}) < \infty$.

Then P is pro-representable by a pro-couple (R, ξ) such that $t_R \simeq H^1(X_0, \mathcal{O}_{X_0})$.

Remark 1.16. Under the condition (i), the condition (ii) is equivalent to $H^0(X_0, \mathcal{O}_{X_0}) = k$. Indeed, the functor $M \mapsto H^0(X, \mathcal{O}_X \otimes M)$ of Λ-module is left exact, since X is flat over Λ. By induction on length and five lemma, the natural map $M \mapsto H^0(X, \mathcal{O}_X \otimes M)$ is an isomorphism for all M of finite length.

Before proving Proposition 1.15, we need two simple lemmas on flatness.

Lemma 1.17

Lett A be a ring, J be a nilpotent ideal in A, and $u : M \to N$ be a homomorphism of A-modules, with N flat over A. If $\overline{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism.

Proof. Let $K = \operatorname{coker} u$. Tensor the exact sequence

$$M \to N \to K \to 0$$

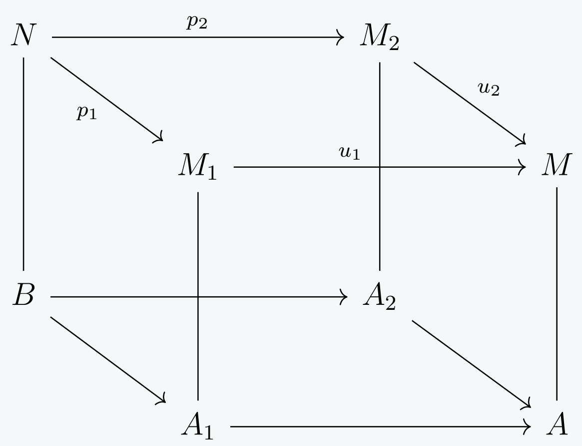
with A/J, we have $K/JK = 0$. Since J is nilpotent, we have $K = 0$. If $K' = \ker u$, then we get an exact sequence

$$0 \to K'/JK' \to M/JM \to N/JN \to 0$$

by the flatness of N. Hence, $K' = 0$. □

Lemma 1.18

Consider a commutative diagram



of compatible ring and module homomorphisms, where $B = A_1 \times_A A_2$, $N = M_1 \times_M M_2$, and M_i is a flat A_i-module for $i = 1, 2$. Suppose that A_1 is an artinian ring and

(i) $A_2/J \xrightarrow{\sim} A$, where J is an nilpotent ideal in A_2.

(ii) u_i induces $M_i \otimes_{A_i} A \xrightarrow{\sim} M$.

Then N is flat over B, and p_i induces $N \otimes_B A_i \xrightarrow{\sim} M_i$.

Proof. Let $\{x_{1,i}\}_{i \in I} \subseteq M_1$ such that $\{\overline{x}_{1,i}\}_{i \in I}$ form a basis of $M_1/\mathfrak{m}_{A_1} M_1$. Then $\coprod_{i \in I} A_1 \to M$ defined by $e_i \mapsto x_{1,i}$ is an isomorphism by Lemma 1.17. Hence, M_1 is a free A_1-module with basis $\{x_{1,i}\}_{i \in I}$. By (ii), M is the free module with basis $\{u_1(x_{1,i})\}_{i \in I}$. Choosing $x_{2,i} \in M_2$ such that $u_2(x_{2,i}) = u_1(x_{1,i})$. Then A_2-module homomorphism $\sum A_2 x_{2,i} \to M_2$ is isomorphism after modulo the ideal J. Again, by Lemma 1.17, M_2 is free module with basis $\{x_{2,i}\}_{i \in I}$, and it is clear that N is free B-module with basis $\{x_{1,i} \times x_{2,i}\}_{i \in I}$. Then the projection p_i induces the isomorphism $N \otimes_B A_i \xrightarrow{\sim} M_i$. □

Corollary 1.19

With the notations in Lemma 1.18. Let L be a B-module satisfies the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{q_2} & M_2 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle u_2} \\ M_1 & \xrightarrow{u_1} & M \end{array}$$

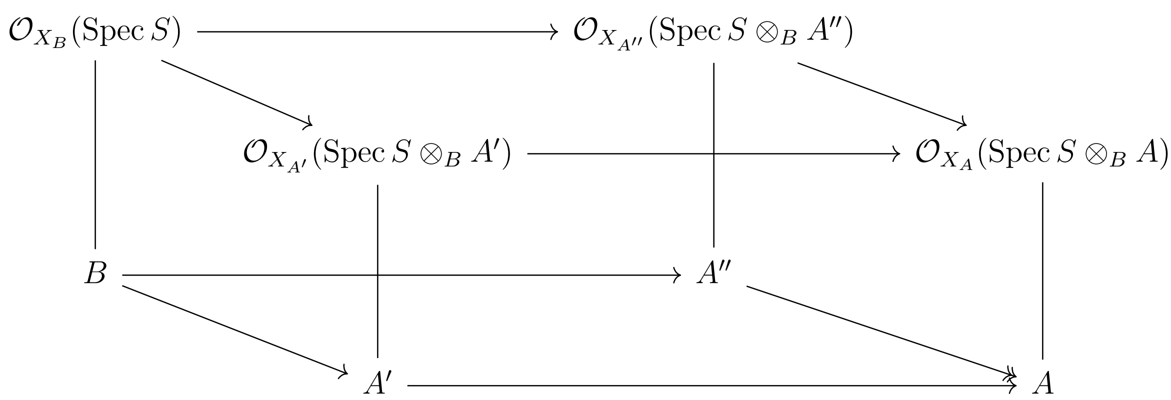
where q_1 induces $L \otimes_B A_1 \xrightarrow{\sim} M_1$. Then the canonical morphism $q_1 \times q_2 : L \to N = M_1 \times_M M_2$ is an isomorphism.

Proof. Apply Lemma 1.17 to the morphism $u = q_1 \times q_2$. □

Proof of Proposition 1.15. Let $u' : (A', \eta') \to (A, \eta)$, $u'' : (A'', \eta'') \to (A, \eta)$ be morphisms of couples, where u'' is a surjection. Let $\mathcal{L}'$, $\mathcal{L}$, $\mathcal{L}''$ be corresponding invertible sheaves on $X_{A'}$, X_A, $X_{A''}$, respectively. Let $B = A' \times_A A''$, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_B} & \longrightarrow & \mathcal{O}_{X_{A''}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X_A} & \longrightarrow & \mathcal{O}_{X_A} \end{array} \tag{4}$$

of sheaves on $|X_0|$. Indeed, if $\operatorname{Spec} S$ is an affine open subset of Z, then the diagram (4) give the commutative diagram



which is compatible with localization map. By Corollary 1.19, there is a canonical isomorphism, there is a canonical isomorphism

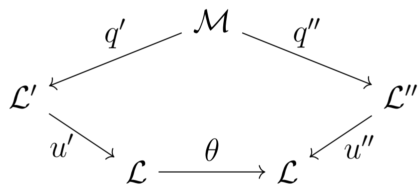
$$\mathcal{O}_{X_B} \xrightarrow{\sim} \mathcal{O}_{X_{A'}} \times_{\mathcal{O}_X} \mathcal{O}_{X''}.$$

Replace $\mathcal{O}$ by $\mathcal{L}$ in the above diagram, by Lemma 1.18, $\mathcal{N} := \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}''$ is locally free sheaf of rank 1 on X_B, and the projections to $\mathcal{L}'$ and $\mathcal{L}''$ induce isomorphisms $\mathcal{N} \otimes_B A' \xrightarrow{\sim} \mathcal{L}'$ and $\mathcal{N} \otimes_B A'' \xrightarrow{\sim} \mathcal{L}''$. Hence, (2) is surjective for any surjection $A'' \to A$.

For the injectivity, if $\mathcal{M}$ is another invertible sheaf on X_B with the isomorphisms

$$\mathcal{M} \otimes_B A' \xrightarrow{\sim} \mathcal{L}', \quad \mathcal{M} \otimes_B A'' \xrightarrow{\sim} \mathcal{L}'',$$

then we have morphisms $q' : \mathcal{M} \to \mathcal{L}'$, $q'' : \mathcal{M} \to \mathcal{L}''$ induce these isomorphisms. Then we have the following commutative diagram



where θ is the automorphism of L given by the composition

$$\mathcal{L} \xrightarrow{\sim} \mathcal{L}' \otimes_{A'} A \xrightarrow{\sim} \mathcal{M} \otimes_B A \xrightarrow{\sim} \mathcal{L}'' \otimes_{A''} A \xrightarrow{\sim} \mathcal{L}.$$

By hypothesis (ii), θ is multiplication by some unit $a \in A$. Take any $a'' \in A''$ such that $u''(a'') = a$. Change q'' to $a''q''$, we may assume that $u'q' = u''q''$. By Corollary 1.19, we conclude that $\mathcal{M} \xrightarrow{\sim} \mathcal{N}$.

Finally, let $Y = X_{k[\varepsilon]}$, we have $\mathcal{O}_Y = \mathcal{O}_{X_0} \oplus \varepsilon \mathcal{O}_{X_0}$, so there is a split exact sequence

$$0 \longrightarrow \mathcal{O}_{X_0} \xrightarrow{\exp} \mathcal{O}_Y^* \longrightarrow \mathcal{O}_{X_0}^* \longrightarrow 1,$$

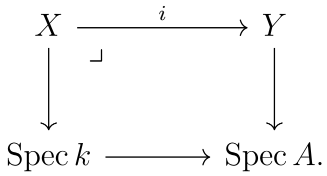
where $\exp(f) = 1 + \varepsilon f$. Hence,

$$F(k[\varepsilon]) \simeq \ker\left(H^1(X_0, \mathcal{O}_Y^*) \longrightarrow H^1(X_0, \mathcal{O}_{X_0}^*)\right) \simeq H^1(X_0, \mathcal{O}_{X_0})$$

has finite dimension by assumption. By Theorem 1.14, P as required. □

1.4 Formal moduli

Fix a scheme X over k. An **(infinitesimal) deformation** of X over $A \in \mathcal{C}$ is a flat scheme Y over $\operatorname{Spec} A$ with an immersion $i : X \hookrightarrow Y$ inducing an isomorphism $X \xrightarrow{\sim} Y \times_A k$



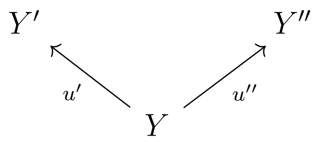
If Y' is another deformation to A, then Y and Y' are **isomorphic** if there exists a morphism $f : Y \to Y'$ over A which induces the identity on the closed fiber X. By Lemma 1.17, f must be an isomorphism of schemes. Given the deformation Y over A and a morphism $A \to B$ in $\mathcal{C}$, one has evidently an induced deformation $Y \otimes_A B$ over B.

Define the deformation functor $\mathsf{D} = \operatorname{Def}_X$ by

$$\mathsf{D}(A) = \{\text{isomorphism classes of deformations of } X/k \text{ to } A\}.$$

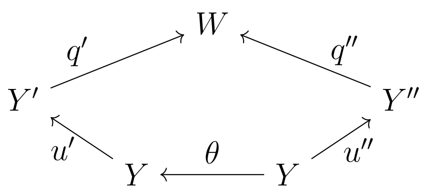
Unfortunately, D is not pro-representable. But in some finiteness restrictions on X, D will have a hull.

Suppose that $(A', \eta') \to (A, \eta)$ and $(A'', \eta'') \to (A, \eta)$ are morphisms of couples, where $A'' \to A$ is a surjection. Let Y', Y, Y'' denote deformations in the class of η', η, η'', respectively. Then we have a diagram



of deformations. As in the proof of Proposition 1.15, the sheaf $\mathcal{O}_{Y'} \times_{\mathcal{O}_Y} \mathcal{O}_{Y''}$ of $A' \times_A A''$-algebras defined a scheme Z on $|X_0|$, which is flat over $B := A' \times_A A''$. Moreover, $Z \in \mathsf{D}(B)$ maps to $(\eta', \eta'') \in \mathsf{D}(A') \times_{\mathsf{D}(A)} \mathsf{D}(A')$. Hence, D satisfies the condition (H_1) in Theorem 1.14.

Suppose that W is another deformation over B with morphism of deformation $q' : Y' \to W$ and $q'' : Y'' \to W$. Then we have the commutative diagram of deformations



where θ is the composition

$$Y \xrightarrow{\sim} Y' \otimes_{A'} A \xrightarrow{\sim} W \otimes_B A \xrightarrow{\sim} Y'' \otimes_{A''} A \xrightarrow{\sim} Y.$$

If θ can be lifted to an automorphism θ' of Y' such that $\theta' u' = u' \theta$, then we can replace q' with $q'\theta'$ and get an isomorphism $W \xrightarrow{\sim} Z$ by Corollary 1.19. In particular, such θ' exists when $A = k$, since $Y = X$ and $\theta = \mathrm{id}$ in this case. Hence, the condition (H_2) is satisfied.

Lemma 1.20

Let $p : (A', \eta') \to (A, \eta)$ be a morphism of couples, where p is a small extension. For each morphism $B \to A$ in $\mathcal{C}$, let

$$\mathsf{D}_\eta(B) = \{\zeta \in \mathsf{D}(B) \mid \zeta \otimes_B A = \eta\}.$$

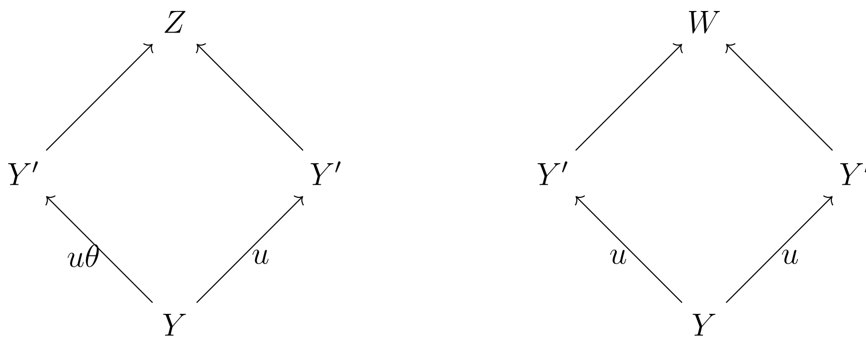
Pick a deformation Y' in the class of η', then the following are equivalent

(i) $\mathsf{D}_\eta(A' \times_A A') \xrightarrow{\sim} \mathsf{D}_\eta(A') \times \mathsf{D}_\eta(A')$.

(ii) Every automorphism of the deformation $Y = Y' \otimes_{A'} A$ is induced by an automorphism of the deformation Y'.

Proof.

- (i) $\implies$ (ii). Let $u : Y \to Y'$ be the induced morphism of deformations. If θ is an automorphism of Y, by (H_1), there exists deformations Z, W over $A' \times_A A'$ fit into the diagram of deformations



Since Z and W have isomorphic projections on both factors, there is an isomorphism $\rho : Z \xrightarrow{\sim} W$ by assumption. Then ρ induces automorphism θ_1 and θ_2 of Y' and an automorphism of Y such that

$$\theta_1 u\theta = u\phi, \quad \theta_2 u = u\phi.$$

Therefore, $u\theta = \theta_1^{-1}\theta_2 u$ implies $\theta_1^{-1}\theta_2$ induces θ.

- By the discussion in (H_2), every automorphism θ of Y has a lifting on Y' by assumption, which guarantee the injectivity.

□

Remark 1.21. Suppose that $H^0(X, \mathcal{T}_X) = 0$, then the condition (ii) in Lemma 1.20 is automatically satisfied. We will show, by induction on $\dim_k A$, that for any deformation Y of X over A, the automorphism group $\mathrm{Aut}(Y/X) = \{\mathrm{id}\}$.

For $A = k$ is clear. Now suppose $p : A \to A'$ be a small extension and let Y' be a deformation of X over A', with $Y = Y' \times_{A'} A$. Then there is an exact sequence

$$\mathrm{Aut}(Y'/Y) \to \mathrm{Aut}(Y'/X) \to \mathrm{Aut}(Y/X) = \{\mathrm{id}\}.$$

By Theorem B.5, we have

$$\mathrm{Aut}(Y'/Y) \simeq H^0(X_0, \mathcal{T}_{X_0}) = 0.$$

It follow that $\mathrm{Aut}(Y'/X) = \{\mathrm{id}\}$ as desired.

To verify condition (H_3), recall that Theorem B.5 states that for any scheme X locally of finite type over k, there is an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{T}_X^0) \longrightarrow t_{\mathsf{D}} \longrightarrow H^0(X, \mathcal{T}_X^1) \longrightarrow H^2(X, \mathcal{T}_X^0).$$

In particular, if X is proper over k, then since $\mathcal{T}_X^i$ are coherent sheaves on a proper scheme, their cohomology groups are finite-dimensional k-vector spaces. It follows that

$$\dim_k t_{\mathsf{D}} \leq \dim_k H^1(X, \mathcal{T}^0) + \dim H^0(X, \mathcal{T}^1) < \infty.$$

If X is nonsingular over k, then $\mathcal{T}_X^0 = \mathcal{T}_X$ and $\mathcal{T}_X^1 = 0$, and thus $t_{\mathsf{D}} \simeq H^1(X, \mathcal{T}_X)$. Affine schemes form another class of examples where t_{D} is more explicitly computable.

Proposition 1.22

If X is affine with only isolated singularities, then D has a hull.

Proof. We remain to verify the Schlessinger condition (H_3).

Let $X = \operatorname{Spec} B$ be an affine scheme with only isolated singularities. By Theorem B.3, $t_{\mathsf{D}} \simeq T^1(B/k, B)$. Since T^i functor compatible with localization and Theorem A.17, it follows that $T^1(B/k, B)$ is supported at finite number of points $\operatorname{Sing} X$. Hence, t_F has finite length. In other word, $\dim_k t_F < \infty$. □

2 Ran-Kawamata T^1-lifting

2.1 T^1-lifting

Let k be a field of characteristic zero. Let $D : \mathcal{C} \to \mathsf{Sets}$ be a deformation functor, i.e., a covariant functor such that $|D(k)| = 1$ and satisfies Schlessinger's conditions (H_1), (H_2), (H_3). We also assume that the obstruction can be calculated by a k-vector space T^2 (cf. Definition B.8).

For $n \in \mathbb{Z}_{\geq 0}$, let

$$A_n = k[t]/(t^{n+1}), \quad B_n = A_n \otimes_k A_1 = k[x,y]/(x^{n+1}, y^2), \quad C_n = k[x,y]/(x^{n+1}, x^n y, y^2).$$

We have natural homomorphism $\alpha_n : A_{n+1} \to A_n$, $\beta_n : B_n \to A_n$, and $\gamma_n : B_n \to C_n$, where $\beta_n(x) = t$, $\beta_n(y) = 0$.

For $X_n \in D(A_n)$, let

$$T^1(X_n/A_n) = \{Y_n \in D(B_n) \mid D(\beta_n)(Y_n) = X_n\}.$$

Example 2.1

Suppose that $D = \mathrm{Def}_X$ is pro-representable, and that X_n is a smooth deformation of X over A_n. By Lemma 1.20, we have an bijection

$$T^1(X_n/A_n) = \mathrm{Def}(X_n/A_n, B_n).$$

By Remark B.6, the deformation space can be computed via the following exact sequence

$$0 \longrightarrow H^1(X_n, \mathcal{T}^0(X_n/A_n, f^*\widetilde{J})) \longrightarrow \mathrm{Def}(X_n/A_n, B_n) \longrightarrow H^0(X_n, \mathcal{T}^1(X_n/A_n, f^*\widetilde{J})),$$

where $f : X_n \to A_n$ is the structure morphism and $J = \ker \beta_n$. Since $\ker \beta_n$ is a free A_n-module and X_n is smooth over A_n, it follows from Theorem A.18 that

$$\mathcal{T}^i(X_n/A_n, f^*\widetilde{J}) = \mathcal{T}^i_{X_n/A_n} = \begin{cases} \mathcal{T}_{X_n/A_n} & \text{if } i = 0, \\ 0 & \text{if } i = 1. \end{cases}$$

So we conclude that

$$T^1(X_n/A_n) = H^1(X_n, \mathcal{T}_{X_n/A_n}).$$

Theorem 2.2 (Version in [Kaw92])

Assume that D is pro-representable by (R, ξ), and that the following T^1 lifting property holds: for any positive integer n and any $X_{n+1} \in D(A_{n+1})$, the natural homomorphism

$$T^1(\alpha_n) : T^1(X_{n+1}/A_{n+1}) \to T^1(X_n/A_n)$$

is surjective for $X_n = D(\alpha_n)(X_{n+1})$. Then D is unobstructed, i.e., the hull of R is smooth.

Proof. It suffices to show that $D(\alpha_n)$ is surjective for all n. Indeed, according to the proof in Schlessinger's criterion, we have $R = k[[x_1, \ldots, x_r]]/I$, where $I \subseteq \mathfrak{m}_R^2$ and $r = \dim t_D$. Since $R \subseteq \mathfrak{m}_R^2$, we can define the morphism $g_2 : R \to A_1$ by $x_i \mapsto a_i t$ for any $a_i \in k$. Since $D(\alpha_n)$ is surjective for all n, there is a lifting $g_n : R \to A_n$ of g_{n-1} by induction for all $n \geq 3$. This define a morphism $g : R \to k[[t]]$ by $x_i \mapsto a_i t$. Suppose that I contains a nonzero power series $f = f_d + f_{d+1} + \cdots$, where f_n is the homogeneous polynomial of degree n, $f_d \neq 0$ and $d \neq 0$. Take a_i such that $f_d(a_1, \ldots, a_r) \neq 0$, then

$$g(f) = t^d f_d(a_1, \ldots, a_r) + t^{d+1} f_{d+1}(a_1, \ldots, a_r) + \cdots \neq 0$$

leading a contradiction. Hence, $R = k[[x_1, \ldots, x_r]]$ is smooth.

Consider the following commutative diagram

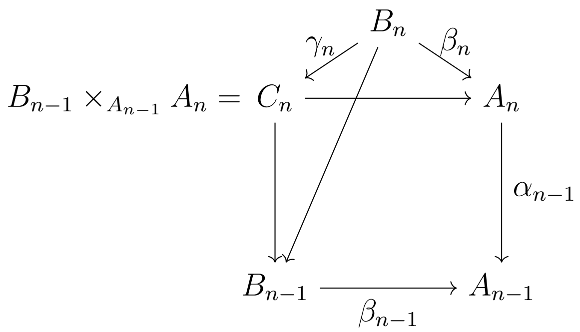
$$\begin{array}{ccccc} D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) & \xrightarrow{\delta_1} & T^2 \otimes (t^{n+1}) \\ \downarrow{\scriptstyle D(\varepsilon_n)} & & \downarrow{\scriptstyle D(\varepsilon'_n)} & & \downarrow{\scriptstyle \varphi} \\ D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) & \xrightarrow{\delta_2} & T^2 \otimes (x^n y) \end{array}$$

where $\varepsilon_n(t) = x + y$ and $\varepsilon'_n(t) = x + y$. By the definition of T^2, we have

$$\operatorname{Im} D(\alpha_n) = \{X_n \in D(A_n) \mid \delta_1(X_n) = 0\}, \quad \operatorname{Im} D(\gamma_n) = \{Y_n \in D(C_n) \mid \delta_2(Y_n) = 0\}.$$

Notice that φ is an isomorphism, since $\varphi : t^{n+1} \mapsto (n+1)x^n y$ and $\operatorname{char} k = 0$. Hence, to show that $D(\alpha_n)$ is surjective, it suffices to show that $D(\gamma_n)$ is surjective.

From the following diagram



and the pro-representability of D, we have

$$D(B_n) \xrightarrow{D(\gamma_n)} D(C_n) \simeq D(B_{n-1}) \times_{D(A_{n-1})} D(A_n).$$

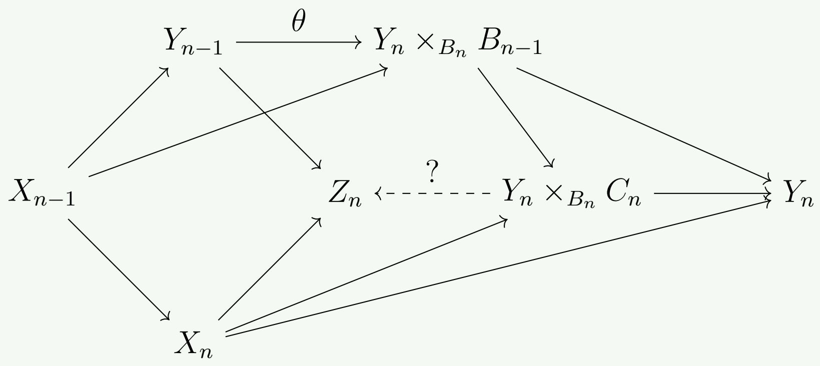
By the T^1 lifting property, for any $Z_n \in D(C_n)$, there exists $Y_n \in D(B_n)$ whose image in $D(B_{n-1})$ and $D(A_n)$ coincide with image of Y_n, respectively. Hence, $D(\gamma_n)(Y_n) = Z_n$. $\square$

Remark 2.3. In [Kaw97], for the situation of $D = \operatorname{Def}_X$, Kawamata replaced the definition of T^1 space by $T^1(X_n/A_n) \coloneqq \operatorname{Def}(X_n/A_n, B_n)$. With this new definition, Theorem 2.2 holds without the assumption that D is pro-representable.

As in the argument above, it suffices to show that the map $D(\gamma_n) : D(B_n) \to D(C_n)$ is surjective. For $Z_n \in D(C_n)$, let

$$Y_{n-1} = Z_n \times_{C_n} B_{n-1}, \quad X_n = Z_n \times_{C_n} A_n, \quad X_{n-1} = Z_n \times_{C_n} A_{n-1}.$$

Let $[Y_n]$ be the equivalent classes in $\mathrm{Def}(X_n/A_n, B_n)$ maps to $[Y_{n-1}] \in \mathrm{Def}(X_{n-1}/A_{n-1}, B_{n-1})$. Then there exists an isomorphism $\sigma : Y_{n-1} \to Y_{n-1} \times_{B_n} B_n$ compatible with the closed immersion X_{n-1}, and we get a diagram such that



By Corollary 1.19, there exists an isomorphism $Y_n \times_{B_n} C_n \xrightarrow{\sim} Z_n$ compatible with the closed immersion X_{n-1}. This show that $D(\gamma_n)(Y_n) = Z_n$.

2.2 Deformation of smooth Calabi-Yau varieties

Before proving the unobstructness of deformations of Calabi-Yau varieties, we need the following essential technique. In fact, this is the generalization of Hodge theorem.

Theorem 2.4 ([Del68, Theorem 5.5])

Let $f : X \to S = \operatorname{Spec} A_n$ be a proper and smooth morphism, where $A_n = \mathbb{C}[t]/(t^{n+1})$. Then $H^q(X, \Omega^p_{X/S})$ is a free A_n-module.

Proof. Let $A = A_n$. Consider the $\mathfrak{m}_A$-adic filtration

$$\Omega^{\bullet,\mathrm{an}}_{X/S} \supseteq \mathfrak{m}_A \Omega^{\bullet,\mathrm{an}}_{X/S} \supseteq \mathfrak{m}_A^2 \Omega^{\bullet,\mathrm{an}}_{X/S} \supseteq \cdots \supseteq \mathfrak{m}_A^N \Omega^{\bullet,\mathrm{an}}_{X/S} = 0$$

of $\Omega^{\bullet,\mathrm{an}}_{X/S}$, which is biregular, and denote the associated graded object by

$$\mathrm{gr}_\bullet \Omega^{\bullet,\mathrm{an}}_{X/S} = \bigoplus \mathrm{gr}_i \Omega^{\bullet,\mathrm{an}}_{X/S}, \quad \text{where } \mathrm{gr}_i \Omega^{\bullet,\mathrm{an}}_{X/S} \coloneqq \mathfrak{m}_A^i \Omega^{\bullet,\mathrm{an}}_{X/S} \Big/ \mathfrak{m}_A^{i+1} \Omega^{\bullet,\mathrm{an}}_{X/S}.$$

Notice that

$$\mathrm{gr}_0 \Omega^{\bullet,\mathrm{an}}_{X/S} = \Omega^{\bullet,\mathrm{an}}_{X/S} \Big/ \mathfrak{m}_A \Omega^{\bullet,\mathrm{an}}_{X/S} = \Omega^{\bullet,\mathrm{an}}_{X_0/\operatorname{Spec}\mathbb{C}}$$

resolves the constant sheaf $\mathbb{C}$, where $X_0 = X_{\mathrm{red}}$ is the smooth fiber of $\operatorname{Spec}\mathbb{C} \hookrightarrow \operatorname{Spec} A$. Then

$$\mathrm{gr}_{\mathfrak{m}_A} A \otimes_{\mathbb{C}} \mathrm{gr}_0 \Omega^{\bullet,\mathrm{an}}_{X/S} = \bigoplus_{n\geq 0} \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} \otimes_{A/\mathfrak{m}_A} \Omega^{\bullet,\mathrm{an}}_{X/S} \Big/ \mathfrak{m}_A \Omega^{\bullet,\mathrm{an}}_{X/S} = \bigoplus_{n\geq 0} \mathfrak{m}_A^n \Omega^{\bullet,\mathrm{an}}_{X/S} \Big/ \mathfrak{m}_A^{n+1} \Omega^{\bullet,\mathrm{an}}_{X/S} = \mathrm{gr}_\bullet \Omega^{\bullet,\mathrm{an}}_{X/S}.$$

resolves $\mathrm{gr}_{\mathfrak{m}_A} A$, and hence $\Omega^{\bullet,\mathrm{an}}_{X/S}$ resolves the constant sheaf A. Since X is compact, by the GAGA principle we have

$$\mathsf{R}^n\Gamma(X, \Omega^\bullet_{X/S}) \simeq \mathsf{R}^n\Gamma(X^{\mathrm{an}}, \Omega^{\bullet,\mathrm{an}}_{X/S}) = H^n(X^{\mathrm{an}}, A) = H^n(X^{\mathrm{an}}, \mathbb{C}) \otimes_{\mathbb{C}} A,$$

and thus

$$\mathrm{length}_A \mathsf{R}^n\Gamma(\Omega^\bullet_{X/S}) = \mathrm{length}_A(A) \cdot \dim_{\mathbb{C}} H^n(X^{\mathrm{an}}, \mathbb{C}). \tag{5}$$

Lemma 2.5 ([Gro63, EGA III, Proposition 6.10.5])

Let $Y = \operatorname{Spec} A$ be a noetherian affine scheme, and let $f : X \to Y$ be a proper morphism. For any complex $\mathscr{F}^\bullet \in \mathsf{D}^+(\operatorname{Coh}(X))$ that is flat over Y, there exists a complex $\mathscr{L}^\bullet \in \mathsf{D}^+(\operatorname{Coh}(X))$, whose terms $\mathscr{L}_i$ has the form $\mathcal{O}_Y^{m_i}$, and a functorial quasi-isomorphism

$$\mathsf{R}f_*(\mathscr{F}^\bullet \otimes \mathsf{L}f^*\mathscr{G}) \longrightarrow \mathscr{L}^\bullet \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathscr{G}^\bullet$$

for any complex $\mathscr{G} \in D^+(\operatorname{Qcoh}(Y))$. Moreover, for arbitrary base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

there is a natural quasi-isomorphism

$$\mathsf{R}f'_*(\mathsf{L}u'^*\mathscr{F}^\bullet \otimes^{\mathsf{L}}_{\mathcal{O}_{X'}} \mathsf{L}f'^*\mathscr{G}) \longrightarrow u^*\mathscr{L}^\bullet \otimes^{\mathsf{L}}_{\mathcal{O}_Y} \mathscr{G}^\bullet.$$

for any $\mathscr{G}^\bullet \in \mathsf{D}^-(\operatorname{Qcoh}(Y'))$.

Since f is smooth, the sheaf $\Omega_{X/S}$ is locally free on X. Applying Lemma 2.5 to the pair $(\mathscr{F}^\bullet, \mathscr{G}^\bullet) = (\Omega^p_{X/S}, \mathcal{O}_Y)$ and the base change morphism $u : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} A$, we obtain

$$\mathsf{R}f_*\Omega^p_{X/S} \simeq \mathscr{L}^\bullet, \quad \mathsf{R}f'_*\Omega^p_{X_0/\operatorname{Spec}\mathbb{C}} \simeq \mathsf{R}f'_*(u'^*\Omega^p_{X/S}) \simeq u^*\mathscr{L}^\bullet.$$

Therefore

$$H^q(X, \Omega^p_{X/S}) = H^0(\operatorname{Spec} A, \mathsf{R}^q f_*\Omega^p_{X/S}) = \Gamma(\operatorname{Spec} A, H^q(\mathscr{L}^\bullet))$$

is finite generated A-module, and the natural isomorphism of base change

$$H^q(X, \Omega^p_{X/S}) \otimes_A \mathbb{C} \simeq \Gamma(\operatorname{Spec}\mathbb{C}, H^q(u^*\mathscr{L}^\bullet)) = H^q(X_0, \Omega^p_{X_0}) = H^q(X_{\text{red}}, \Omega^p_{X_{\text{red}}}).$$

Lemma 2.6

Let $(A, \mathfrak{m}_A)$ be the noetherian local ring with residue field k. For any finite generated A-module M, we have the inequality

$$\operatorname{length}_A(M) \le \operatorname{length}_A(A) \cdot \dim_k(M \otimes_A k).$$

Equality holds if and only if M is free A-module.

Proof. Let $d = \dim_k(M \otimes_A k)$. By Nakayama's lemma, there exists a surjection $f : A^d \twoheadrightarrow M$. Then

$$\operatorname{length}_A(M) = d \cdot \operatorname{length}_A(A) + \operatorname{length}_A(\ker f)$$

and $\operatorname{length}_A(\ker f) = 0$ if and only if $\ker f = 0$. □

By the algebra lemma as above, we have

$$\operatorname{length}_A H^q(X, \Omega^p_{X/S}) \leq \operatorname{length}_A(A) \cdot \dim H^q(X_{\mathrm{red}}, \Omega^p_{X_{\mathrm{red}}}) \tag{6}$$

and the equality holds if and only if $H^q(X, \Omega^p_{X/S})$ is free A-module. Consider Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/S}) \Longrightarrow \mathsf{R}^{p+q}\Gamma(\Omega^\bullet_{X/S}).$$

Since $E_\infty^{p,q}$ is the subquotient of $E_1^{p,q}$, we deduce that

$$\operatorname{length}_A \mathsf{R}^n\Gamma(\Omega^\bullet_{X/S}) = \sum_{p+q=n} \operatorname{length}_A E_\infty^{p,q} \leq \sum_{p+q=n} \operatorname{length}_A H^q(X, \Omega^p_{X/S}). \tag{7}$$

Finally, combining inequalities (5), (6), and (7), we obtain that

$$\begin{aligned} \operatorname{length}_A(A) \dim_{\mathbb{C}} H^n(X^{\mathrm{an}}, \mathbb{C}) &= \operatorname{length}_A \mathsf{R}^n\Gamma(\Omega^\bullet_{X/S}) \\ &\leq \sum_{p+q=n} \operatorname{length}_A H^q(X, \Omega^p_{X/S}) \\ &\leq \operatorname{length}_A(A) \sum_{p+q=n} \dim H^q(X_{\mathrm{red}}, \Omega^p_{X_{\mathrm{red}}}) \end{aligned} \tag{8}$$

By Hodge decomposition in analytic setting and the GAGA principle, we have

$$H^n(X^{\mathrm{an}}, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X^{\mathrm{an}}, \Omega^{p,\mathrm{an}}_{X_{\mathrm{red}}}) \simeq \bigoplus_{p+q=n} H^q(X, \Omega^\bullet_{X_{\mathrm{red}}}),$$

and hence equality holds throughout in (8). It follows that each

$$H^q(X, \Omega^p_{X/S}) = \Gamma(\operatorname{Spec} A, \mathsf{R}^q f_* \Omega^p_{X/S})$$

is a free A-module. □

Theorem 2.7

Let X be a smooth projective variety over $\mathbb{C}$. If $\omega_X \simeq \mathcal{O}_X$, then the flat deformations of X are unobstructed.

Proof. By Remark 2.3, it suffices to show that

$$\operatorname{Def}(X_{n+1}/A_{n+1}, B_{n+1}) \longrightarrow \operatorname{Def}(X_n/A_n, B_n)$$

is surjective for all positive integer n and $X_n = D(\alpha_n)(X_{n+1})$. By Example 2.1, the problem becomes to show that

$$H^1(X_{n+1}, \mathcal{T}_{X_{n+1}/A_{n+1}}) \longrightarrow H^1(X_n, \mathcal{T}_{X_n/A_n}). \tag{9}$$

is surjective.

Let $f : X_n \to \operatorname{Spec} A_n$ be the deformation of X over A_n, which is proper by Stack Project. By Theorem 2.4, it follows that

$$H^q(X_n, \Omega^p_{X_n/A_n})$$

is a free A_n-module of rank $h^{p,q}_n$. By cohomology and base change, we have the isomorphisms

$$H^q(X_n, \Omega^p_{X_n/A_n}) \otimes_{A_n} k \xrightarrow{\ \sim\ } H^q(X_0, \Omega^p_X),$$

and thus $h^{p,q}_n = h^{p,q}(X)$ is independent on n. In particular,

$$H^0(X, \omega_X) = k \implies H^0(X_n, \omega_{X_n/\operatorname{Spec} A_n}) = A_n.$$

Take a non-trivial section $s : \mathcal{O}_{X_n} \to \omega_{X_n/A_n}$, then $s \otimes_{X_n} X$ is an isomorphism on X. Since ω_{X_n/A_n} is flat over $\operatorname{Spec} A_n$, by Lemma 1.17, s is an isomorphism. From the perfect pairing

$$\Omega^1_{X_n/A_n} \otimes \Omega^{d-1}_{X_n/A_n} \longrightarrow \omega_{X_n/A_n} \simeq \mathcal{O}_{X_n},$$

we have $\mathcal{T}_{X_n/A_n} \simeq \Omega^{d-1}_{X_n/A_n}$. Now, we may rewrite (9) to

$$A_{n+1}^{h^{d-1,1}} \longrightarrow A_n^{h^{d-1,1}},$$

which is a surjection as desired. □

A First order deformations

The main reference for Appendix is [Har10].

A.1 The T^i Functors

Let $A \to B$ be a homomorphism of rings and let M be a B-module. Choose the set of variables $x = \{x_i\}$ and a free R-module F such that

$$0 \longrightarrow I \longrightarrow R \longrightarrow B \longrightarrow 0, \qquad 0 \longrightarrow Q \longrightarrow F \xrightarrow{j} I \longrightarrow 0$$

and let

$$F_0 = \langle j(a)b - j(b)a \mid a, b \in F \rangle_R \subseteq F.$$

Define a complex of B-modules, called the **cotangent complex**

$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

as follows.

- Take $L_2 = Q/F_0$, which is a B-module. Indeed, if $x \in I$ and $a \in Q$. Write $x = j(x')$ for some $x' \in F$, then
$$xa = j(x')a \equiv j(a)x' = 0 \pmod{F_0}.$$
- Take $L_1 = F \otimes_R B = F/IF$, and let $d_2 : L_2 \to L_1$ be the map induced from the inclusion $Q \to F$.
- Take $L_0 = \Omega_{R/A} \otimes_R B$, where $\Omega_{R/A}$ is the module of relative Kähler differential. Let d_1 be the composition of
$$\begin{array}{ccccc} L_1 & \twoheadrightarrow & I/I^2 & \longrightarrow & \Omega_{R/A} \otimes_R B \\ & & x & \longmapsto & d_{R/A}(x) \otimes 1 \end{array}.$$
It is clear that $d_1 d_2 = 0$, so we defined a complex of B-modules. Notice that L_0, L_1 are free B-module, since R is a polynomial ring over A.
- Indeed, the second fundamental exact sequence of Kähler differential
$$I/I^2 \xrightarrow{d} \Omega_{R/A} \otimes_R B \longrightarrow \Omega_{B/A} \longrightarrow 0$$
give the exact sequence
$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \longrightarrow \Omega_{B/A} \longrightarrow 0. \tag{10}$$

Lemma A.1

For any B-module M, define

$$T^i(B/A, M) \coloneqq H^i(\mathrm{Hom}_B(L_\bullet, M)),$$

which is independent of the choice of F and R.

Proof. First, we fixed the polynomial R. If F and F' are two choices of free R-modules mapping onto I, then $F \oplus F'$ gives the third choice. Since F' is free, the map $j' : F' \to I$ factors through F, i.e., $j' = jp$ for some map $p : F' \to F$. Replacing each generator e' of $F' \subseteq F \oplus F'$ by $e' - p(e')$, we may assume that the map $F \oplus F' \to I$ is just $(j, 0)$. Then $\ker(j, 0) = Q \oplus F'$ and thus $(F \oplus F')_0 = F_0 + IF'$. Let $L'_\bullet$ be the complex obtained from $(F \oplus F', R)$, then

$$L'_2 = L_2 \oplus F'/IF', \quad L'_1 = L_1 \oplus (F' \otimes_R B), \quad L'_0 = L_0.$$

Hence, $L'_\bullet$ is obtained by the direct sum of $L_\bullet$ with the free acyclic complex $F' \otimes_R B \to F' \otimes_R B \to 0$, which induces same cohomology after taking Hom functor.

Second, it suffices to compare $R = A[x]$ with $R' = A[x, y]$. Again, the map $A[y] \to B$ can be factored through $A[x]$ by a homomorphism $p : A[y] \to A[x]$. Replacing each $y_i \in A[x, y]$ by $y_i - p(y_i)$, we may assume the ring homomorphism $A[x, y] \to B$ maps y_i to 0, then it has kernel $IR' + yR'$. Since T^i is independent of the choice of F, we may take $F' = F \otimes_R R'$ and G be a free R'-module on the index set of the y variables. Then we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q' & \longrightarrow & F' \oplus G' & \longrightarrow & IR' + yR' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

and it is clear that

$$Q' = QR' + \langle y_i a - j(a) e_i \mid a \in F \rangle_{R'} + \langle y_i e_j - y_j e_i \rangle_{R'}.$$

Since $\langle y_i a - j(a) e_i \mid a \in F \rangle_{R'} + \langle y_i e_j - y_j e_i \rangle_{R''} \subseteq (F' \oplus G')_0$, we have $L'_2 = L_2$.

On the other hand, $L'_1 = L_1 \oplus (G' \otimes_{R'} B)$ and $L'_0 = L_0 \oplus (\Omega_{A[y]/A} \otimes B)$. Then $L'_\bullet$ is obtained by the direct sum of $L_\bullet$ with the free acyclic complex

$$\begin{array}{ccccc} G' \otimes_{R'} B & \longrightarrow & \Omega_{A[y]/A} \otimes B & \longrightarrow & 0 \\ e_i & \longmapsto & dy_i, & & \end{array}$$

and thus induces same cohomology after taking Hom functor. □

Remark A.2. The proof also show that $L_\bullet$ is a well-defined element in the derived category of the category of B-modules.

Theorem A.3

Let $A \to B$ be a homomorphism of rings. Then $T^i(B/A, -)$ is covariant, additive functor from $\mathcal{M}od_B$ to itself. If

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules, then there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow T^0(B/A, M') \longrightarrow T^0(B/A, M) \longrightarrow T^0(B/A, M'') \\ \longrightarrow T^1(B/A, M') \longrightarrow T^1(B/A, M) \longrightarrow T^1(B/A, M'') \\ \longrightarrow T^2(B/A, M') \longrightarrow T^2(B/A, M) \longrightarrow T^2(B/A, M''). \end{aligned}$$

Proof. By construction, $T^i(B/A, -)$ is a covariant additive functors. Since the terms L_1 and L_0 of the complex $L_\bullet$ are free, we get a sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_B(L_\bullet, M') \longrightarrow \mathrm{Hom}_B(L_\bullet, M) \longrightarrow \mathrm{Hom}_B(L_\bullet, M'') \longrightarrow 0$$

that is exact except possibly for the map

$$\mathrm{Hom}_B(L_2, M) \to \mathrm{Hom}_B(L_2, M'')$$

may not be surjective. This sequence of complexes gives the long exact sequence of cohomology above. □

Theorem A.4

Let $A \to B \to C$ be homomorphisms of rings, and let M be a C-module. Then there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow T^0(C/B, M) \longrightarrow T^0(C/A, M) \longrightarrow T^0(B/A, M) \\ \longrightarrow T^1(C/B, M) \longrightarrow T^1(C/A, M) \longrightarrow T^1(B/A, M) \\ \longrightarrow T^2(C/B, M) \longrightarrow T^2(C/A, M) \longrightarrow T^2(B/A, M). \end{aligned}$$

Proof. Choose

$$0 \longrightarrow I \longrightarrow A[x] \longrightarrow B \longrightarrow 0, \qquad 0 \longrightarrow Q \longrightarrow F \xrightarrow{j} I \longrightarrow 0$$

$$0 \longrightarrow J \longrightarrow B[y] \longrightarrow C \longrightarrow 0, \qquad 0 \longrightarrow P \longrightarrow G \longrightarrow J \longrightarrow 0$$

to calculate $T^i(B/A, M)$ and $T^i(C/B, M)$. Then $A[x, y] \to B[y] \to C$ is surjective, and denoted it kernel by K. It induces an exact sequence

$$0 \longrightarrow I[y] \longrightarrow K \longrightarrow J \longrightarrow 0$$

of A-module. Let F' and G' be free $A[x, y]$-modules on the same index sets as F and G, respectively. Choose a lifting of the map $G \to J$ to a map $G' \to K$. From the map $F \to I$, we get the natural map $F' \to I[y]$. Hence, we get the surjection $F' \oplus G' \to K$, and let S be it kernel. Now, we can calculate $T^i(C/A, M)$ by

$$0 \longrightarrow K \longrightarrow A[x,y] \longrightarrow C \longrightarrow 0, \qquad 0 \longrightarrow S \longrightarrow F' \oplus G' \xrightarrow{j} K \longrightarrow 0.$$

Now, we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q[y] & \longrightarrow & F' & \longrightarrow & I[y] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & F' \oplus G' & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & J & \longrightarrow & 0 \end{array}$$

which is exact for each horizontal sequence, but not exact for vertical sequence. It induces the maps of complexes

$$L_\bullet(B/A) \otimes_B C \longrightarrow L_\bullet(C/A) \longrightarrow L_\bullet(C/B).$$

Indeed, on the degree 0 level we have split exact sequence

$$\begin{array}{ccccc} \Omega_{A[x]/A} \otimes_{A[x]} C & \longrightarrow & \Omega_{A[x,y]/A} \otimes_{A[x,y]} C & \longrightarrow & \Omega_{B[y]/B} \otimes_{B[y]} C \\ dx_i & \longrightarrow & dx_i & & \\ & & dx_i,\ dy_j & \longrightarrow & 0,\ dy_j \end{array}$$

On the degree 1 level we have split exact sequence

$$F \otimes_{A[x]} C \longrightarrow (F' \oplus G') \otimes_{A[x,y]} C \longrightarrow G \otimes_{B[y]} C$$

of free C-modules. On the degree 2 level we have

$$(Q/F_0) \otimes_B C \longrightarrow S/(F' \oplus G')_0 \longrightarrow P/G_0. \tag{11}$$

We claim that $S \to P$ is surjective. Indeed, if $p \in P$. Take $(f', g') \in F' \oplus G'$ maps to p. Then the image of (f', g') in K is contained in $I[y]$. Since $F' \to I[y]$ is surjective, there exists $\widetilde{f} \in F'$ has same image in K with (f', g'). Hence, $(f' - \widetilde{f}, g') \in S$ maps to p.

To complete the proof, we remain to show the exactness of (11) in the middle. Let $s = f + g$ be an element of $S \subseteq F' \oplus G'$ such that it image in P is contained in G_0. Then the image g of g' in G belongs to G_0, write $g = j(a)b - j(b)a$ for some a, $b \in G$. Lift a, b to elements a', b' in G', then $j(a')b' - j(b')a' \in S$. Let $g'' = g - j(a')b' + j(b')a'$, then $g'' \in \ker(G' \to G) = IG'$. Say $g'' = xh$ and $x = j(x')$ for some $x \in I$, $h \in G'$, and $x' \in F$. Then

$$xh = j(x')h \equiv j(h)x' \pmod{(F' + G')_0}.$$

Therefore,

$$s \equiv f' + g'' \equiv f' + j(h)x' \pmod{(F' + G')_0}$$

and $f' + j(h)x' \in F' \cap S = Q[y]$. □

Proposition A.5

For any morphism $A \to B$ and $M \in \mathcal{M}od_B$,

$$T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M).$$

In particular,

$$T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B) = T_{B/A}$$

is the tangent module of B over A.

Proof. Taking $\mathrm{Hom}(-, M)$ to the exact sequence (10), we have the exact sequence

$$0 \to \mathrm{Hom}(\Omega_{B/A}, M) \to \mathrm{Hom}(L_0, M) \to \mathrm{Hom}(L_1, M).$$

□

Proposition A.6

If B is a polynomial ring over A, then $T^i(B/A, M) = 0$ for $i = 1$, 2 and for all M.

Proof. We may take $R = B$ and $F = 0$ in our construction. Hence, the cotangent complex only has L_0 term. □

Proposition A.7

If $A \to B$ is a surjective ring homomorphism with kernel I, then

$$T^0(B/A, M) = 0, \quad T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$$

for any $M \in \mathcal{M}ob_B$. In particular, $T^1(B/A, B) = \mathrm{Hom}_B(I/I^2, B)$ is the normal module $N_{B/A}$ of $\mathrm{Spec}\, B$ in $\mathrm{Spec}\, A$.

Proof. In this case, we can take $R = A$, and thus $L_0 = 0$. Hence, $T^0 = 0$ for any module M. Furthermore, the exact sequence

$$0 \longrightarrow Q \longrightarrow F \xrightarrow{j} I \longrightarrow 0$$

tensored with B with get

$$\begin{array}{ccccccc} Q \otimes_A B & \longrightarrow & F \otimes_A B & \xrightarrow{j} & I/I^2 & \longrightarrow & 0. \\ \big\downarrow & & \big\downarrow \wr & & & & \\ Q/F_0 & \longrightarrow & F/IF & & & & \end{array}$$

Hence, we have an exact sequence

$$L_2 \longrightarrow L_1 \longrightarrow I/I^2 \longrightarrow 0,$$

and thus $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$. □

Corollary A.8

Suppose that A is a local ring and $B = A/I$, where I is generated by a regular sequence $a_1, \ldots, a_r$. Then $T^2(B/A, M) = 0$ for all M.

Proof. Since the Koszul complex of regular sequence is exact, we conclude that $Q = F_0$. Hence, $L_2 = 0$ and $T^2(B/A, M) = 0$ for all M. □

Proposition A.9

Suppose that $A = k[x_1, \ldots, x_n]$ and $B = A/I$. Then for any M, there is an exact sequence

$$0 \longrightarrow T^0(B/k, M) \longrightarrow \mathrm{Hom}_A(\Omega_{A/k}, M) \longrightarrow \mathrm{Hom}_B(I/I^2, M) \longrightarrow T^1(B/k, M) \longrightarrow 0$$

and an isomorphism

$$T^2(B/A, M) \xrightarrow{\quad\sim\quad} T^2(B/k, M).$$

Proof. It follows from the long exact sequence of T^i-functors for $k \to A \to B$ and Proposition A.5, A.6, and A.7. □

Remark A.10. In this section, we have not made any finiteness assumptions on the rings and modules. It is clear that if A is a noetherian ring, B is a finitely generated A-algebra, and M is a finitely generated B-module, then $T^i(B/A, M)$ are finitely generated B-modules.

Lemma A.11

The construction of the T^i functors is compatible with localization, and thus we may defined sheaves $\mathcal{T}^i(X/Y, \mathcal{F})$ for any morphism of schemes $f : X \to Y$ and any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, such that for any open affine $\mathrm{Spec}\, A \subseteq Y$ and any open affine $\mathrm{Spec}\, B \subseteq f^{-1}(\mathrm{Spec}\, A)$, where $\mathcal{F}|_{\mathrm{Spec}\, B} = \widetilde{M}$, the section $\mathcal{T}^i(X/Y, \mathcal{F})(\mathrm{Spec}\, B) = T^i(B/A, M)$.

For the sake of convenience, denote the modules $T^i(B/A, B)$ and $T^i(B/k, B)$ by $T^i_{B/A}$ and $T^i_{B/k}$ (or T^i_B) if there is no confusion will happen. Furthermore, $T^0_{B/A}$ will be written the tangent module $T_{B/A}$ of B over A. Similarly for the sheaves $\mathcal{T}^i(X/Y, \mathcal{O}_X)$ and $\mathcal{T}^i(X/k, \mathcal{O}_X)$ will write $\mathcal{T}^i_{X/Y}$ and $\mathcal{T}^i_{X/k}$ (or $\mathcal{T}^i_X$). The sheaf $\mathcal{T}^0_X$ will be written the tangent sheaf $\mathcal{T}_X$ of X.

Proposition A.12 (Base change)

Assume that A is noetherian and B is a finitely generated A-algebra.

(1) Let $M \in \mathcal{M}ob_B$ and $A \to A'$ be a flat morphism. Let $B' = B \otimes_A A'$ and $M' = M \otimes_B B'$. Then $T^i(B/A, M) \otimes_A A' \simeq T^i(B'/A', M')$ for each i.

(2) Let $A \to A'$ be arbitrary ring homomorphism and B is flat over A. Let $B' = B \otimes_A A'$ and $M' \in \mathcal{M}ob_{B'}$. Then $T^i(B/A, M') = T^i(B'/A', M')$ for each i.

A.2 The infinitesimal Lifting Property

Proposition A.13 (Infinitesimal Lifting Property)

Let X be a nonsingular affine scheme of finite type over k, let $f : Y \to X$ be a morphism from an affine scheme Y over k, and let $Y \subseteq Y'$ be an infinitesimal thickening of Y. Then the morphism f lifts to a morphism $g : Y' \to X$ such that $g|_Y = f$.

Proof. Since $Y'_{\text{red}} = Y_{\text{red}}$ is affine, we also have Y' is affine. We may transfer the problem to algebra setting. Let $f : A \to B$ be the ring homomorphism over k corresponding to $f : Y \to X$, where $B = B'/I$ with $I^n = 0$ for some n. We want to find a homomorphism $g : A \to B'$ lifting f.

Consider the sequence $B' = B'/I^n \to B'/I^{n-1} \to \cdots \to B'/I^2 \to B'/I$, it will suffice to lift one step at time. So we may assume that $I^2 = 0$.

Since X is finite type over k, we may write $A = k[x_1, \ldots, x_n]/J$. Consider any lifting $h : k[x_1, \ldots, x_n] \to B'$ of $k[x_1, \ldots, x_n] \to A \to B$. Then h induces a map $J \to I$, and thus induces $\overline{h} : J/J^2 \to I$ since $I^2 = 0$.

Embed $X \hookrightarrow \mathbb{A}^n_k$ by $P = k[x_1, \ldots, x_n] \to A$. Since X is smooth, we obtain an exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega^1_{P/k} \otimes_P A \longrightarrow \Omega^1_{A/k} \longrightarrow 0.$$

Notice that those modules correspond to locally free sheaves on X, and thus are projective A-modules. Via the maps h, f, we get a P-module structure on B', and A-module structures on B, I. Applying $\mathrm{Hom}_A(-, I)$ will get another exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^1_{A/k}, I) \longrightarrow \mathrm{Hom}_P(\Omega^1_{P/k}, I) \longrightarrow \mathrm{Hom}_A(J/J^2, I) \longrightarrow 0.$$

Let $\theta \in \mathrm{Hom}_P(\Omega^1_{P/k}, I) = \mathrm{Der}_k(P, I)$ be an element whose image is $\overline{h} \in \mathrm{Hom}_A(J/J^2, I)$. We claim that $h' := h - \theta : P \to B'$ is a ring homomorphism lifting of f with $h'(J) = 0$. The first assertion follows by the following easy lemma.

Lemma A.14

Let $B' \to B$ be a surjective homomorphism of k-algebras with kernel I is square zero. Let $R \to B$ be a homomorphism of k-algebras.

(a) If f, $g : R \to B'$ are two liftings of the map $R \to B$ to B', then $\theta = g - f$ is a k-derivation of R to I.

(b) Conversely, if $f : R \to B'$ is one lifting, and $\theta : R \to I$ is a derivation, then $g = f + \theta$ is another homomorphism of R to B' lifting the given map $R \to B$.

In other words, if it is nonempty, the set of liftings $R \to B$ to k-algebra homomorphisms of

$R \to B'$ is a principal homogeneous space under the $\mathrm{Der}_k(R, I) = \mathrm{Hom}_R(\Omega_{R/k}, I)$ action.

Since $h'(J) \subseteq I$, we have $h'(J^2) \subseteq I^2 = 0$. Then h' descends to $P/J^2 \to B'$, and thus $h'|_{J/J^2} = h - h = 0$ by the choice of θ. Hence, h' descends to the desired homomorphism $g : A \to B'$ lifting f. ☐

For the converse statement of Proposition A.13, we only need to check special cases of the infinitesimal lifting property.

Proposition A.15

Let X be a scheme of finite type over algebraically closed field k. Suppose that for every morphism $f : Y \to X$ finite over k, where $Y = \operatorname{Spec} R$ is the spectrum of a local artinian ring R, and for every infinitesimal thickening $Y \subseteq Y'$ with ideal sheaf of square zero, there is a lifting $g : Y' \to X$. Then X is nonsingular.

Proof. It suffices to show that the local ring $\mathcal{O}_{X,x}$ is a regular local ring for every closed point $x \in X$, so the problem is local and can reformulate to the algebra problem. Let $(A, \mathfrak{m})$ be a local k-algebra, essentially of finite type over k with residue field k. Assume that for every homomorphism $f : A \to B$, where B is a local artinian k-algebra, and for every thickening

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

with $I^2 = 0$, there is a lifting $g : A \to B'$. Then we want to show that A is a regular local ring.

Let $a_1, \dots, a_n \in A$ such that $\{\overline{a}_i\}_{i=1}^n$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$. By the structure theorem of complete local ring, there is a surjective homomorphism $f : P := k[[x_1, \dots, x_n]] \to \widehat{A}$ defined by $x_i \mapsto a_i$, which induces an isomorphism of $P/\mathfrak{m}_P^2 \to A/\mathfrak{m}^2$.

Consider the thickening $P/\mathfrak{m}_P^{n+1} \to P/\mathfrak{m}_P^n$ and start with the map $A \to A/\mathfrak{m}^2 \simeq P/\mathfrak{m}_P^2$, we can lift to the maps $A \to P/\mathfrak{m}_P^n$ for every n by assumption, and thus get a map $g : \widehat{A} \to P$. Then the morphism $P \xrightarrow{f} \widehat{A} \xrightarrow{g} P$ induces an isomorphism on $P/\mathfrak{m}_P^2$.

> **Claim.** Let $(A, \mathfrak{m})$ be a local noetherian k-algebra with residue field k. Let $f \in \mathrm{End}_k A$ induces an isomorphism on $A/\mathfrak{m}^2$, then f is an isomorphism.

subproof. If $\operatorname{Im} f \subseteq \mathfrak{m}$, then $A = \operatorname{Im} f + \mathfrak{m}^2 \subseteq \mathfrak{m}$ gives a contradiction. Tensoring $A/\mathfrak{m}$, we have

$$0 \longrightarrow \ker f/\mathfrak{m} \ker f \longrightarrow A/\mathfrak{m} \xrightarrow{f} A/\mathfrak{m} \longrightarrow 0,$$

and thus $\ker f = 0$ by Nakayama's lemma. ☐

By Claim, $g \circ f$ is an automorphism, and hence f is injective. This shows that f is an isomorphism and $\widehat{A}$ is regular, which implies A is also regular. ☐

Corollary A.16

Let X be a nonsingular affine scheme over k. Let A be a local artinian ring over k, and let X' be a scheme flat over A such that $X' \times_A k$ is isomorphic to X. Then X' is isomorphic to the trivial deformation $X \times_k A$ of X over A.

Proof. Apply Proposition A.13 to $\mathrm{id}_X : X \to X$ and the infinitesimal thickening $i : X \to X'$, there exists a lifting $p : X' \to X$ such that $p \circ i = \mathrm{id}_X$. This define a map $f : X' \to X \times_k A$ of flats scheme over A, which induces identity map on closed fiber X. By the following claim, f is an isomorphism.

> **Claim.** Let A be a local artinian k-algebra. Let $f : X_1 \to X_2$ be an A-morphism between two finite type flat scheme over A, which induces an isomorphism of closed fibers $f \otimes_A k : X_1 \times_A k \to X_2 \times_A k$. Then f is an isomorphism.

subproof. Since $X_i \times_A k \to X_i$ is an isomorphism as the topological spaces, we remain to show that f induces isomorphism on each stalk of structure sheaf. We may check this locally. Let $f : B_2 \to B_1$ be the ring homomorphism corresponding to the open affine neighborhood of x and $f(x)$ respectively. Let $\mathfrak{p} \in \operatorname{Spec} B_1$ corresponding to x and $\mathfrak{q} = f^{-1}(\mathfrak{p})$. Then f induces a morphism $g : (B_2)_\mathfrak{q} \to (B_1)_\mathfrak{p}$, which induces an isomorphism $g \otimes_A A/\mathfrak{m}_A$ by assumption. By long exact sequence of Tor, we have $\operatorname{coker} g = 0$. Since B_i are flat over A, we have

$$0 = \operatorname{Tor}_1^A((B_1)_\mathfrak{p}, A/\mathfrak{m}_A) \longrightarrow \ker g \otimes_A A/\mathfrak{m}_A \longrightarrow (B_2)_q \otimes_A A/\mathfrak{m}_A \longrightarrow (B_1)_p \otimes_A A/\mathfrak{m}_A \longrightarrow 0,$$

and thus $\ker g = 0$ by Nakayama's lemma. □

□

Next, we investigate the relation between nonsingulairity and the T^i functors.

Theorem A.17

Let $X = \operatorname{Spec} B$ be an affine scheme over algebraically closed field k. Then X is nonsingular if and only if $T^1(B/k, M) = 0$ for all $M \in \mathcal{M}od_B$. Furthermore, if X is nonsingular, then $T^2(B/k, M) = 0$ for all M.

Proof. Write B as a quotient of a polynomial ring $A = k[x_1, \ldots, x_n]$ over k. Then X is nonsingular if and only if the conormal sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow 0 \tag{12}$$

is exact and $\Omega_{B/k}$ is locally free, i.e., a projective B-module. Since $\Omega_{A/k}$ is a free A-module, we conclude that X is nonsingular if and only if (12) is split exact. From the four term exact sequence in Proposition A.9

$$0 \longrightarrow T^0(B/k, M) \longrightarrow \operatorname{Hom}(\Omega_{A/k}, M) \longrightarrow \operatorname{Hom}(I/I^2, M) \longrightarrow T^1(B/k, M) \longrightarrow 0,$$

$T^1(B/k, M) = 0$ for all $M \in \mathcal{M}ob_B$ if and only if the map

$$\operatorname{Hom}(\Omega_{A/k}, M) \longrightarrow \operatorname{Hom}(I/I^2, M)$$

is surjective for all M, which is equivalent to the split exactness by considering $M = I/I^2$.

Again, by Proposition A.9 we have $T^2(B/k, M) = T^2(B/A, M)$. Since X is smooth, the localizing ideal I_x at $x \in X$ is generated by regular sequence in the regular local ring A_x. By Corollary A.8, $T^2(B_x/A_x, M) = 0$ for all $M \in \mathcal{M}ob_{B_x}$. By Lemma A.11, we conclude that $T^2(B/A, M) = 0$ for all $M \in \mathcal{M}ob_B$. □

Theorem A.18

A morphism of finite type $f : X \to Y$ of noetherian schemes is smooth if and only if it is flat, and $\mathcal{T}^1(X/Y, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F} \in \mathcal{C}oh(X)$. Furthermore, if f is smooth, then $\mathcal{T}^2(X/Y, \mathcal{F}) = 0$ for all $\mathcal{F}$.

Proof. The problem is local, so we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine and f is given by a ring homomorphism $A \to B$.

First suppose B is flat over A and $T^1(B/A, M) = 0$ for all $M \in \mathcal{M}ob_B$. Let $y \in Y$ corresponding to $\mathfrak{p} \in \operatorname{Spec} A$ and let $k = k(y)$ be its residue field. Let $A' = A/\mathfrak{p}$ and $B' = B \otimes_A A' = B/\mathfrak{p}B$, then for any $M \in \mathcal{M}ob_{B'}$ we obtain $T^1(B'/A', M) = T^1(B/A, M) = 0$ by base change A.12. Write $B' = A'[x_1, \ldots, x_n]/I$, then $T^1(B'/A', I/I^2) = 0$.

Now consider the flat base extension from A' to $\overline{k} = \overline{\text{Frac(A')}}$. By base change A.12, we have

$$T^1(B' \otimes_{A'} \overline{k}/\overline{k}, (I/I^2) \otimes \overline{k}) = 0.$$

Since $\overline{k}$ is flat over A', we have $B' \otimes \overline{k} = \overline{k}[x_1, \ldots, x_n]/\overline{I}$ and $(I/I^2) \otimes \overline{I} = \overline{I}/\overline{I}^2$, where $\overline{I} = I \otimes \overline{k}$. Then the proof of A.17 shows that $\operatorname{Spec} B' \otimes \overline{k}$ is nonsingular over $\overline{k}$. Hence, the geometric fibers of f are nonsingular, and thus f is smooth.

Conversely, suppose that B is smooth over A.

> **Claim.** $T^1(B/A, B/\mathfrak{m}) = 0$ for all maximal $\mathfrak{m}$ of B.

subproof. Let $\mathfrak{m}$ correspond to the point $x \in \operatorname{Spec} B$, let $f(x) = y$, and let $k = k(y)$. Apply base change on $B/\mathfrak{m} \in \mathcal{M}ob_{B \otimes_A k}$, we obtain

$$T^1(B/A, B/\mathfrak{m}) \otimes_k \overline{k} = T^1(B \otimes_A k/k, B/\mathfrak{m}) \otimes_k \overline{k} = T^1(B \otimes_A \overline{k}/\overline{k}, (B/\mathfrak{m}) \otimes_k \overline{k}) = 0$$

since the geometric fibers are nonsingular. Since $k \to \overline{k}$ is faithfully flat, we conclude that $T^1(B/A, B/\mathfrak{m}) = 0$. □

Lemma A.19 (Dévissage)

Let B be a noetherian ring, and let F be a **semi-exact** (that is, F(short exact sequence) exact in middle) additive functor from finitely generated B-modules to itself. Assume that $F(B/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of B. Then $F(M) = 0$ for all finitely generated B-modules.

subproof. Let $0 = M_n \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$ be a composition series such that $M_i/M_{i+1} \simeq B/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} B$. By semi-exactness, it suffices to prove the case for $M = B/\mathfrak{p}$.

We induct on $\dim \operatorname{Supp} M$. If $\dim \operatorname{Supp} M = 0$, then $M = B/\mathfrak{m}$ for some maximal ideal, and thus $F(M) = 0$ by hypothesis. For general case, let $M = B/\mathfrak{p}$. For any maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}$, pick an element $t \in \mathfrak{m} \setminus \mathfrak{p}$. Then t is a non-zero divisor for M and $M' := \operatorname{coker}(M \xrightarrow{t} M)$ has support of dimension $< \dim \operatorname{Supp} M$. By induction hypothesis, $F(M) \xrightarrow{t} F(M)$ is surjective. By Nakayama's lemma, $F(M)_{\mathfrak{m}} = 0$. Hence, $F(M) = 0$. $\square$

Since $T^1(B/A, -)$ satisfies the condition in Dévissage lemma, we have $T^1(B/A, M) = 0$ for all finitely generated B-modules, and hence for all B-modules, since T^i commute with direct sum. The same argument show also that $T^2(B/A, M) = 0$ for all M. $\square$

Theorem A.20

Let A be a regular local k-algebra with residue field $k = \overline{k}$, and $B = A/I$. Then B is a local complete intersection in A if and only if $T^2(B/k, M) = 0$ for all $M \in \mathcal{M}ob_B$.

Proof. Since A is regular, we have $T^1(A/k, M) = 0$ for $i = 1, 2$ and all M by Theorem A.17. From nine term exact sequence, we have $T^2(B/k, M) = T^2(B/A, M) = 0$ by Corollary A.8.

Conversely, suppose that $T^2(B/k, M) = 0$ for all M. As above, this implies $T^2(B/A, M) = 0$ for all M. To compute this group, we may take $R = A$, $I = I$, and let F map to a minimal set of generators $(a_1, \dots, a_s)$ of I, with kernel Q. Then the hypothesis $T^2(B/A, M) = 0$ for all M implies that

$$\operatorname{Hom}(F/IF, M) \twoheadrightarrow \operatorname{Hom}(Q/F_0, M)$$

for all M. In particular for $M = Q/F_0$ and the map $d_2 : Q/F_0 \to F/IF$, there exists $p : F/IF \to Q/F_0$ such that $p \circ d_2 = \operatorname{id}_{Q/F_0}$. By Nakayama's lemma, $Q \subseteq \mathfrak{m}_A F$. Then the identity map $p \circ f_2$ sends Q/F_0 into $\mathfrak{m}_A(Q/F_0)$, and thus $Q/F_0 = 0$ by Nakayama's lemma. But Q/F_0 actually is the first homology group of the Koszul complex $K_\bullet(a_1, \dots, a_s)$ over A, and the vanishing of this group is equivalent to $a_1, \dots, a_s$ being a regular sequence. $\square$

Remark A.21. If we define a relative local complete intersection morphism $f : X \to Y$ be the flat morphism whose geometric fibers are local complete intersection schemes, then f is relative local complete intersection morphism if and only if $\mathcal{T}^i(X/Y, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X.

A.3 Deformations of Rings

Proposition A.22 (local criterion of flatness)

Let $A' \to A$ be a surjective homomorphism of noetherian rings whose kernel J is square zero. Then an A'-module M' is flat over A' if and only if

(1) $M = M' \otimes_{A'} A$ is flat over A, and

(2) the natural map $M \otimes_A J \to M'$ is injective.

Proof. Since $J^2 = 0$, we may regard it as A-module and identify $M' \otimes_{A'} J$ with $M \otimes_A J$.

If M' is flat over A', then (1) follows by base extension, and (2) follows by tensoring M' with the exact sequence

$$0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0.$$

Conversely, suppose that M' satisfies conditions (1) and (2). It suffices to prove $\operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}') = 0$ for every $\mathfrak{p}' \subseteq A'$. Since J is nilpotent, it is contained in $\mathfrak{p}'$. Let $\mathfrak{p} = \mathfrak{p}'/J \in \operatorname{Spec} A$, then we have the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & \mathfrak{p}' & \longrightarrow & \mathfrak{p} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A'/\mathfrak{p}' & & A/\mathfrak{p} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Tensoring with M', we obtain

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}') & \longrightarrow & \operatorname{Tor}_1^{A}(M, A/\mathfrak{p}) & & \\
 & & \downarrow & & \downarrow & & \\
M \otimes_A J & \longrightarrow & M' \otimes_{A'} \mathfrak{p}' & \longrightarrow & M \otimes_A \mathfrak{p} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
M \otimes_A J & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & M' \otimes_{A'} A'/\mathfrak{p}' & & M \otimes_A A/\mathfrak{p} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

By condition (2), the second (and thus also the first) horizontal sequence is exact on the left. By snake lemma, $\operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}') \simeq \operatorname{Tor}_1^{A}(M, A/\mathfrak{p}) = 0$ by condition (1). □

We start by considering deformations (see the definition in subsection 1.4) of affine schemes. Let B be a k-algebra. A deformation of $\operatorname{Spec} B$ over the $k[\varepsilon]$ is a flat $k[\varepsilon]$-algebra B', together with a homomorphism $B' \to B$ inducing an isomorphism $B' \otimes_{k[\varepsilon]} k \to B$. By Proposition A.22, the flatness of B' over $k[\varepsilon]$ if and only if the exactness of

$$0 \longrightarrow B \xrightarrow{\varepsilon} B' \longrightarrow B \longrightarrow 0. \tag{13}$$

Here we think of B' and B on the right as rings, and B on the left as an ideal of B' with square zero, which is a B-module. Furthermore, B' is a $k[\varepsilon]$-algebra and B is a k-algebra. On the other hand, give an exact sequence (13) as k-algebra, we can recover the $k[\varepsilon]$-algebra structure of B' via $\varepsilon : B' \to B \xrightarrow{\varepsilon} B'$, which is the unique way compatible with the original exact sequence. This given a 1-1 correspondence between $\mathrm{Def}_X(k[\varepsilon])$ and the equivalent class of extensions as k-algebras of the k-algebra B by the B-module B.

Theorem A.23 (Grothendieck)

Let X be a nonsingular variety over k. Then the deformations of X over the dual numbers are in natural one-to-one correspondence with the elements of the group $H^1(X, \mathcal{T}_X)$.

Proof. Let X' be a deformation of X, and let $\mathcal{U} = \{U_i\}$ be an open affine covering covering of X. By Corollary A.16, the induced deformation U_i' of U_i is trivial. Let $\varphi_i : U_i \times_k k[\varepsilon] \xrightarrow{\sim} U_i'$, then we get an automorphism $\psi_{ij} = \varphi_j^{-1}\varphi_i$ of $U_{ij} \times_k k[\varepsilon]$, which induces identity map on closed fiber. For any affine subscheme $\operatorname{Spec} A$ of U_{ij}, ψ_{ij} induces an automorphism $A \otimes_k k[\varepsilon] \to A \otimes_k k[\varepsilon]$, which induces an identity map after modulo the ideal $A \otimes_k (\varepsilon)$. That is, we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A \otimes_k k[\varepsilon] & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi_{ij}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A \otimes_k k[\varepsilon] & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Since $A \otimes_k k[\varepsilon] = A \oplus A\varepsilon$ as an A-module, we may describe ψ_{ij} by $\psi_{ij}(a_1 + a_2\varepsilon) = a_1 + a_2\varepsilon + \delta(a_1)\varepsilon$ for some $\delta : A \to A\varepsilon$. To let ψ_{ij} be an algebra homomorphism, we need

$$\psi_{ij}((a_1 + a_2\varepsilon)(a_1' + a_2'\varepsilon)) = \psi_{ij}(a_1 a_1' + (a_1 a_2' + a_1' a_2)\varepsilon),$$

that is, $\delta \in \mathrm{Der}_k(A, A) \simeq \mathrm{Hom}_A(\Omega_{A/k}, A) \simeq \mathcal{T}_X(\operatorname{Spec} A)$. This identification is compatible with localization, so we may glue $\psi_{ij} \longleftrightarrow \theta_{ij} \in H^0(U_{ij}, \mathcal{T}_X)$. By construction, $(\theta_{ij}) \in Z^1(\mathcal{U}, \mathcal{T}_X)$. If we replace the original chosen isomorphisms $\varphi_i : U_i \times_k k[\varepsilon] \xrightarrow{\sim} U_i'$ by some others φ_i', then $\varphi_i^{-1}\varphi_i$ will coming from a section $\alpha_i \in H^0(U_i, \mathcal{T}_X)$, and the new cocycle $\theta_{ij}' = \theta_{ij} + \alpha_i - \alpha_j$. Hence, $(\theta_{ij}) \in \check{H}^1(\mathcal{U}, \mathcal{T}_X) = H^1(X, \mathcal{T}_X)$ is well-defined.

Reversing the process will get another direction. □

B Obstruction theory

B.1 Obstructions to deformations of schemes

Let X_0 be a scheme over k, and let $X_0 \hookrightarrow X$ be a deformation of X_0 over $C \in \mathsf{Art}_k$. Suppose C' is another artinian k-algebra, equipped with a surjective map $C' \to C$. In this section, we will assume the kernel J is annihilated by $\mathfrak{m}_{C'}$.

An **extension** of X over C' is a deformation X' of X_0 over C', together with a closed immersion $X \hookrightarrow X'$ inducing an isomorphism $X \xrightarrow{\sim} X' \times_{C'} C$. Two such extensions X' and X'' are **equivalent** if there is an isomorphism of deformations $X' \xrightarrow{\sim} X''$ compatible with the respective closed immersions of X into X' and X''. Let $\mathrm{Def}(X/C, C')$ be the set of equivalence classes of such extensions of X' over C'.

Remark B.1. There is a surjective map

$$\mathrm{Def}(X/C, C') \longrightarrow \mathrm{Def}(C' \to C)^{-1}(X),$$

which is not injective in general, since there may exist an automorphism of X' that restricts to the identity on X_0 but does not extend to an automorphism of X. This map becomes bijective if Schlessinger's condition (H_4) holds, see Lemma 1.20.

Theorem B.2

In the situation described above, with the additional structure as follows: let $Y_0 \subseteq X_0$ and $Y \subseteq X$ be closed subschemes such that $Y_0 = Y \times_C k$ under the identification $X_0 \xrightarrow{\sim} X \times_C k$. Fixed an extension X' of X over C'. An **(embedding) extension** of Y over C' in X' is a closed subscheme $Y' \subseteq X'$, flat over C', such that $Y' \times_{C'} C = Y$. Then the set of such extension of Y over C' in X' form a $H^0(Y_0, \mathcal{N}_{Y_0/X_0} \otimes_k J)$-pseudotorsor.

Proof. First we consider the affine case $X = \operatorname{Spec} B$, $X' = \operatorname{Spec} B'$, $Y = \operatorname{Spec} A$, and $A = B/I$. Suppose that $Y' = \operatorname{Spec} A'$ is given by $A' = B'/I'$. Tensoring the short exact sequence

$$0 \longrightarrow I' \longrightarrow B' \longrightarrow A' \longrightarrow 0$$

with the sequence $0 \to J \to C' \to C \to 0$, we will get a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C I & \longrightarrow & I' & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C B & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes_C A & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Proposition A.22, the exactness of the bottom two rows is equivalent to the flatness of B' and A' over C, respectively. The exactness of the first column follows from the flatness of A over C. Now we seek to classify the possible A' (resp. I') full in the diagram.

Suppose that I' and I'' are two choices of I' to fill in the diagram. Given $x \in I$, lift it to $x' \in I'$ and $x'' \in I''$. Then $x'' - x' \in B'$ maps to 0 in B, and thus $x'' - x' \in J \otimes_C B$. Denote its image in $J \otimes_C A$ by $\varphi(x)$. Notice that the choices of x' and x'' may differ by elements in $J \otimes_C I$, which maps to 0 in $J \otimes_C A$. So φ is well-defined additive map. In fact, it is an B-linear homomorphism $\varphi \in \mathrm{Hom}_B(I, J \otimes_C A)$.

Conversely, given I' and $\varphi \in \mathrm{Hom}_B(I, J \otimes_C A)$, we can define another ideal I'' solving the extension problem as follows. I'' is the set of $x'' \in B'$ whose image in B is $x \in I$, and such that for any lifting $x' \in I'$ of x, the image $x'' - x$ in $J \otimes_C A$ is equal to $\varphi(x)$.

If I', I'', I''' are three choices of solution, and if φ_1 is defined by I', I'' as above, φ_2 defined by I'', I''', and φ_3 defined by I', I'', then $\varphi_3 = \varphi_1 + \varphi_2$. So we conclude that

$$\begin{aligned} \mathrm{Hom}_B(I, J \otimes_C A) \times \{\text{deformation of } Y \text{ over } C' \text{ in } X'\} &\longrightarrow \{\text{deformation of } Y \text{ over } C' \text{ in } X'\} \\ (I', \varphi) &\longmapsto I'' \end{aligned}$$

is an proper transitive group action. Since we can't guarantee the existence, it only a pesudo-torsor. In this case,

$$\mathrm{Hom}_B(I, J \otimes_C A) = \mathrm{Hom}_B(I, J \otimes_k A_0).$$

Since J is a k-vector space, this term becomes to

$$\mathrm{Hom}_{B_0}(I_0, J \otimes_k A_0) = \mathrm{Hom}_{A_0}(I_0 \otimes_{B_0} A_0, A_0 \otimes_k J) = H^0(Y_0, \mathcal{N}_{Y_0/X_0} \otimes_k J).$$

In general case, the group action $H^0(Y_0, \mathcal{N}_{Y_0/X_0} \otimes_k J)$ on the set of Y' can be defined locally and functorial. So we may glue it to the global action. □

We back to the abstract deformation and consider the affine case first. Given a finite generated k-algebra B_0, and a deformation B of B_0 over C. We want to describe the set $\mathrm{Def}(B/C, C') \coloneqq \mathrm{Def}(\mathrm{Spec}\, B/C, C')$.

Theorem B.3

In the situation as beginning.

(a) There is an element $\delta \in T^2(B_0/k, B_0 \otimes J)$, called the **obstruction**, with the property that $\delta = 0$ if and only if an extension B' of B exists.

(b) If extensions exists, then $\mathrm{Def}(B/C, C')$ form a $T^1(B_0/k, B_0 \otimes J)$-torsor.

Proof. Take a presentation $R = C[x_1, \dots, x_n] \to B$ with kernel $I = \langle f_1, \dots, f_r \rangle_C$. Define a morphism $F \coloneqq R^r \to I$ by $e_i \mapsto f_i$, and denoted the kernel by Q. Lift each f_i to an element $f_i' \in R' \coloneqq C'[x_1, \dots, x_n]$, and define $I' = \langle f_1', \dots, f_r' \rangle$, $B' = R'/I'$. Now, consider

$$0 \longrightarrow I' \longrightarrow R' \longrightarrow B' \longrightarrow 0 \qquad \begin{aligned} 0 \longrightarrow Q' \longrightarrow F' &\longrightarrow I' \longrightarrow 0. \\ e_i' &\longmapsto f_i' \end{aligned}$$

Tensoring C over C', we have

$$B' \otimes_{C'} C \simeq \operatorname{coker}(I' \otimes_{C'} C \to R' \otimes_{C'} C) = B.$$

Tensoring with $0 \to J \to C' \to C \to 0$ over C', we get a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q' & \longrightarrow & Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F\otimes_C J & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & R_0\otimes_k J & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & B_0\otimes_k J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where we use $-\otimes_{C'} J = -\otimes_{C'} k \otimes_k J$ and $R_0 = k[x_1, \dots, x_n]$. By snake lemma, there is a map $\delta_0 : B_0 \otimes_k J$ depending on the lifting f' of f. By local criterion of flatness A.22, B' is flat over C' if and only if $\delta_0 = 0$.

Recall that in the definition of cotangent complex, any element in F_0 is of the form $f_j e_i - f_i e_j \in F$, and lifts to $f'_j e'_i - f'_i e'_j \in Q'$. So the map δ_0 factors through $\delta_1 : Q/F_0 \to B_0 \otimes_k J$. By definition of T^i functor and base change A.12, we get

$$
\begin{array}{ccccc}
\operatorname{coker}(\operatorname{Hom}(F/IF, B_0 \otimes_k J) \to \operatorname{Hom}(Q/F_0, B_0 \otimes J)) & = & T^2(B/C, B_0 \otimes J) & = & T^2(B_0/k, B_0 \otimes_k J) \\
\bar{\delta}_1 & & \longrightarrow & & \delta
\end{array}
$$

We claim that δ is independent of all the choices made. If we make a different choice of lifting f''_i of the f_i, then $f'_i - f''_i$ define a map from F' to $R_0 \otimes_k J$, and hence from $F/IF \to B_0 \otimes_k J$, and these go to zero in T^2. Suppose we choose a different polynomial ring $R^* \to B$. As in the proof of Lemma A.1, we may reduce to the case $R^* = R[y_1, \dots, y_s]$, and $y_i \mapsto 0 \in B$. Then the $\delta_0^*(y_i) = 0$ implies $\delta = \delta^*$ in $T^2(B_0/k, B_0 \otimes_k J)$.

If the extension B'/C' exists, take a polynomial ring R' over C' maps surjectively to B'. Consider

$$0 \longrightarrow I' \longrightarrow R' \longrightarrow B' \longrightarrow 0 \qquad 0 \longrightarrow Q' \longrightarrow F' \longrightarrow I' \longrightarrow 0. \tag{14}$$

Since B' and R' are flat over C', we also have I' is flat over C'. Let $R = R' \otimes_{C'} C$, $I = I' \otimes_{C'} C$, and $Q = Q' \otimes_{C'} C$. Then (14) $\otimes_{C'} C$ will give a presentation to calculate cotangent complex of B over C. Since $Q' \to Q$ is surjective, it follows that $\delta_0 = 0$, and thus $\delta = 0$.

Conversely, suppose that $\delta = 0$. We claim that for a suitable choose of lifting f'_i, the induced map $\delta_0 : Q \to B_0 \otimes J$ is zero. By assumption, the map $\delta_1 \in \operatorname{Hom}(Q/F_0, B_0 \otimes J)$ lifts to a map $\gamma : F/IF \to B_0 \otimes J$, and defines a map $F \to B_0 \otimes J$. Since F is free, it lifts to a map $F \to R_0 \otimes_k J$. Denote the image of e_i by $g_i \in R_0 \otimes_k J$. Now take $f_i'' = f_i' - g_i$, then the new δ_0 is zero, and thus the new B' is flat over C'. This complete the proof of (a).

Suppose one such extension B_1' exists. Take a presentation $0 \to I_0 \to R_0 \to B_0 \to 0$ as usual. Lift the surjection $R_0 \to B_0$ to a map $R \to B$, and to a map $R' \to B_1'$, which compatible with $B_1' \to B \to B_0$. By Nakayama's lemma, $R \to B$ and $R' \to B_1'$ are surjective.

For any another extension B_2' of B, the map $R \to B$ lifts to a map $R' \to B_2'$. Thus, every abstract deformation is also an embedded deformation $X \hookrightarrow \mathbb{A}_k^n$. By Theorem B.2, the embedded deformation is a $\mathrm{Hom}_B(I/I^2, B \otimes_C J)$-torsor. Suppose that we have two embedding deformation B_2' and B_3' are equivalent as abstract extensions of B. Choose an isomorphism $B_2' \simeq B_3'$ as extension of B over C'. Then we obtain two maps $R' \to B_2'$, which can be regard as the lifting of $R' \to R \to B$. By Lemma A.14, the differ of this two map is a C'-derivation of R' to $B_2' \otimes_{C'} J \simeq B \otimes_C J$, which can be regard as an element of

$$\mathrm{Hom}_{R'}(\Omega_{R'/C'}, B \otimes_C J) = \mathrm{Hom}_R(\Omega_{R'/C'} \otimes_{C'} C, B \otimes_C J) = \mathrm{Hom}_R(\Omega_{R/C}, B \otimes_C J).$$

Using the exact sequence A.9

$$\mathrm{Hom}_R(\Omega_{R/C}, B \otimes_C J) \longrightarrow \mathrm{Hom}(I/I^2, B \otimes_C J) \longrightarrow T^1(B/C, B \otimes_C J) \longrightarrow 0,$$

we observe that the ambiguity of embedding is exactly resolved by the image of the derivations. We conclude that $\mathrm{Def}(X/C, C')$ is a torsor under $T^1(B/C, B_0 \otimes_k J) = T^1(B_0/k, B_0 \otimes_k J)$ by base change A.12. □

Remark B.4. Given an extension B' of B over C', the automorphism group of B' as an extension of B over C is naturally isomorphic to the group $T^0(B_0/k, J \otimes_k B_0)$. Indeed, take $k = C'$ and $R = B'$ in Lemma A.14, we have an isomorphism

$$\begin{array}{ccc} \mathrm{Der}_{C'}(B', J \otimes_{C'} B') & \xrightarrow{\ \sim\ } & \mathrm{Aut}(B'/B) = \mathrm{Deck}_{C'}(B'/B) \\ \theta & \longrightarrow & \mathrm{id}_{B'} + \theta \end{array}$$

where $\theta_1 + \theta_2 \mapsto (\mathrm{id}_{B'} + \theta_1)(\mathrm{id}_{B'} + \theta_2) = \mathrm{id}_{B'} + \theta_1 + \theta_2$ since $J \otimes_{C'} B'$ is square zero ideal. Moreover, we have

$$\begin{aligned} \mathrm{Der}_{C'}(B', J \otimes_{C'} B') &= \mathrm{Hom}_{B'}(\Omega_{B'/C'}, J \otimes_{C'} B') = \mathrm{Hom}_{B_0}(\Omega_{B'/C'} \otimes_{B'} B_0, J \otimes_k B_0) \\ &= \mathrm{Hom}_{B_0}(\Omega_{B_0/k}, J \otimes_k B_0) = T^0(B_0/k, B_0 \otimes_k J). \end{aligned}$$

For general case, we recall the notation $\mathcal{T}^i_{X_0} = \mathcal{T}^i(X_0/k, \mathcal{O}_{X_0})$.

Theorem B.5

In the situation as beginning.

(a) There are three successive obstructions for the existence of an extension X' of X over C'

- $\delta_1 \in H^0(X_0, \mathcal{T}^2_{X_0} \otimes J)$ for the existence of local extension.
- $\delta_2 \in H^1(X_0, \mathcal{T}^1_{X_0} \otimes J)$ for the compatible isomorphism type of local extensions.
- $\delta_3 \in H^2(X_0, \mathcal{T}^0_{X_0} \otimes J)$ for the obstruction of gluing.

(b) Fixed an extension X'_1 of X over C', there is an exact sequence

$$0 \longrightarrow H^1(X_0, \mathcal{T}^0_{X_0} \otimes J) \longrightarrow \mathrm{Def}(X/C, C') \longrightarrow H^0(X_0, \mathcal{T}^1_{X_0} \otimes J) \longrightarrow H^2(X_0, \mathcal{T}^0_{X_0} \otimes J).$$

(c) The automorphism group of X'_1 as an extension of X over C' is naturally isomorphic to the group $H^0(X_0, \mathcal{T}_{X_0} \otimes J)$.

Proof. For each open affine subscheme $U_i \subseteq X$, by Theorem B.3 (a), there is an obstruction lying in $H^0(U_i, \mathcal{T}^2_{U_i} \otimes J)$ for the existence of an extension U'_i of U_i over C'. Since the definition of obstruction in Theorem B.3 (a) is compatible with localization, we may glue the data to a global obstruction $\delta_1 \in H^0(X_0, \mathcal{T}^2_{X_0} \otimes J)$.

If $\delta_1 = 0$, then for each U_i, there exists an extension U'_i of U_i over C'. For each $U_{ij} = U_i \cap U_j$, we have two deformations $U'_i|_{U_{ij}}$ and $U'_j|_{U_{ij}}$. By Theorem B.3 (b), their difference gives an element in $H^0(U_{ij}, \mathcal{T}^1 \otimes J)$. The difference of three of these is zero on U_{ijk}, so we get the second obstruction $\delta_2 \in H^1(X_0, \mathcal{T}^1_{X_0} \otimes J)$.

If $\delta_2 = 0$, then there exists isomorphism $\varphi_{ij} : U'_i|_{U_{ij}} \xrightarrow{\sim} U'_j|_{U_{ij}}$ as the extension of U_{ij} over C' for each i, j. Then $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki}|_{U_{ijk}}$ gives an automorphism of $U'_i|_{U_{ijk}}$. By Remark B.4, this gives an element in $H^0(U_{ijk}, \mathcal{T}^0 \otimes J)$, and hence glue to the third obstruction $\delta_3 \in H^2(X_0, \mathcal{T}^\circ_{X_0} \otimes J)$.

If $\delta_3 = 0$, then we can glue the extensions U'_i to get a global extension X' by gluing lemma.

For (b), fixed an extension X'_1 of X over C'. If X'_2 is another extension, by Theorem B.3, their difference on each open affine subscheme U_i gives an element of $H^0(U_i, \mathcal{T}^1_{X_0} \otimes J)$. These glue to a global element of $H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$. Conversely, a global element of $H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$ will give extensions of U_i that are isomorphic on the intersection U_{ij}, and will there is an obstruction in $H^2(X_0, \mathcal{T}^0_{X_0} \otimes J)$ to glue these to a global extension.

Suppose that two extension X'_2 and X'_3 give the same element in $H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$. Then there is an isomorphism $\varphi_i : X'_2|_{U_i} \xrightarrow{\sim} X'_3|_{U_i}$ as extension of U_i. Then $\psi_{ij} := \varphi_j^{-1} \circ \varphi_i$ is an automorphism of $X'_2|_{U_{ij}}$, which defines a element in $H^0(U_{ij}, \mathcal{T}^0_{X_0} \otimes J)$. These elements agree on U_{ijk}, so we get an element of $H^1(X_0, \mathcal{T}^0_{X_0} \otimes J)$. The vanishing of this element is equivalent to the existence of automorphism $\theta_i : X'_2|_{U_i} \to X'_2|_{U_i}$ as the extension of U_i such that $\psi_{ij} = \theta_j^{-1}\theta_i$. Then $\varphi_i \circ \theta_i$ glue to an isomorphism between X'_2 and X'_3 as the extension of X. Hence, we get the desired exact sequence for $\mathrm{Def}(X/C, C')$.

For (c), the automorphism group of $X'_1|_{\mathrm{Spec}\,B}$ is naturally isomorphic to

$$T^0(B_0/k, B_0 \otimes J) = \mathrm{Hom}_k(\Omega_{B_0/k}, B_0 \otimes J) = T_{B_0/k} \otimes J = (\mathcal{T}^0_{X_0} \otimes J)(\mathrm{Spec}\,B).$$

Hence, the automorphism group of X'_1 is naturally isomorphic to $H^0(X_0, T^0_{X_0} \otimes J)$ by gluing. ☐

Remark B.6. The assumption $mJ = 0$ is too strong for the discussion in the future. In fact, in most situations we only assume that $J^2 = 0$. The statements of the theorems above can be modified accordingly as follows:

(1) In the proof of Theorem B.2, we show that the embedding extension of Y over C' in X' form a

$$\mathrm{Hom}_B(I, J \otimes_C A) = \mathrm{Hom}_A(I/I^2, J \otimes_C A) = H^0(\operatorname{Spec} A, \mathcal{N}_{Y/X} \otimes f^*\widetilde{J})$$

pseudotorsor, where $f : Y \to \operatorname{Spec} C$ is the structure map.

(2) In the proof of Theorem B.3, the obstruction live in

$$T^2(B/C, B_0 \otimes_k J) = T^2(B/C, B \otimes_C J) = H^0(\operatorname{Spec} B, \mathcal{T}^2(\operatorname{Spec} B/\operatorname{Spec} C, f^*\widetilde{J}))$$

and we show that $\mathrm{Def}(B/C, C')$ form a

$$T^1(B/C, B \otimes_C J) = H^0(\operatorname{Spec} B, \mathcal{T}^1(\operatorname{Spec} B/\operatorname{Spec} C, f^*\widetilde{J}))$$

pseudotorsor, where $f : \operatorname{Spec} B \to \operatorname{Spec} C$ is the structure map.

(3) Finally, Theorem B.5 may be reformulated by replacing each cohomology group of the form $H^q(X_0, \mathcal{T}^p_{X_0} \otimes J)$ with

$$H^q(X, \mathcal{T}^p(X/\operatorname{Spec} C, f^*\widetilde{J})),$$

where $f : X \to \operatorname{Spec} C$ is the structure map. Or more concisely, we may denote it by $H^q(X, \mathcal{T}^p(X/C, f^*\widetilde{J}))$.

Corollary B.7

If X_0 is nonsingular, then

(a) There is just one obstruction in $H^2(X_0, \mathcal{T}_{X_0} \otimes J)$ for the existence of an extension X' of X over C'.

(b) If such extension exist, their equivalence classes form a $H^1(X_0, \mathcal{T}_{X_0} \otimes J)$-torsor.

Proof. By Theorem A.17, the sheaves $\mathcal{T}^1_{X_0}$ and $\mathcal{T}^2_{X_0}$ are zero. Then the assertion follows from Theorem B.5. □

Definition B.8. An **obstruction theory** for a functor $F : \mathcal{C} \to \mathsf{Sets}$ is a k-vector space V, together with, for every exact sequence $0 \to J \to C' \to C \to 0$ such that $\mathfrak{m}_{C'} J = 0$, and for every $u \in F(C)$, an element $\varphi(u, C') \in V \otimes_k J$ such that

(1) $\varphi(u, C') = 0$ if and only if $u \in \operatorname{Im} F(C' \to C)$.

(2) φ is functorial in the sense that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1 & \longrightarrow & C_1' & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_2 & \longrightarrow & C_2' & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

is a commutative diagram such that $\mathfrak{m}_{C_i'} J_i = 0$, then we have the following commutative diagram

$$\begin{array}{ccccc} F(C_1') & \longrightarrow & F(C_1) & \longrightarrow & V \otimes_k J_1 \\ \downarrow & & \downarrow & & \downarrow \\ F(C_2') & \longrightarrow & F(C_2) & \longrightarrow & V \otimes_k J_2. \end{array}$$

Corollary B.9

The deformation of smooth scheme X over k has obstruction theory.

Proof. It follows from Corollary B.7 and take $V = H^2(X, \mathcal{T}_X)$. □

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