Kodaira vanishing and Kodaira embedding

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This report mainly follows [2], except for the proof of Lemmas 4 and 5, which are taken from [1].

1 Kodaira vanishing theorem

An important problem in complex geometry is the computation of the cohomology groups of a holomorphic vector bundle. The Hirzebruch–Riemann–Roch theorem states that for a holomorphic vector bundle over a compact complex manifold of dimension n,

$$\sum_{j=0}^{n}(-1)^j \dim H^j(X,E) = \int_X \mathrm{ch}(E)\mathrm{td}(X),$$

where $\mathrm{ch}(E)$ is the Chern character of E and $\mathrm{td}(X)$ is the Todd class of the holomorphic tangent bundle of X. There are many vanishing theorems stating that under certain conditions, the higher cohomology groups vanish. Combined with the Hirzebruch–Riemann–Roch theorem, this allows us to compute $\dim H^0(X,E)$.

The Kodaira vanishing theorem is a vanishing theorem for positive line bundles.

Definition 1. A line bundle L on a complex manifold X is said to be *positive* if its first Chern class $c_1(L) \in H^2(X,\mathbf{R})$ can be represented by a closed positive real $(1,1)$-form.

Note that a closed positive real $(1,1)$-form is the same as a Kähler form. Therefore a complex manifold admitting a positive line bundle is Kähler.

On a Hermitian holomorphic vector bundle (E,h) on a complex manifold X, there exists a unique connection ∇, called the *Chern connection*, such that ∇ is compatible with the metric and $\nabla^{0,1} = \bar{\partial}_E$. The curvature $R = \nabla \circ \nabla$ of the Chern connection is of type $(1,1)$, i.e. $R \in A^{1,1}(X,\mathrm{End}(E))$.

For a line bundle L, $\mathrm{End}(L)$ is trivial, so R may be regarded as a scalar-valued $(1,1)$-form, which is also denoted by Ω. It can be shown that $\frac{i}{2\pi}\Omega$ is a closed real $(1,1)$-form representing the first Chern class of L.

Lemma 1. *Let L be a line bundle on a compact Kähler manifold X. Let α be a closed real $(1,1)$-form representing $c_1(L)$. Then there exists a Hermitian metric h on L such that $\frac{i}{2\pi}\Omega = \alpha$.*

Proof. Fix an arbitrary Hermitian metric h_0 on L, and denote the curvature of its Chern connection by Ω_0. If $h = e^f h_0$ is another Hermitian metric, where f is a real-valued function, then the curvature of its Chern connection is given by

$$\Omega = \bar{\partial}\partial \log(e^f h_0) = \bar{\partial}\partial f + \bar{\partial}\partial \log(h_0) = \bar{\partial}\partial f + \Omega_0.$$

Now α and $\frac{i}{2\pi}\Omega_0$ both represent $c_1(L)$, hence $\alpha - \frac{i}{2\pi}\Omega_0$ is a d-exact real $(1,1)$-form. By the $\partial\bar{\partial}$ lemma, there exists a function g such that

$$\bar{\partial}\partial g = \alpha - \frac{i}{2\pi}\Omega_0.$$

Since $\bar{\partial}\partial g$ is real,

$$\bar{\partial}\partial(g + \bar{g}) = \bar{\partial}\partial g - \overline{\bar{\partial}\partial g} = 0.$$

By the Kähler identities,

$$\begin{aligned} -i\partial^*\partial(g+\bar{g}) &= [\Lambda, \bar{\partial}]\partial(g+\bar{g}) \\ &= \Lambda\bar{\partial}\partial(g+\bar{g}) \text{ since } \partial(g+\bar{g}) \text{ is of degree 1} \\ &= 0. \end{aligned}$$

Thus $\mathrm{Re}(g) = \frac{1}{2}(g+\bar{g})$ is harmonic. Since X is compact, it is constant, and we may assume it is 0. Then $f = -2\pi i g$ is real and

$$\bar{\partial}\partial f = -2\pi i\alpha - \Omega_0.$$

Hence $h = e^f h_0$ satisfies

$$\frac{i}{2\pi}\Omega = \frac{i}{2\pi}(\bar{\partial}\partial f + \Omega_0) = \alpha.$$

□

Corollary 2. *A line bundle L on a compact Kähler manifold is positive if and only if it admits a Hermitian metric whose Chern connection has positive curvature.*

Example 1. The standard example of a positive line bundle is $\mathcal{O}_{\mathbf{P}^n}(1)$. Define a Hermitian metric h on $\mathcal{O}(1)$ as follows: On $U_i = \{z_i \neq 0\}$, z_i is a holomorphic trivializing section of $\mathcal{O}(1)$. Define h on U_i by

$$h(z_i, z_i) = \frac{1}{\sum_{l=0}^n |\frac{z_l}{z_i}|^2}.$$

On $U_i \cap U_j$, we have

$$\left|\frac{z_j}{z_i}\right|^2 \frac{1}{\sum_{l=0}^n |\frac{z_l}{z_i}|^2} = \frac{1}{\sum_{l=0}^n |\frac{z_l}{z_j}|^2}.$$

Therefore these Hermitian metrics glue to a globally defined Hermitian metric on $\mathcal{O}(1)$.

On U_i, the curvature of the Chern connection is given by

$$\Omega = \bar{\partial}\partial \log \left(\frac{1}{\sum_{l=0}^{n} |\frac{z_l}{z_i}|^2} \right) = \partial\bar{\partial} \log \left(\sum_{l=0}^{n} |\frac{z_l}{z_i}|^2 \right).$$

Thus $\frac{i}{2\pi}\Omega$ is the Fubini–Study Kähler form on $\mathbf{P}^n$, and hence $\mathcal{O}(1)$ is positive.

The fact that $\mathcal{O}(1)$ is positive will be used in the proof of the Kodaira embedding theorem.

Theorem 3 (Kodaira–Akizuki–Nakano vanishing theorem)**.** *Let L be a positive line bundle on a compact Kähler manifold X of dimension n. Then for $p+q > n$,*

$$H^q(X, \Omega_X^p \otimes L) = 0.$$

To prove the theorem, we need a commutation relation generalizing the Kähler identities. It is valid on any Hermitian holomorphic vector bundle.

Let (E, h) be a Hermitian holomorphic vector bundle of rank r on a compact Kähler manifold (X, g) of dimension n. Denote the induced Hermitian metric on $\Lambda^{p,q}X \otimes E$ by $\langle \cdot, \cdot \rangle$.

To simplify computations with differential operators on $A^{p,q}(X, E)$, we first prove the existence of a holomorphic local frame for E that is orthonormal up to $O(|z|^2)$.

Lemma 4. *Let $x_0 \in X$, and let $z = (z_1, \ldots, z_n)$ be holomorphic local coordinates centered at x_0. There exist a holomorphic local frame $\{e_\lambda\}_{\lambda=1}^r$ for E and constants $b_{jk\lambda\mu}$ such that*

$$h(e_\lambda, e_\mu) = \delta_{\lambda\mu} + \sum_{j,k=1}^{n} b_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3).$$

Proof. First choose an arbitrary holomorphic local frame $\{g_\lambda\}$ for E. By a linear transformation with constant coefficients, we may assume $\{g_\lambda\}$ is orthonormal at x_0. Thus

$$h(g_\lambda, g_\mu) = \delta_{\lambda\mu} + \sum_{j=1}^{n} (a_{j\lambda\mu} z_j + a'_{j\lambda\mu} \bar{z}_j) + O(|z|^2),$$

for constants $a_{j\lambda\mu}, a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu} = \overline{a_{j\mu\lambda}}$.

Set

$$f_\lambda = g_\lambda - \sum_{j=1}^{n} \sum_{\nu=1}^{r} a_{j\lambda\nu} z_j g_\nu.$$

Then $\{f_\lambda\}$ is a holomorphic frame in a neighborhood of x_0, and

$$\begin{aligned} h(f_\lambda, f_\mu) &= h(g_\lambda, g_\mu) - h\left(g_\lambda, \sum_j \sum_\nu a_{j\mu\nu} z_j g_\nu\right) \\ &\quad - h\left(\sum_j \sum_\nu a_{j\lambda\nu} z_j g_\nu, g_\mu\right) + O(|z|^2) \\ &= \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu} z_j + a'_{j\lambda\mu} \bar{z}_j) - \sum_j \overline{a_{j\mu\lambda}} \bar{z}_j - \sum_j a_{j\lambda\mu} z_j + O(|z|^2) \\ &= \delta_{\lambda\mu} + O(|z|^2) \\ &= \delta_{\lambda\mu} + \sum_{j,k} (b_{jk\lambda\mu} z_j \bar{z}_k + b'_{jk\lambda\mu} z_j z_k + b''_{jk\lambda\mu} \bar{z}_j \bar{z}_k) + O(|z|^3), \end{aligned}$$

for some constants $b_{jk\lambda\mu}, b'_{jk\lambda\mu}, b''_{jk\lambda\mu}$ such that $b''_{jk\lambda\mu} = \overline{b'_{jk\mu\lambda}}$.

Now set

$$e_\lambda = f_\lambda - \sum_{j,k=1}^{n} \sum_{\nu=1}^{r} b'_{jk\lambda\nu} z_j z_k f_\nu.$$

Then $\{e_\lambda\}$ is a holomorphic frame in a neighborhood of x_0, and

$$\begin{aligned} h(e_\lambda, e_\mu) &= h(f_\lambda, f_\mu) - h\left(f_\lambda, \sum_{j,k} \sum_\nu b'_{jk\mu\nu} z_j z_k f_\nu\right) \\ &\quad - h\left(\sum_{j,k} \sum_\nu b'_{jk\lambda\nu} z_j z_k f_\nu, f_\mu\right) + O(|z|^3) \\ &= \delta_{\lambda\mu} + \sum_{j,k} (b_{jk\lambda\mu} z_j \bar{z}_k + b'_{jk\lambda\mu} z_j z_k + b''_{jk\lambda\mu} \bar{z}_j \bar{z}_k) \\ &\quad - \sum_{j,k} \overline{b'_{jk\mu\lambda}} \bar{z}_j \bar{z}_k - \sum_{j,k} b'_{jk\lambda\mu} z_j z_k + O(|z|^3) \\ &= \delta_{\lambda\mu} + \sum_{j,k} b_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3). \end{aligned}$$

□

Define a Hermitian inner product on $A^{p,q}(X, E)$ by

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle \mathrm{vol},$$

where vol is the volume form induced by g. Let ∇ be the Chern connection on E. Let $\partial^* : A^{p,q}(X) \to A^{p-1,q}(X)$ denote the formal adjoint of ∂ with respect to the inner product induced by g and vol, and let $(\nabla^{1,0})^* : A^{p,q}(X, E) \to A^{p-1,q}(X, E)$ denote the formal adjoint of $\nabla^{1,0}$ with respect to $(\cdot, \cdot)$. Extend the

Lefschetz operator L and dual Lefschetz operator Λ to $\Lambda^{p,q}X \otimes E$ by $L = L \otimes \mathrm{id}$ and $\Lambda = \Lambda \otimes \mathrm{id}$. We have the following generalization of the Kähler identity $[\Lambda, \bar{\partial}] = -i\partial^*$.

Lemma 5. *Let E be a Hermitian holomorphic vector bundle on a compact Kähler manifold X. Then*

$$[\Lambda, \bar{\partial}_E] = -i(\nabla^{1,0})^*$$

Proof. Fix $x_0 \in X$. By Lemma 4, there exists a holomorphic local frame $\{e_\lambda\}$ in a neighborhood U of x_0 such that

$$h(e_\lambda, e_\mu) = \delta_{\lambda\mu} + \sum_{j,k=1}^{n} b_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3).$$

Set $h_{\lambda\mu} = h(e_\lambda, e_\mu)$ and let H be the Hermitian matrix $H = (h_{\lambda\mu})_{\lambda,\mu=1}^{r}$. Then the Chern connection is given by

$$\nabla e_\lambda = \sum_{\nu=1}^{r} \omega_{\nu\lambda} \otimes e_\nu,$$

where $(\omega_{\nu\lambda}) = \bar{H}^{-1}\partial(\bar{H})$. Thus for

$$\alpha = \sum_{\lambda=1}^{r} \alpha_\lambda \otimes e_\lambda \in A^{p,q}(X, E),$$

we have

$$\begin{aligned} \nabla\alpha &= \sum_{\lambda=1}^{r} (d\alpha_\lambda \otimes e_\lambda + (-1)^{p+q} \alpha_\lambda \wedge \nabla e_\lambda) \\ &= \sum_{\lambda=1}^{r} \left(d\alpha_\lambda \otimes e_\lambda + \sum_{\nu=1}^{r} \omega_{\nu\lambda} \wedge \alpha_\lambda \otimes e_\nu \right). \end{aligned}$$

Thus

$$\nabla^{1,0}\alpha = \sum_{\lambda=1}^{r} \left(\partial\alpha_\lambda \otimes e_\lambda + \sum_{\nu=1}^{r} \omega_{\nu\lambda} \wedge \alpha_\lambda \otimes e_\nu \right).$$

To compute the formal adjoint of $\nabla^{1,0}$, let $\beta = \sum_\mu \beta_\mu \otimes e_\mu$ be a smooth section of $\Lambda^{p+1,q}X \otimes E$ with compact support in U. We have

$$\begin{aligned}
(\nabla^{1,0}\alpha, \beta) &= \int_U \langle \nabla^{1,0}\alpha, \beta\rangle \mathrm{vol} \\
&= \int_U \left\langle \sum_\lambda \left(\partial\alpha_\lambda \otimes e_\lambda + \sum_\nu \omega_{\nu\lambda} \wedge \alpha_\lambda \otimes e_\nu \right), \sum_\mu \beta_\mu \otimes e_\mu \right\rangle \mathrm{vol} \\
&= \int_U \sum_{\lambda,\mu} g(\partial\alpha_\lambda, \beta_\mu) h_{\lambda\mu} + \sum_{\lambda,\nu,\mu} g(\omega_{\nu\lambda} \wedge \alpha_\lambda, \beta_\mu) h_{\nu\mu} \mathrm{vol} \\
&= \int_U \sum_{\lambda,\mu} g(\alpha_\lambda, \partial^*(\overline{h_{\lambda\mu}}\beta_\mu)) + \sum_{\lambda,\nu,\mu} g(\omega_{\nu\lambda} \wedge \alpha_\lambda, \beta_\mu) h_{\nu\mu} \mathrm{vol},
\end{aligned}$$

since ∂^* is the formal adjoint of ∂. Now write

$$\begin{aligned}
&\sum_{\lambda,\mu} g(\alpha_\lambda, \partial^*(\overline{h_{\lambda\mu}}\beta_\mu)) + \sum_{\lambda,\nu,\mu} g(\omega_{\nu\lambda} \wedge \alpha_\lambda, \beta_\mu) h_{\nu\mu} = \\
&\sum_{\lambda,\mu} g(\alpha_\lambda, \overline{h_{\lambda\mu}}\partial^*\beta_\mu) + g(\alpha_\lambda, [\partial^*, \overline{h_{\lambda\mu}}]\beta_\mu) + \sum_{\lambda,\nu,\mu} g(\omega_{\nu\lambda} \wedge \alpha_\lambda, \beta_\mu) h_{\nu\mu}.
\end{aligned}$$

Since ∂^* is a first-order operator, $[\partial^*, \overline{h_{\lambda\mu}}]$ is a zeroth-order operator, i.e. a linear operator. Thus

$$g(\alpha_\lambda, [\partial^*, \overline{h_{\lambda\mu}}]\beta_\mu) + \sum_{\lambda,\nu,\mu} g(\omega_{\nu\lambda} \wedge \alpha_\lambda, \beta_\mu) h_{\nu\mu}$$

is sesquilinear in α and β. Using the Hermitian metric $\langle \cdot, \cdot \rangle$, we can write it as $\langle \alpha, P\beta \rangle$ for some linear operator P. Then

$$\begin{aligned}
(\nabla^{1,0}\alpha, \beta) &= \int_U \left(\sum_{\lambda,\mu} g(\alpha_\lambda, \overline{h_{\lambda\mu}}\partial^*\beta_\mu) + \langle \alpha, P\beta \rangle \right) \mathrm{vol} \\
&= \int_U \left(\left\langle \sum_\lambda \alpha_\lambda \otimes e_\lambda, \sum_\mu \partial^*\beta_\mu \otimes e_\mu \right\rangle + \langle \alpha, P\beta \rangle \right) \mathrm{vol} \\
&= (\alpha, \partial^*\beta) + (\alpha, P\beta),
\end{aligned}$$

and hence

$$(\nabla^{1,0})^* = \partial^* + P.$$

Note that

$$\begin{aligned}
[\partial^*, \overline{h_{\lambda\mu}}] &= -* \circ \bar\partial \circ * \circ \overline{h_{\lambda\mu}} - \overline{h_{\lambda\mu}} \circ \partial^* \\
&= -* \circ \bar\partial \circ \overline{h_{\lambda\mu}} \circ * - \overline{h_{\lambda\mu}} \circ \partial^* \\
&= -* \circ (\bar\partial \overline{h_{\lambda\mu}} \wedge) \circ * - * \circ \overline{h_{\lambda\mu}} \circ \bar\partial \circ * - \overline{h_{\lambda\mu}} \circ \partial^* \\
&= -* \circ (\overline{\partial h_{\lambda\mu}} \wedge) \circ *.
\end{aligned}$$

Since $h_{\lambda\mu} = \delta_{\lambda\mu} + O(|z|^2)$, $\overline{\partial h_{\lambda\mu}} = O(|z|)$. Also, $(\omega_{\nu\lambda}) = \bar{H}^{-1}\partial(\bar{H}) = O(|z|)$. It follows that $P = O(|z|)$.

By the Kähler identities on $A^{p,q}(U)$,

$$[\Lambda, \bar{\partial}_E] = [\Lambda, \bar{\partial}] = -i\partial^* = -i((\nabla^{1,0})^* - P).$$

Thus P is in fact a globally defined linear operator. The above argument shows that P vanishes at every point. Hence

$$[\Lambda, \bar{\partial}_E] = -i(\nabla^{1,0})^*.$$

□

Lemma 6. *Let E be a Hermitian holomorphic vector bundle on a compact Kähler manifold X. Then for every harmonic form $\alpha \in \mathcal{H}^{p,q}(X, E)$,*

$$\frac{i}{2\pi}(R\Lambda\alpha, \alpha) \leq 0$$

and

$$\frac{i}{2\pi}(\Lambda R\alpha, \alpha) \geq 0.$$

Proof. Recall that the curvature of the Chern connection is of type $(1,1)$. Then

$$\begin{aligned} R &= \nabla \circ \nabla \\ &= (\nabla^{1,0} + \bar{\partial}_E) \circ (\nabla^{1,0} + \bar{\partial}_E) \\ &= \nabla^{1,0} \circ \bar{\partial}_E + \bar{\partial}_E \circ \nabla^{1,0}, \end{aligned}$$

since the $(2,0)$-part must be zero.

We have

$$\begin{aligned} i(R\Lambda\alpha, \alpha) &= i(\nabla^{1,0}\bar{\partial}_E\Lambda\alpha, \alpha) + i(\bar{\partial}_E\nabla^{1,0}\Lambda\alpha, \alpha) \\ &= i(\bar{\partial}_E\Lambda\alpha, (\nabla^{1,0})^*\alpha) + i(\nabla^{1,0}\Lambda\alpha, \bar{\partial}_E^*\alpha) \\ &= (\bar{\partial}_E\Lambda\alpha, -i(\nabla^{1,0})^*\alpha) \text{ since } \alpha \text{ is harmonic} \\ &= (\bar{\partial}_E\Lambda\alpha, [\Lambda, \bar{\partial}_E]\alpha) \text{ by Lemma 5} \\ &= -(\bar{\partial}_E\Lambda\alpha, \bar{\partial}_E\Lambda\alpha) \text{ since } \alpha \text{ is harmonic} \\ &\leq 0. \end{aligned}$$

For the second inequality,

$$\begin{aligned} i(\Lambda R\alpha, \alpha) &= i(\Lambda\nabla^{1,0}\bar{\partial}_E\alpha, \alpha) + i(\Lambda\bar{\partial}_E\nabla^{1,0}\alpha, \alpha) \\ &= i(\Lambda\bar{\partial}_E\nabla^{1,0}\alpha, \alpha) \text{ since } \alpha \text{ is harmonic} \\ &= i([\Lambda, \bar{\partial}_E]\nabla^{1,0}\alpha, \alpha) + i(\bar{\partial}_E\Lambda\nabla^{1,0}\alpha, \alpha) \\ &= i(-i(\nabla^{1,0})^*\nabla^{1,0}\alpha, \alpha) + i(\Lambda\nabla^{1,0}\alpha, \bar{\partial}_E^*\alpha) \text{ by Lemma 5} \\ &= (\nabla^{1,0}\alpha, \nabla^{1,0}\alpha) \text{ since } \alpha \text{ is harmonic} \\ &\geq 0. \end{aligned}$$

□

Proof of Theorem 3. Since L is positive, by Lemma 2 there exists a Hermitian metric on L such that the real $(1,1)$-form

$$\omega = \frac{i}{2\pi}\Omega$$

is a Kähler form.

Now take ω to be the Kähler form on X. For a (p,q)-form α and a local section s of L, we have

$$L(\alpha \otimes s) = (\frac{i}{2\pi}\Omega \wedge \alpha) \otimes s = \frac{i}{2\pi}R(\alpha \otimes s),$$

where on the right-hand side we view R as a linear operator $A^{p,q}(X,E) \to A^{p+1,q+1}(X,E)$. Thus

$$L = \frac{i}{2\pi}R$$

as operators.

Suppose

$$p + q > n.$$

By Lemma 6, for every harmonic form $\alpha \in \mathcal{H}^{p,q}(X,L)$,

$$0 \leq (\frac{i}{2\pi}[\Lambda, R]\alpha, \alpha) = ([\Lambda, L]\alpha, \alpha) = (n - (p+q))||\alpha||^2,$$

where the last equality follows from the commutation relation for L and Λ (part of the $\mathfrak{sl}(2)$ representation). This implies $||\alpha||^2 = 0$, and hence $\alpha = 0$. Since every class in $H^q(X, \Omega_X^p \otimes L) \cong H^{p,q}(X,L)$ can be represented by a harmonic form, we conclude that

$$H^q(X, \Omega_X^p \otimes L) = 0.$$

□

Example 2. We know that for $k \geq 0$, $H^0(\mathbf{P}^n, \mathcal{O}(k))$ can be identified with the space $\mathbf{C}[z_0, \ldots, z_n]_k$ of homogeneous polynomials of degree k. Using Serre duality and the fact that $K_{\mathbf{P}^n} \cong \mathcal{O}(-n-1)$, we have

$$H^n(\mathbf{P}^n, \mathcal{O}(k)) \cong H^0(\mathbf{P}^n, \mathcal{O}(-n-1-k))^* \cong \mathbf{C}[z_0, \ldots, z_n]^*_{-n-1-k}$$

for $k \leq -n-1$.

Using Kodaira vanishing and Serre duality, we can show that $H^q(\mathbf{P}^n, \mathcal{O}(k))$ vanishes for all other q, k. By example 1, $\mathcal{O}(1)$ is positive, and hence for every $k \geq 1$, $\mathcal{O}(k)$ is positive. By Theorem 3, $H^q(\mathbf{P}^n, K_{\mathbf{P}^n} \otimes \mathcal{O}(k)) = 0$ for $n + q > n$, i.e.

$$H^q(\mathbf{P}^n, \mathcal{O}(k)) = 0 \text{ for } q > 0 \text{ and } k > -n-1.$$

By Serre duality, for $0 < q < n$ and $k \leq -n-1$,

$$H^q(\mathbf{P}^n, \mathcal{O}(k)) \cong H^{n-q}(\mathbf{P}^n, \mathcal{O}(-n-1-k))^* = 0,$$

since $-n-1-k > -n-1$. Finally, since a nontrivial line bundle and its dual cannot both have nonzero global sections,

$$H^0(\mathbf{P}^n, \mathcal{O}(k)) = 0 \text{ for } k < 0.$$

A similar proof gives Serre's vanishing theorem.

Theorem 7 (Serre's vanishing theorem)**.** *Let L be a positive line bundle on a compact Kähler manifold X, and let E be a holomorphic vector bundle on X. Then there exists a constant m_0 such that for all $m \geq m_0$ and $q > 0$,*

$$H^q(X, E \otimes L^m) = 0.$$

Proof. As in the proof of theorem 3, choose a Hermitian metric on L such that

$$\omega = \frac{i}{2\pi}\Omega_L$$

is a Kähler form, and give X the corresponding Kähler structure. Set

$$E' = E \otimes K_X^*,$$

and give E' an arbitrary Hermitian metric. The Chern connection on $E' \otimes L^m$ is given by

$$\nabla = \nabla_{E'} \otimes \mathrm{id}_{L^m} + \mathrm{id}_{E'} \otimes \nabla_{L^m},$$

and its curvature is given by

$$R = R_{E'} \otimes \mathrm{id}_{L^m} + \mathrm{id}_{E'} \otimes R_{L^m}.$$

Since $\Omega_{L^m} = m\Omega_L$,

$$\frac{i}{2\pi}R_{L^m} = mL_\omega$$

on L^m-valued forms, where L_ω is the Lefschetz operator. Thus on $A^{p,q}(X, E' \otimes L^m)$,

$$\begin{aligned}\frac{i}{2\pi}[\Lambda, R] &= \frac{i}{2\pi}([\Lambda, R_{E'}] \otimes \mathrm{id}_{L^m}) + m(\mathrm{id}_{E'} \otimes [\Lambda, L_\omega]) \\ &= \frac{i}{2\pi}([\Lambda, R_{E'}] \otimes \mathrm{id}_{L^m}) + m(\mathrm{id}_{E'} \otimes (n - (p+q))\mathrm{id}_{L^m}),\end{aligned}$$

by the commutation relation for L and Λ.

Let $\alpha \in \mathcal{H}^{p,q}(X, E' \otimes L^m)$. By Lemma 6,

$$\begin{aligned}0 &\leq \frac{i}{2\pi}([\Lambda, R]\alpha, \alpha) \\ &= \frac{i}{2\pi}(([\Lambda, R_{E'}] \otimes \mathrm{id}_{L^m})\alpha, \alpha) + m(n - (p+q))||\alpha||^2.\end{aligned}$$

Since $[\Lambda, R_{E'}]$ is a linear operator and X is compact, there exists a constant $C > 0$ independent of α and m such that

$$|(([\Lambda, R_{E'}] \otimes \mathrm{id}_{L^m})\alpha, \alpha)| \leq C||\alpha||^2,$$

and hence

$$0 \leq \left(\frac{C}{2\pi} + m(n - (p+q))\right)||\alpha||^2. \tag{1}$$

Choose $m_0 > \frac{C}{2\pi}$. Then for all $m \geq m_0$ and $q > 0$, (1) forces every harmonic form in $\mathcal{H}^{n,q}(X, E' \otimes L^m)$ to be equal to zero. Thus

$$H^q(X, E \otimes L^m) \cong H^q(X, E' \otimes K_X \otimes L^m) \cong \mathcal{H}^{n,q}(X, E' \otimes L^m) = 0.$$

□

2 Kodaira embedding theorem

The Kodaira embedding theorem provides a criterion deciding whether a compact Kähler manifold is projective. It also shows that for a line bundle on a compact Kähler manifold, positivity is equivalent to ampleness.

Let L be a line bundle on a compact complex manifold X. Let $s_0, \ldots, s_N$ be a basis of $H^0(X, L)$. Then we have a map $\phi_L : X \backslash \mathrm{Bs}(L) \to \mathbf{P}^N$ defined by

$$\phi_L(x) = (s_0(x) : \cdots : s_N(x)).$$

For a different choice of basis of $H^0(X, L)$, the resulting map differs by a linear transformation of $\mathbf{P}^N$. Therefore, whether ϕ_L is an embedding is independent of the choice of basis.

ϕ_L is defined on all of X if and only if $\mathrm{Bs}(L) = \emptyset$. Equivalently, for every $x \in X$, the restriction map $H^0(X, L) \to L(x)$ is surjective.

ϕ_L is injective, or *separates points*, if and only if for every pair of distinct points $x_1, x_2 \in X$, there exists $s \in H^0(X, L)$ such that $s(x_1) \neq 0$ and $s(x_2) = 0$. Equivalently, for every pair of distinct points $x_1, x_2 \in X$, the restriction map $H^0(X, L) \to L(x_1) \oplus L(x_2)$ is surjective.

Next we determine a criterion for ϕ_L to be an immersion, or *separates tangents*. Let $x \in X \backslash \mathrm{Bs}(L)$. We can choose a basis $s_0, \ldots, s_N$ for $H^0(X, L)$ such that $s_0(x) \neq 0$ and $s_i(x) = 0$ for $1 \leq i \leq N$. Then $t_i(y) = \frac{s_i(y)}{s_0(y)}, 1 \leq i \leq N$ are well-defined functions near x, and ϕ_L is locally given by $y \to (t_1(y), \ldots, t_N(y))$. Thus $d\phi_L, x$ is injective if and only if $dt_{1,x}, \ldots, dt_{N,x}$ span $\Lambda^1_x X$.

Note that s_0 is a local trivialization section for L. Also, $s_1, \ldots, s_N$ is a basis for the the subspace of sections in $H^0(X, L)$ vanishing at x, which may be identified with $H^0(X, L \otimes \mathcal{I}_{\{x\}})$. Define

$$d_x : H^0(X, L \otimes \mathcal{I}_{\{x\}}) \to L(x) \otimes \Lambda^1_x X$$

by

$$d_x(s) = s_0(x) \otimes d(\frac{s}{s_0})_x.$$

This is well defined, since a different choice of s_0 multiplies it by a nonvanishing function λ, and $d(\frac{s}{\lambda s_0})_x = \frac{1}{\lambda(x)} d(\frac{s}{s_0})_x$ for s vanishing at x. Now $d\phi_L, x$ is injective if and only if $dt_{1,x}, \ldots, dt_{N,x}$ span $\Lambda^1_x X$ if and only if d_x is surjective.

Definition 2. A line bundle L on a compact complex manifold X is said to be *ample* if for some positive integer k, the kth tensor power L^k defines an embedding $\phi_{L^k} : X \hookrightarrow \mathbf{P}^N$.

Lemma 8. *A compact complex manifold is projective if and only if it admits an ample line bundle.*

Proof. By definition, a compact complex manifold admitting an ample line bundle is projective.

Conversely, suppose $\phi : X \to \mathbf{P}^N$ is an embedding. Consider the line bundle

$$L = \phi^* \mathcal{O}(1).$$

Let $z_0, \dots, z_N$ be homogeneous coordinates on $\mathbf{P}^N$. These can be regarded as sections of $\mathcal{O}(1)$, and pull back to $s_i = \phi^*(z_i) \in H^0(X, L)$, $0 \le i \le N$. Then the embedding ϕ is given by $\phi(x) = (s_0(x) : \cdots : s_N(x))$. Note that $\{s_i\}$ is not necessarily a basis of $H^0(X, L)$, but it is clear that ϕ_L is also an embedding. Thus L is ample. $\square$

Theorem 9 (Kodaira embedding theorem)**.** *Let L be a line bundle on a compact Kähler manifold X. Then L is positive if and only if L is ample.*

To prove the theorem, we will use blow-ups to transform points in X into codimension-1 hypersurfaces. Thus we need to study how positive line bundles behave under blow-ups. Let X be a compact complex manifold. Let $\sigma : \hat{X} \to X$ be the blow-up of X along a finite number of points $x_1, \dots, x_l$. Denote the exceptional divisors $\sigma^{-1}(\{x_j\})$ by E_j.

Lemma 10. *Let X be a compact complex manifold. Let $\sigma : \hat{X} \to X$ be the blow-up of X as above. Let L be a positive line bundle on X, and let M be an arbitrary line bundle on X. For any postive integers $n_1, \dots, n_l$, the line bundle*

$$\sigma^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_{j=1}^{l} n_j E_j)$$

on $\hat{X}$ is positive for sufficiently large k.

Proof. Consider a small coordinate neighborhood $U_j \subseteq X$ of x_j. Let $\sigma_{\mathbf{C}^n} : \mathcal{O}(-1) \subseteq \mathbf{C}^n \times \mathbf{P}^{n-1} \to \mathbf{C}^n$ be the projection. By the construction of $\hat{X}$, we may identify U_j with an open set in $\mathbf{C}^n$, and identify $\hat{U}_j = \sigma^{-1}(U_j)$ with $\sigma_{\mathbf{C}^n}^{-1}(U_j)$. Let $\pi : \hat{U}_j \subseteq \mathbf{C}^n \times \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$ be the second projection. The fiber of $\pi^*\mathcal{O}(-1)$ over $(z, l) \in \hat{U}_j$ is l, and by definition of $\mathcal{O}(-1)$, $z \in l$. Thus $(z, l) \mapsto z$ is a section of $\pi^*\mathcal{O}(-1)$. It vanishes along the exceptional divisor E_j with multiplicity one. It follows that on $\hat{U}_j$, $\mathcal{O}(E_j) \cong \pi^*\mathcal{O}(-1)$, and hence

$$\mathcal{O}(-E_j) \cong \pi^*\mathcal{O}(1).$$

By Example 1, the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^{n-1}$ admits a Hermitian metric with positive curvature. Pulling back via π gives a Hermitian metric on $\mathcal{O}(-E_j)|_{\hat{U}_j}$ whose curvature is semipositive, and strictly positive for tangent directions of E_j, since π is an isomorphism on E_j. This induces a Hermitian metric on $\mathcal{O}(-n_j E_j)|_{\hat{U}_j}$ with the same properties.

Using a partition of unity, these Hermitian metrics can be glued to a Hermitian metric h on $\mathcal{O}(-\sum_{j=1}^{l} n_j E_j)$ that agrees with h_j in a neighborhood of E_j.

Let Ω be the curvature form of the Chern connection of $\mathcal{O}(-\sum_j n_j E_j)$ with respect to h. Let $\theta = \frac{i}{2\pi}\Omega$. By hypothesis, $c_1(L)$ can be represented by a positive form α. Let β be a real $(1,1)$-form representing $c_1(M)$. Then

$$\sigma^*(k\alpha + \beta) + \theta$$

represents the first Chern class of $\sigma^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_j n_j E_j)$, and we claim that it is a positive form on $\hat{X}$ for sufficiently large k.

By the construction of h, there exists an open set $V \subseteq \hat{X}$ containing $\cup_j E_j$ such that θ is semipositive on V. Since X is compact, $k\alpha + \beta$ is a positive form on X for sufficiently large k. Since σ is an isomorphism outside $\cup_j E_j$, the pullback of a positive form is semipositive on $\hat{X}$ and strictly positive outside $\cup_j E_j$. Since $X \backslash V$ is compact, it follows that $\sigma^*(k\alpha + \beta) + \theta$ is positive on $X \backslash \cup_j E_j$ for sufficiently large k.

If $x \in \cup_j E_j$ and $v \in T_x^{1,0}\hat{X}$ is nonzero, then

$$-i[\sigma^*(k\alpha + \beta) + \theta](v, \bar{v}) = -i(k\alpha + \beta)(d\sigma(v), d\sigma(\bar{v})) - i\theta(v, \bar{v}),$$

where both terms are nonnegative. Since $k\alpha + \beta$ is positive, this will be positive if $d\sigma(v) \neq 0$. On the other hand, if $d\sigma(v) = 0$, then v is tangent to some E_j. Since by construction h has strictly positive curvature for tangent directions of E_j, we have

$$-i\theta(v, \bar{v}) > 0.$$

Hence $\sigma^*(k\alpha + \beta) + \theta$ is positive on all of $\hat{X}$. □

Lemma 11. *With the same hypotheses as in Lemma 10, for any postive integers* $n_1, \ldots, n_l$, *we have*

$$H^1(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-\sum_{j=1}^{l} n_j E_j)) = 0$$

for sufficently large k.

Proof. The canonical bundle of $\hat{X}$ is given by

$$K_{\hat{X}} \cong \sigma^*(K_X) \otimes \mathcal{O}(\sum_{j=1}^{l}(n-1)E_j).$$

We have

$$\sigma^*(L^k) \otimes \mathcal{O}(-\sum_{j=1}^{l} n_j E_j) \cong K_{\hat{X}} \otimes \sigma^*(L^k \otimes K_X^*) \otimes \mathcal{O}(-\sum_{j=1}^{l}(n - 1 + n_j)E_j).$$

Applying Lemma 10 to $M = K_X^*$, the line bundle $L' = \sigma^*(L^k \otimes K_X^*) \otimes \mathcal{O}(-\sum_j (n-1+n_j)E_j)$ is positive for sufficiently large k. By Theorem 3,

$$H^1(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-\sum_{j=1}^{l} n_j E_j)) \cong H^1(\hat{X}, K_{\hat{X}} \otimes L') = 0.$$

□

Proof of Theorem 9. ($\Leftarrow$) Assume L is ample. Then for some positive integer k, the complete linear system $|L^k|$ defines an embedding $\phi_{L^k} : X \hookrightarrow \mathbf{P}^N$. Thus $\phi_{L^k}^* \mathcal{O}(1) \cong L^k$. Since $\mathcal{O}(1)$ is positive, so is L^k. Since $c_1(L^k) = kc_1(L)$, it follows that L is positive.

($\Rightarrow$) Assume L is positive.

Step 1: For fixed $x \in X$, there exists $k_0(x)$ such that for all $k \geq k_0(x)$, the restriction map $H^0(X, L^k) \to L^k(x)$ is surjective.

Let $\sigma : \hat{X} \to X$ be the blowup of X along x, with exceptional divisor E. We have a commutative diagram

$$\begin{array}{ccc} H^0(X, L^k) & \longrightarrow & L^k(x) \\ {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow \\ H^0(\hat{X}, \sigma^*(L^k)) & \longrightarrow & H^0(E, \sigma^*(L^k) \otimes \mathcal{O}_E), \end{array}$$

where the vertical maps are given by pullback and the horizontal maps are given by restriction. Since σ maps E to x, $H^0(E, \sigma^*(L^k) \otimes \mathcal{O}_E) \cong L^k(x) \otimes H^0(E, \mathcal{O}_E) \cong L^k(x)$. Thus the vertical map on the right is an isomorphism.

We claim that α is also an isomorphism. α is clearly injective. If $n = 1$, then σ is an isomorphism, and clearly α is bijective. Suppose $n \geq 2$. Let $s \in H^0(\hat{X}, \sigma^*(L^k))$. Since σ restricts to an isomorphism $\hat{X} \backslash E \cong X \backslash \{x\}$, $s|_{\hat{X} \backslash E}$ is the pullback of a section $t \in H^0(X \backslash \{x\}, L^k)$. By Hartogs' theorem, t extends to $\tilde{t} \in H^0(X, L^k)$. By continuity, the pullback $\alpha(\tilde{t}) = s$. Hence α is bijective in this case as well.

Therefore, it suffices to show that $H^0(\hat{X}, \sigma^*(L^k)) \to H^0(E, \sigma^*(L^k) \otimes \mathcal{O}_E)$ is surjective. This is part of the long exact sequence induced by

$$0 \to \sigma^*(L^k) \otimes \mathcal{O}(-E) \to \sigma^*(L^k) \to \sigma^*(L^k) \otimes \mathcal{O}_E \to 0.$$

By Lemma 11, for sufficiently large k

$$H^1(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-E)) = 0,$$

and hence $H^0(\hat{X}, \sigma^*(L^k)) \to H^0(E, \sigma^*(L^k) \otimes \mathcal{O}_E)$ is surjective.

Step 2: For fixed $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $k_1(x_1, x_2)$ such that for all $k \geq k_1(x_1, x_2)$, the restriction map $H^0(X, L^k) \to L^k(x_1) \oplus L^k(x_2)$ is surjective.

Let $\sigma : \hat{X} \to X$ be the blow-up of X along x_1, x_2, with exceptional divisors E_1, E_2. As in Step 1, we have a commutative diagram

$$\begin{array}{ccc} H^0(X, L^k) & \longrightarrow & L^k(x_1) \oplus L^k(x_2) \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \sigma^*(L^k)) & \longrightarrow & H^0(E_1, \sigma^*(L^k) \otimes \mathcal{O}_{E_1}) \oplus H^0(E_2, \sigma^*(L^k) \otimes \mathcal{O}_{E_2}), \end{array}$$

where the vertical maps are isomorphisms. Thus it suffices to show that the bottom map is surjective.

The bottom map is induced by the short exact sequence

$$0 \to \sigma^*(L^k) \otimes \mathcal{O}(-E_1 - E_2) \to \sigma^*(L^k) \to \sigma^*(L^k) \otimes \mathcal{O}_{E_1+E_2}.$$

By Lemma 11, for sufficiently large k

$$H^1(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-E_1 - E_2)) = 0,$$

which proves the surjectivity.

Step 3: For fixed $x \in X$, there exists $k_2(x)$ such that for all $k \geq k_2(x)$, $d_x : H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) \to L^k(x) \otimes \Lambda^1_x X$ is surjective.

Consider the two short exact sequences

$$0 \to \mathcal{I}^2_{\{x\}} \to \mathcal{I}_{\{x\}} \to \Lambda^1_x X \to 0 \tag{2}$$

and

$$0 \to \mathcal{O}(-2E) \to \mathcal{O}(-E) \to \mathcal{O}_E(-E) \to 0. \tag{3}$$

The quotient maps $\mathcal{I}_{\{x\}} \to \Lambda^1_x X$ and $\mathcal{O}(-E) \to \mathcal{O}_E(-E)$ are given by the differential. Here, we use the isomorphism $\mathcal{O}_E(-E) \cong N^*_{E/\hat{X}}$.

Since σ maps E to x, σ pulls back functions vanishing to order k at x to functions vanishing to order k along E. Thus pullback via σ gives a commutative diagram

$$\begin{array}{ccc} \sigma^*\mathcal{I}^2_{\{x\}} & \longrightarrow & \sigma^*\mathcal{I}_{\{x\}} \\ \downarrow & & \downarrow \\ \mathcal{O}(-2E) & \longrightarrow & \mathcal{O}(-E). \end{array}$$

This induces a commutative diagram on the quotients

$$\begin{array}{ccc} \sigma^*\mathcal{I}_{\{x\}} & \longrightarrow & \sigma^*\Lambda^1_x X \\ \downarrow & & \downarrow \\ \mathcal{O}(-E) & \longrightarrow & \mathcal{O}_E(-E). \end{array}$$

Tensoring with $\sigma^* L^k$ then gives a commutative diagram

$$\begin{array}{ccc} \sigma^*(L^k \otimes \mathcal{I}_{\{x\}}) & \longrightarrow & \sigma^*(L^k(x) \otimes \Lambda^1_x X) \\ \downarrow & & \downarrow \\ \sigma^*(L^k) \otimes \mathcal{O}(-E) & \longrightarrow & \sigma^*(L^k) \otimes \mathcal{O}_E(-E). \end{array}$$

Therefore on the level of global sections, we have a commutative diagram

$$\begin{array}{ccc} H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) & \longrightarrow & L^k(x) \otimes \Lambda^1_x X \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \sigma^*(L^k \otimes \mathcal{I}_{\{x\}})) & \longrightarrow & H^0(\hat{X}, \sigma^*(L^k(x) \otimes \Lambda^1_x X)) \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-E)) & \longrightarrow & H^0(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}_E(-E)). \end{array}$$

Ignoring the middle row, we can write it as

$$\begin{array}{ccc} H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) & \xrightarrow{d_x} & L^k(x) \otimes \Lambda^1_x X \\ \downarrow & & \downarrow \\ H^0(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-E)) & \longrightarrow & H^0(E, L^k(x) \otimes \mathcal{O}_E(-E)). \end{array} \tag{4}$$

It follows from the definition of d_x that the top map is indeed d_x.

The vertical map on the left is clearly injective. We may identify $s \in H^0(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-E))$ with a section in $H^0(\hat{X}, \sigma^*(L^k))$ vanishing along E. As before, s is the pullback of some $t \in H^0(X, L^k)$. Since s vanishes along E, t must vanish at x. Thus the vertical map on the left is bijective.

Recall that

$$\mathcal{O}_E(E) \cong N_{E/\hat{X}} \cong \mathcal{O}_{\mathbf{P}(N_{\{x\}/X})}(-1) \cong \mathcal{O}_{\mathbf{P}(T_x X)}(-1).$$

In fact, the isomorphism is obtained as follows. Let $l \in E$. Note that $d\sigma_l$ maps $T_l E$ into $T_x\{x\} = 0$, and therefore induces a map $d\sigma_l^N : N_{E/\hat{X},l} \to T_x X$. The fiber of $\mathcal{O}_{\mathbf{P}(T_x X)}(-1)$ over $l \in E \cong \mathbf{P}(T_x X)$ is l. Using the construction of the blow-up, one can check that $d\sigma_l^N$ is an isomorphism of $N_{E/\hat{X},l}$ onto l. Thus σ^* gives an isomorphism $\mathcal{O}_{\mathbf{P}(T_x X)}(1) \cong N^*_{E/\hat{X}} \cong \mathcal{O}_E(-E)$, which induces an isomorphism

$$\Lambda^1_x X \cong T^*_x X \cong H^0(\mathbf{P}(T_x X), \mathcal{O}(1)) \cong H^0(E, \mathcal{O}_E(-E)). \tag{5}$$

In the sequences (2) and (3), the quotient maps are given by the differential. Thus in (4), the vertical map on the right is given by the pullback of differentials, hence is obtained by tensoring (5) with the identity map on $L^k(x)$. Therefore, it is an isomorphism.

To show that $d_x : H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) \to L^k(x) \otimes \Lambda^1_x X$ is surjective, it suffices to show that the bottom map in (4) is surjective. This map is induced by the short exact sequence

$$0 \to \sigma^*(L^k) \otimes \mathcal{O}(-2E) \to \sigma^*(L^k) \otimes \mathcal{O}(-E) \to \sigma^*(L^k) \otimes \mathcal{O}_E(-E) \to 0.$$

Hence it suffices to show that

$$H^1(\hat{X}, \sigma^*(L^k) \otimes \mathcal{O}(-2E)) = 0.$$

By Lemma 11, this holds for sufficiently large k.

Step 4: We can find a k independent of the point(s) in X.

Considering the map $H^0(X, L^{2^l}) \to H^0(X, L^{2^{l+1}})$ given by $s \mapsto s^2$ shows that we have a decreasing sequence of compact sets

$$Bs(L) \supseteq Bs(L^2) \supseteq \cdots \supseteq Bs(L^{2^l}) \supseteq \cdots .$$

By Step 1, the sequence has empty intersection. Therefore for some l_0, $Bs(L^{2^l}) = \emptyset$ for all $l \geq l_0$.

Next, note that if x is not a base point of L^{2^l} and L^{2^l} separates tangents at x, then $L^{2^{l+1}}$ separates tangents at x. For, let $v \otimes w \in L^{2^{l+1}}(x)$, where $v, w \in L^{2^l}(x)$, and let $\omega \in \Lambda^1_x X$. There exists $s \in H^0(X, L^{2^l})$ such that $s(x) = v$, and there exists $t \in H^0(X, L^{2^l} \otimes \mathcal{I}_{\{x\}})$ such that $d_x(t) = w \otimes \omega$. Then $s \otimes t \in H^0(X, L^{2^{l+1}} \otimes \mathcal{I}_{\{x\}})$, and

$$d_x(s \otimes t) = s(x) \otimes d_x(t) = v \otimes w \otimes \omega.$$

The set of points $S(L^k)$ where L^k does not separate tangents is the set of points where $d\phi_{L^k}$ is not injective, hence closed. Thus we have a decreasing sequence of compact sets

$$S(L^{2^{l_0}}) \supseteq S(L^{2^{l_0+1}}) \supseteq \cdots ,$$

which has empty intersection by Step 3. Therefore for some $l_1 \geq l_0$, $S(L^{2^l}) = \emptyset$ for all $l \geq l_1$.

Finally, note that if L^{2^l} separates x and y, then so does $L^{2^{l+1}}$. For, if $s \in H^0(X, L^{2^l})$ vanishes at one point but not the other, then the same holds for $s^2 \in H^0(X, L^{2^{l+1}})$.

By way of contradiction, suppose there is no $l_2 \geq l_1$ such that $L^{2^{l_2}}$ separates all pairs of distinct points in X. Then for every $l \geq l_1$, there exist distinct points $x_l, y_l \in X$ not separated by L^{2^l}. By compactness, a subsequence $\{(x_{l_j}, y_{l_j})\}$ converges to some $(x, y) \in X \times X$.

If $x \neq y$, then by Step 2, there exists $m = m(x, y)$ such that L^{2^m} separates x and y. Then $\phi_{L^{2^m}}$ maps x and y to distinct points, hence there are open neighborhoods U of x and V of y such that $\phi_{L^{2^m}}(U)$ and $\phi_{L^{2^m}}(V)$ are disjoint. For large j, we have $x_{l_j} \in U$ and $y_{l_j} \in V$, hence they are separated by $\phi_{L^{2^m}}$. But if $l_j \geq m$, this implies they are also separated by $\phi_{L^{2^{l_j}}}$, a contradiction.

If $x = y$, then by Step 3, there exists $m = m(x)$ such that L^{2^m} separates tangents at x. Then the differential of $\phi_{L^{2^m}}$ is injective at x. Thus $\phi_{L^{2^m}}$ is injective in a neighborhood U of x. For large j, we have $x_{l_j}, y_{l_j} \in U$, which again leads to a contradiction.

Therefore there exists l_2 such that $L^{2^{l_2}}$ is base-point free, separates tangents, and separates points. Hence L is ample.

□

Corollary 12. *A compact Kähler manifold is projective if and only if it admits a positive line bundle.*

We can restate the projectivity criterion using the notion of Hodge metrics.

Definition 3. A Kähler metric is said to be a *Hodge metric* if its Kähler class $[\omega] \in H^2(X, \mathbf{C})$ belongs to the image of

$$H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C}).$$

A Kähler manifold is said to be a *Hodge manifold* if it admits a Hodge metric. Equivalently, the image of $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})$ contains a Kähler class.

Corollary 13. *A compact Kähler manifold is projective if and only if it is a Hodge manifold.*

Proof. ($\Rightarrow$) Suppose $\phi : X \to \mathbf{P}^N$ is an embedding. From the definition of the first Chern class using the exponential sequence, it is immediate that $c_1(\phi^*\mathcal{O}(1))$ belongs to the image of $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})$. Since $\phi^*\mathcal{O}(1)$ is positive, $c_1(\phi^*\mathcal{O}(1))$ is a Kähler class.

($\Leftarrow$) Suppose the image of $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})$ contains a Kähler class α. Then

$$\alpha \in \mathrm{Im}(H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{C})) \cap H^{1,1}(X).$$

By the Lefschetz theorem on $(1,1)$-classes, there exists a line bundle L on X such that $c_1(L) = \alpha$. Then L is positive. By Corollary 12, L is projective. $\square$

We now give some applications of the Kodaira embedding theorem.

Definition 4. Let $\Lambda \subseteq \mathbf{C}^n$ be a lattice. A *Riemann form* for Λ is an alternating $\mathbf{R}$-bilinear form $\omega : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{R}$ such that

1. $\omega(\cdot, i\cdot)$ is a positive definite symmetric $\mathbf{R}$-bilinear form, and
2. for all $u, v \in \Lambda$, $\omega(u, v) \in \mathbf{Z}$.

Corollary 14. *Let $\Lambda \subseteq \mathbf{C}^n$ be a lattice. Then the complex torus $X = \mathbf{C}^n/\Lambda$ is projective if and only if there exists a Riemann form for Λ.*

Proof. An alternating $\mathbf{R}$-bilinear form on $\mathbf{C}^n$ descends to a closed real 2-form on X. This gives a map $\phi : \Lambda^2(\mathbf{C}^n)^* \to H^2(X, \mathbf{R})$. We claim that ϕ is an isomorphism.

Let $\{a_1, \ldots, a_{2n}\}$ be a $\mathbf{Z}$-basis for Λ. We have real 2-tori $T_{jk} = (\mathbf{R}/\mathbf{Z})a_j \oplus (\mathbf{R}/\mathbf{Z})a_k \subseteq X$, $1 \le j < k \le 2n$. A direct computation gives

$$\omega(a_j, a_k) = \int_{T_{jk}} \omega.$$

If $\phi(\omega) = 0$, i.e. ω descends to an exact 2-form, then by Stokes' theorem, for all $j < k$, $\omega(a_j, a_k) = 0$. Since $\{a_1, \ldots, a_{2n}\}$ is an $\mathbf{R}$-basis for $\mathbf{C}^n$, this implies

$\omega = 0$. Thus ϕ is injective. Since $\dim_{\mathbf{R}} \Lambda^2(\mathbf{C}^n)^* = \dim_{\mathbf{R}} H^2(X, \mathbf{R}) = \binom{2n}{2}$, ϕ is an isomorphism.

Now we prove the corollary.

($\Leftarrow$) Assume that there exists a Riemann form ω for Λ. Condition 1 in Definition 4 implies that ω descends to a Kähler form on X. Condition 2 implies that for all $j < k$,

$$\int_{T_{jk}} \omega = \omega(a_j, a_k) \in \mathbf{Z}.$$

Thus the restriction of ω to T_{jk} belongs to $H^2(T_{jk}, \mathbf{Z})$. By the Künneth formula,

$$H^2(X, \mathbf{Z}) = \bigoplus_{j<k} H^2(T_{jk}, \mathbf{Z}).$$

Hence $\omega \in H^2(X, \mathbf{Z})$. By Corollary 13, X is projective.

($\Rightarrow$) Assume that X is projective. By Corollary 13, there exists a Kähler form $\tilde{\omega}$ on X such that $[\tilde{\omega}] \in H^2(X, \mathbf{Z})$. Then $[\tilde{\omega}] = \phi(\omega)$ for some $\omega \in \Lambda^2(\mathbf{C}^n)^*$.

To show that ω is a Riemann form, we determine the inverse of ϕ explicitly. Define

$$\omega = \frac{1}{\mathrm{vol}(X)} \int_{x \in X} (\tau_x^* \tilde{\omega}) dx,$$

where $\tau_x(y) = y - x$ is translation by $-x$. Since τ_x is homotopic to the identity and $\tilde{\omega}$ is closed, for all $j < k$ we have

$$\int_{T_{jk}} \tau_x^* \tilde{\omega} = \int_{T_{jk}} \tilde{\omega}.$$

Therefore

$$\int_{T_{jk}} \omega = \frac{1}{\mathrm{vol}(X)} \int_{x \in X} \left[\int_{T_{jk}} \tau_x^* \tilde{\omega} \right] dx = \int_{T_{jk}} \tilde{\omega},$$

and hence ω is cohomologous to $\tilde{\omega}$. It is clear that ω is a translation-invariant Kähler form on X. Hence ω lifts to a Kähler form on $\mathbf{C}^n$, which is in fact a Riemann form, since $[\omega] \in H^2(X, \mathbf{Z})$ implies $\omega(a_j, a_k) = \int_{T_{jk}} \omega \in \mathbf{Z}$. $\square$

Corollary 15. *Every compact Kähler manifold with $H^2(X, \mathcal{O}) = 0$ is projective.*

Proof. $H^2(X, \mathcal{O}) = 0$ implies $H^{0,2}(X) = H^{2,0}(X) = 0$. Thus

$$H^2(X, \mathbf{C}) = H^{1,1}(X). \tag{6}$$

Let ω be a Kähler form on X. Then $[\omega]$ is a linear combination

$$[\omega] = \sum_{j=1}^{N} r_j [\alpha_j],$$

where $\{[\alpha_j]\}$ is a $\mathbf{Z}$-basis for the free part of $H^2(X, \mathbf{Z})$. Since ω is real, the r_j are real numbers. Taking harmonic representatives,

$$\omega = \sum_{j=1}^{N} r_j \alpha_j$$

as forms, since ω is harmonic. Since the $[\alpha_j]$ are real classes, the α_j are real, and by (6), they are $(1, 1)$-forms.

The positivity of ω and the compactness of X imply that, if s_j are rational numbers sufficiently close to r_j, then $\sum_j s_j \alpha_j$ is a positive real $(1, 1)$-form. Multiplying by a common denominator, we get a Kähler form whose cohomology class belongs to the image of $H^2(X, \mathbf{Z})$. By Corollary 13, X is projective. $\square$

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