COMPLEX ANALYSIS 2024
MIDTERM EXAM

1. (a) State and prove Cauchy's theorem for a rectangle (or triangle), then for piecewise C^1 curves inside a disk. (b) Define the winding number and prove the Cauchy integral formula for a disk.

2. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}, \qquad \frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{\operatorname{Im} z},$$

and the equality implies that $f(z)$ is a Möbius transformation.

3. Evaluate the following integrals by the method of residues:

$$\text{(a)} \quad \int_0^{\pi/2} \frac{dx}{a + \sin^2 x}, \quad |a| > 1, a \in \mathbb{R}, \qquad \text{(b)} \quad \int_0^{\infty} \frac{x^{1/3}\, dx}{1 + x^2}.$$

4. (a) Let $f(z)$ be analytic in $R_1 < |z - a| < R_2$, prove the existence and uniqueness of Laurent development of it. (b) Let $f(z)$ be analytic on Ω with $\Omega^c = E_1 \cup E_2$, E_i connected. Show that $f(z) = f_1(z) + f_2(z)$ where $f_i(z)$ is analytic outside E_i, $i = 1, 2$.

5. Let $f(z)$ be analytic in $A : r_1 < |z| < r_2$ and continuous on $\overline{A}$. Denote $M(r) = \max_{|z|=r} |f(z)|$. Show that $M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-\alpha}$ where $\alpha = \log(r_2/r)/\log(r_2/r_1)$. Discuss cases of equality.

6. Let $a > 1$. Show that $-\frac{1}{2}|t|^a + 2\pi|zt| \leq c|z|^{a/(a-1)}$ for some constant c by comparing $|t|^{a-1}$ with $A|z|$ for suitable A. Then use it to show that

$$f(z) := \int_{-\infty}^{\infty} e^{-|t|^a} e^{2\pi i z t}\, dt$$

is entire of growth order $a/(a-1)$.

7. (a) Show that $e^{-\pi x^2}$ is its own Fourier transform by Cauchy's theorem. (b) Prove Poisson summation formula using Fourier series, and deduce the modularity of theta series $\vartheta(t) = t^{-1/2}\vartheta(1/t)$ where $\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$. (c) For $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, show that $\xi(s) = \xi(1 - s)$.

8. (Bonus) Present a topic taught in the class but not shown above.

Date: November 5, pm 12:20 - 3:20 at Room 101. Each problem in **1 - 6** = 15 pts, **7** = 20 pts. Give your works in details. An NTU honors course delivered by Chin-Lung Wang.