Brieskorn–Hirzebruch Exotic Sphere

Hsin, Wei-Hsuan

June 3, 2024

1 Introduction

The following theorem is due to Kervaire and Milnor's landmark work on exotic spheres.

Definition 1.1. Let Θ_n be the group of manifolds that are homotopy n-spheres, modulo the h-cobordant relation, with the connected sum as the operator. Denote by $bP_{n+1} \subset \Theta_n$ the subgroup consisting of s-parallelizable homotopy n-spheres.

Theorem 1.2 (Kervaire-Milnor)**.** Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$, $m > 1$, which bound s-parallelizable manifolds M_1 and M_2 respectively. Then Σ_1 is h-cobordant to Σ_2 if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m},$$

where

$$\sigma_m = \frac{3+(-1)^{m+1}}{2} 2^{2m-2}(2^{2m-1}-1)\,\mathrm{numerator}(4B_m/m).$$

Brieskorn has constructed a series of $(4m-1)$-dimensional spheres of the form:

$$\Sigma_a = \{z \in \mathbb{C}^n \mid z_1^{a_1} + \cdots + z_n^{a_n} = 0\} \cap S^{2n-1}, \quad n = 2m+1.$$

These spheres bound parallelizable manifolds with different signatures. This provides a representation for each class of bP_{4m} and shows that bP_{4m} is a cyclic group of order $\sigma_m/8$. The construction proceeds in two steps. First, we verify that Σ_a is a topological sphere under certain conditions and identify it as the boundary of a parallelizable manifold. Second, we compute the signature of the bounding manifold.

2 Some Setup

Let $n \geq 2$ be an integer, $a = (a_1, \ldots, a_n)$ be a n-tuple of integers with $a_i > 1$ for all i. We introduce the following notations.

(1) $f(z) = z_1^{a_1} + \cdots + z_n^{a_n}$;

(2) $\Sigma_a = \Sigma(a_1, \ldots, a_n) \coloneqq V(f) \cap S^{2n-1}$;

(3) $\Xi_a(t) \coloneqq \{z \in \mathbb{C}^n \mid f(z) = t\}$. In particular, set $\Xi_a = \Xi_a(1)$.

Proposition 2.1. The space Σ_a is a smooth closed manifold of dimension $2n-3$.

Proof. It suffices to show that 0 is a regular value of $f : S^{2n-1} \to \mathbb{C}$. Consider the hermitian vector space $\mathbb{C}^n$ as a euclidean vector space $\mathbb{R}^{2n}$, defining the euclidean inner product of two vectors u, v to be the real part

$$\langle u, v\rangle_{\text{Eucl}} = \text{Re}\langle u, v\rangle = \text{Re}\langle u, v\rangle.$$

Then the tangent space of S^{2n-1} can be identified as the orthogonal complement

$$T_z(S^{2n-1}) = \{z\}^{\perp_{\text{Eucl}}} = \{u \in \mathbb{C}^n \mid \text{Re}\langle u, z\rangle = 0\}.$$

Along a curve $z = p(t)$ on S^{2n-1}, we have

$$\frac{df(p(t))}{dt} = \langle dp/dt, \text{grad} f\rangle,$$

where

$$\text{grad} f = \left(\overline{\frac{\partial f}{\partial z_1}}, \ldots, \overline{\frac{\partial f}{\partial z_n}}\right).$$

Under this identification, the differential $df : T_z S^{2n-1} \to T_{f(z)}\mathbb{C}$ is taking the hermitian product with $\text{grad} f$. Notice that $T_z S^{2n-1}$ contains a $\mathbb{C}$ vector subspace $\{z\}^\perp$. Therefore, if $z \in \Sigma_a$ is a critical point, then $\text{grad} f$ is a complex multiple of z, i.e.,

$$\text{grad} f = (a_1 \bar{z}_1^{a_1-1}, \ldots, a_n \bar{z}^{a_n-1}) = c(z_1, \ldots, z_n)$$

for some $c \in \mathbb{C}$. While,

$$0 = \sum_{i=1}^n z_i^{a_i} = c \sum_{i=1}^n \frac{1}{a_i}|z_i|^2.$$

Since $z \neq 0$ and $c \neq 0$, it leads to a contradiction. □

3 The Milnor Fibration $S^{2n-1} \setminus \Sigma_a$ over S^1

Consider the Milnor map $\phi : S^{2n-1} \setminus \Sigma_a \to S^1$ defined by

$$\phi(z) = \frac{f(z)}{|f(z)|}.$$

The idea is that $S^{2n-1} \setminus \Sigma_a$ forms a fiber bundle over S^1. The closure of each fiber is a smooth parallelizable manifold with boundary Σ_a, and the interior of each fiber is isomorphic to Ξ_a.

Theorem 3.1. The space $S^{2n-1} \setminus \Sigma_a$ is a smooth fiber bundle over S^1 with the projection mapping ϕ.

Remark 3.2. Milnor has studied such a map for general polynomial f. The theorem above is an adjustment to Milnor's fibration theorem which states that for any polynomial f, the map $f/|f| : S_\varepsilon \setminus V(f) \to S^1$ is a fiber bundle for sufficiently small ε. We use the ideal of the proof of Milnor's fibration theorem

To prove the theorem, we will use Morse theory.

Lemma 3.3. The critical points of $\phi : S^{2n-1} \setminus \Sigma_a \to S^1$ are precisely those points $z \in S^{2n-1} \setminus \Sigma_a$ for which the vector $i\, \text{grad} \log f(z)$ is a real multiple of the vector z.

Proof. Using the local coordinate $e^{i\theta}$ for S^1, we have

$$i\theta = \log(f/|f|) = \log f - \log|f| = \mathrm{Re}(-i\log f).$$

Along a curve $z = p(t)$, we obtain

$$\begin{aligned} d\theta(p(t))/dt &= \mathrm{Re}\left(d(-i\log f(p(t)))/dt\right) \\ &= \mathrm{Re}\langle dp/dt, \mathrm{grad}(-i\log f)\rangle \\ &= \mathrm{Re}\langle dp/dt, i\,\mathrm{grad}(\log f)\rangle \end{aligned}$$

As in Proposition 2.1, the tangent space of $S^{2n-1} \setminus \Sigma_a$ can be identified as

$$T_z(S^{2n-1} \setminus \Sigma_a) = \{z\}^{\perp_{\mathrm{Eucl}}} = \{u \in \mathbb{C}^n \mid \mathrm{Re}\langle u, z\rangle = 0\}.$$

Under this identification, the differential $d\phi : T_z(S^{2n-1} \setminus \Sigma_a) \to T_{\phi(z)}S^1$ is just taking the euclidean inner product with $i\,\mathrm{grad}\log f$. Therefore, we conclude that z is a critical point if and only if $i\,\mathrm{grad}\log f$ is a real multiple of the vector z, as desired. ☐

Proof of Theorem 3.1. Applying Morse theory to the pre-image of $[\theta - \varepsilon, \theta + \varepsilon]$, it then suffices to show that ϕ has no critical points. Assume for the sake of correctness that there is a critical point $z \in S^{2n-1} \setminus \Sigma_a$. By Lemma 3.3, we have

$$\frac{i}{f(z)}\left(a_1 z^{a_1 - 1}, \ldots, a_n z^{a_n - 1}\right) = c(\bar{z}_1, \ldots, \bar{z}_n)$$

for some $c \in \mathbb{R}$. Then

$$c\sum_{i=1}^{n} \frac{1}{a_i}|z_i|^2 = \sum_{i=1}^{n} \frac{i}{f(z)} z^{a_i} = i.$$

Since the left hand side is real and the right hand side is purely imaginary, it derives a contradiction. ☐

For each $e^{i\theta} \in S^1$, denote the fiber by

$$F_\theta = \phi^{-1}(e^{i\theta}) = \left\{z \in S^{2n-1} \setminus \Sigma_a \mid \arg f(z) = \theta\right\}.$$

It is a $(2n-2)$-dimensional manifold without boundary.

Proposition 3.4. The fiber F_θ is diffeomorphic to Ξ_a.

Proof. Consider the map $F_0 \to \Xi_a$ by

$$(z_1, \ldots, z_n) \mapsto \left(\frac{z_1}{f(z)^{1/a_1}}, \ldots, \frac{z_n}{f(z)^{1/a_n}}\right).$$

We conclude that all the fiber $F_\theta \simeq F_0$ are diffeomorphic to Ξ_a. ☐

Proposition 3.5. The closure of each fiber F_θ in S^{2n-1} is a smooth $(2n-2)$-dimensional manifold with boundary, the interior of this manifold being F_θ and the boundary being precisely Σ_a.

Proof. In the proof of Proposition 2.1, we have shown that 0 is a regular value of f. Let z_0 be a point of Σ_a. Choose a real local coordinate system $u_1, \ldots, u_{2n-1}$ for S^{2n-1} in a neighborhood U of z_0 so that

$$f(z) = u_1(z) + iu_2(z)$$

for all $z \in U$. Note that a point of U belongs to the fiber $F_0 = \phi^{-1}(1)$ if and only if

$$u_1 > 0,\ u_2 = 0.$$

Hence the closure $\overline{F}_0$ intersects U in the set

$$u_1 \geq 0,\ u_2 = 0.$$

Clearly it is a smooth $2n$-dimensional manifold, with $F_0 \cap U$ as interior and with $\Sigma_a \cap U$ as boundary. The discussion for other fibers F_θ is similar. This completes the proof. □

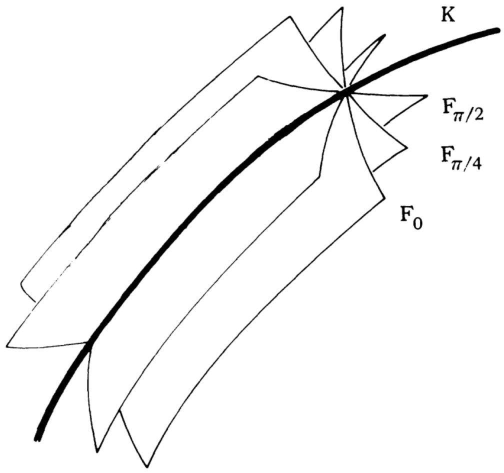

Figure 1: The Milnor Fiber F_θ

4 The Singular Homology of Ξ_a

For $t \neq 0$, $\Xi_a(t)$ and Ξ_a are diffeomorphic. On Ξ_a, there is an automorphism ω_k, namely, multiplying the kth coordinate by $\xi_k = e^{2\pi i/a_k}$. These ω_k's generate a group Ω_a, which is the direct product of cyclic groups:

$$\Omega_a = \prod \langle \omega_k \rangle \simeq \prod_{k=1}^{n} \mathbb{Z}_{a_k}.$$

Let $J_a = \mathbb{Z}[\Omega_a]$ be the group ring of Ω_a and let I_a be the ideal of J_a generated by the elements $1 + \omega_k + \cdots + \omega_k^{a_k - 1}$ for $k = 1, \ldots, n$. Let $\mathbf{e}$ be the subset of Ξ_a defined by

$$\mathbf{e} = \left\{ (z_1, \ldots, z_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{k=1}^{n} z_k^{a_k} = 1 \right\}.$$

It is homeomorphic to the standard simplex Δ_{n-1}

$$\Delta_{n-1} = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{k=1}^{n} y_k = 1 \right\}$$

under the map $(z_1, \ldots, z_n) \mapsto (z_1^{1/a_1}, \ldots, z_n^{1/a_n})$. Let

$$\mathscr{E} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; z_i^{a_i} \in \mathbb{R}_{\geq 0},\ \sum_{k=1}^{n} z_k^{a_k} = 1 \right\} = \Omega_a \mathbf{e}.$$

This collection of cells forms a simplicial complex.

Lemma 4.1. The space $\mathscr{E}$ is a deformation retract of Ξ_a under a retraction compatible with the group action of Ω_a.

Proof. Consider the real hypersurfaces

$$X = \left\{ (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \eta_i = 1 \right\},$$
$$S_i = \{ \eta \in X \mid \eta_i = 0 \},$$

and construct a deformation retraction from the system of hyperplanes $(X, S_1, \ldots, S_n)$ to $(\Delta_{n-1}, \partial_1 \Delta_{n-1}, \ldots, \partial_n \Delta_{n-1})$ as follows: this can be done by combining the deformation retraction from complex to its real part and the deformation and the deformation retraction on Δ_{n-1} symbolized by Figure 2. Explicitly, for any point $(\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, $\sum_{i=1}^{n} \eta_i = 1$, deform it linearly to the point $c(\varepsilon_1 \eta_1, \varepsilon_2 \eta_2, \ldots, \varepsilon_n \eta_n)$, where

$$\varepsilon_i = \begin{cases} 0 & , \eta_i \leq 0 \\ 1 & , \eta_i > 0, \end{cases}$$

and c is the constant such that $c \sum \varepsilon_i \eta_i = 1$.

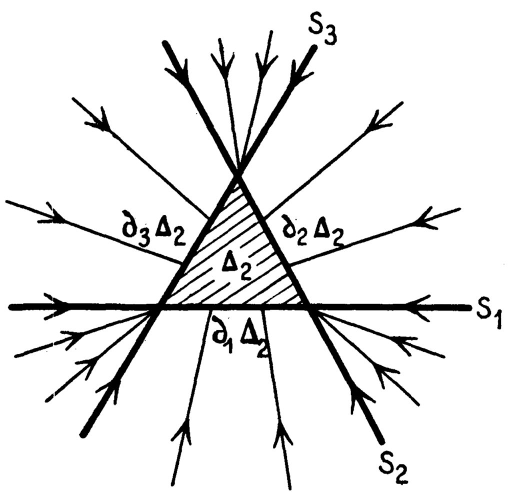

Figure 2: Deformation retraction to the simplicial system

Back to the original problem. Divide Ξ_a into $a_1 \cdots a_n$ parts

$$X_{i_1 \ldots i_k} = \left\{ z \in \Xi_a \;\middle|\; \frac{2\pi i_k}{a_k} - \frac{\pi}{2a_k} \leq \arg z_k < \frac{2\pi i_k}{a_k} + \frac{3\pi}{2a_k} \text{ or } z_k = 0 \right\},$$

where $0 \leq i_k \leq a_k - 1$. On each $X_{i_1 \ldots i_n}$, consider the change of variables $z_k = \eta_k^{1/a_k}$ with the branch $i\mathbb{R}_{\leq 0}$. We obtain a deformation retraction from $X_{i_1 \ldots i_k}$ to $\prod \omega_k^{i_k} \mathbf{e}$. Notice that if $\arg z_k \equiv -\pi/2a_k \mod 2\pi/a_k$ or $z_k = 0$, then $z_k^{a_k}$ is purely imaginary. So z is mapped to a point $z' \in \mathbb{R}^n$ with $z'_k = 0$. This shows that the deformations on $X_{i_1 \ldots i_n}$ glued to a deformation from Ξ_a to $\mathscr{E}$, as desired. $\square$

Proposition 4.2 (Pham)**.** The singular homology $H_i(\Xi_a, \mathbb{Z})$ vanishes for $i \neq 0, n-1$, and $H_{n-1}(\Xi_a, \mathbb{Z}) \simeq J_a/I_a$.

Proof. By Lemma 4.1, it suffices to compute the simplicial homology of $\mathscr{E}$. Observe that $\omega_{i_1}, \ldots, \omega_{i_k}$ act trivially on the simplex $\partial_{i_1} \cdots \partial_{i_k}\mathbf{e}$. So the annihilator ideal of $\partial_{i_1} \cdots \partial_{i_k}\mathbf{e}$ is the ideal generated by $1-\omega_{i_1}$, ..., $1-\omega_{i_k}$. So the simplicial complex sequence is

$$0 \to J_a\mathbf{e} \to \bigoplus_i J_{a_1,\ldots,\hat{a}_i,\ldots,a_n}\{\partial_i\mathbf{e}\} \to \bigoplus_{i<j} J_{a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_n}\{\partial_i\partial_j e\} \to \cdots$$

where

$$J_{a_{i_1},\ldots,a_{i_k}} = J_a/(1-\omega_{i_{k+1}}, \ldots, 1-\omega_{i_n}), \qquad \{i_1, \ldots, i_n\} = [n].$$

To ease the notation, we translate the sequence in the language of Čech cohomology. Let $X = [n] \cup \{0\}$ be the topological space with the basis $\{\{0\}, \{0,1\}, \ldots, \{0,n\}\}$. Define a presheaf of abelian group $\mathscr{F}$ on X by

$$\mathscr{F}(\{i_1, \ldots, i_k\} \cup \{0\}) = J_{a_{i_1},\ldots,a_{i_k}}.$$

We set the restriction map $\mathscr{F}(\{i_1, \ldots, i_{k+1}\} \cup \{0\}) \to \mathscr{F}(\{i_1, \ldots, i_k\} \cup \{0\})$ to be the quotient map with the sign $(-1)^{i_{k+1}-1}$. Consider an open covering $\mathfrak{U} = \{U_i\}$ of X, where $U_i = [n] \cup \{0\} \setminus \{i\}$. To give the desired result, it suffices to show that the sequence

$$\Gamma(X, \mathscr{F}) \xrightarrow{\varepsilon} C^0(\mathfrak{U}, \mathscr{F}) \to \cdots \to C^n(\mathfrak{U}, \mathscr{F}) \to 0$$

is exact and that

$$\ker \varepsilon = J_a(1-\omega_1) \cdots (1-\omega_n).$$

We prove by induction on n. The base case $n = 2$ is can be done by hand. For the inductive step, let $\mathfrak{V} = \mathfrak{U} \setminus \{U_n\}$ be an open covering of X, and let $\mathfrak{W} = \{U_i \cap U_n\}$, $i \neq n$, be an open covering of U_n. We have the decomposition

$$C^k(\mathfrak{U}, \mathscr{F}) = C^k(\mathfrak{V}, \mathscr{F}) \oplus C^{k-1}(\mathfrak{W}, \mathscr{F}|_{U_n}),$$

and

$$\begin{array}{ccccccccccc}
\Gamma(X,\mathscr{F}) & \to & C^0(\mathfrak{V},\mathscr{F}) & \longrightarrow & C^1(\mathfrak{V},\mathscr{F}) & \longrightarrow & \cdots & \longrightarrow & C^{n-1}(\mathfrak{V},\mathscr{F}) & \longrightarrow & 0 \\
& \searrow & \oplus & \searrow & \oplus & \searrow & & \searrow & \oplus & \searrow & \oplus \\
& & \Gamma(U_n,\mathscr{F}|_{U_n}) & \xrightarrow{\varepsilon'} & C^0(\mathfrak{W},\mathscr{F}|_{U_n}) & \to & \cdots & \to & C^{n-2}(\mathfrak{W},\mathscr{F}|_{U_n}) & \to & C^{n-1}(\mathfrak{W},\mathscr{F}|_{U_n}) \to 0
\end{array}$$

We show that two horizontal sequences are exact. Let $\mathscr{G}$ be the presheaf on the topological space $Y = [n-1] \cup \{0\}$ defined in the same way.

- Obviously, $\mathscr{G} \simeq \mathscr{F}|_{U_n}$. So the lower sequence is exact by the inductive hypothesis.
- Consider the map $f : X \to Y$ by $n \mapsto 0$ and $k \mapsto k$ for $k < n$. Then

  $$f_*\mathscr{F} \simeq \mathscr{G}^{\oplus a_n}.$$

  The map $f : X \to Y$ is open, so let $\mathfrak{V}'$ denote the image of $\mathfrak{V}$ in Y. Then we have the isomorphisms

  $$\Gamma(X, \mathscr{F}) = \Gamma(Y, f_*\mathscr{F}) \simeq \Gamma(Y, \mathscr{G})^{\oplus a_n},$$
  $$C^k(\mathfrak{V}, \mathscr{F}) = C^k(\mathfrak{V}', f_*\mathscr{F}) \simeq C^k(\mathfrak{V}', \mathscr{G})^{\oplus a_n}.$$

  Notice that the direct sum of exact sequences is exact. By the inductive hypothesis, we have the exact sequence

  $$\Gamma(X, \mathscr{F}) \to C^0(\mathfrak{V}, \mathscr{F}) \to \cdots \to C^{n-1}(\mathfrak{V}, \mathscr{F}) \to 0.$$

From the exactness of the two sequences, we deduces that the original sequence is exact at $C^1(\mathfrak{U},\mathscr{F}),\dots,C^n(\mathfrak{U},\mathscr{F})$. The exactness at $C^0(\mathfrak{U},\mathscr{F})$ follows from the fact that the kernel of ε' is generated by $(1-\omega_1)\cdots(1-\omega_{n-1})$, which maps to the zero element in $C^0(\mathfrak{V},\mathscr{F})$. Finally, we have

$$\ker\varepsilon = ((1-\omega_1)\cdots(1-\omega_{n-1}))\cap(1-\omega_n) = ((1-\omega_1)\cdots(1-\omega_n)).$$

We complete the proof. □

Remark 4.3. $H_{n-1}(\Xi_a,\mathbb{Z})\simeq J_a/I_a$ is a free $\mathbb{Z}$-module of rank $\prod_{k=1}^n(a_k-1)$.

Proposition 4.4. For $n\geq 3$, Ξ_a is simply connected, and therefore $(n-2)$-connected.

Proof. By Lemma 4.1, it suffices to show that $\mathscr{E}_2$ is simply connected. The vertices of $\mathscr{E}_2$ are $p_k^s=(0,\dots,\xi_k^s,0,\dots,0)$, where ξ_k is the primitive a_kth root of unity and $0\leq s<a_k$. There is exactly one edge connecting p_i^r, p_k^s for $i\neq k$ and exactly one 2-simplex connecting p_i^r, p_j^s, p_k^t for distinct i, j, k. Notice that

- an edge path connecting p_i^r, p_j^s, p_k^t is homotopic to the edge connecting p_i^r, p_k^t;
- an edge path connecting $p_i^{r_1}p_k^{t_1}p_i^{r_2}p_k^{t_2}$ is homotopic to the edge path connecting $p_i^{r_1}$, p_j^s, $p_k^{t_2}$ for any $j\neq i,k$.

Both operations reduce the number of edges of a path by 1. Therefore, one can convert every closed edge path in $\mathscr{E}_2$ into a null homotopic path by repeatedly using them. So $\mathscr{E}_2$ and thus also Ξ_a is simply connected. □

Proposition 4.5. The space Ξ_a is parallelizable.

Proof. By Lemma 4.1, Ξ_a has a homotopy type of a CW-complex of dimension $n-1<\dim\Xi_a$. Recall that $\Xi_a\simeq F_\theta$. So it suffices to show that TF_θ is stably trivial. Note that ϕ is locally trivial, so the normal bundle of F_θ in $S^{2n-1}\setminus\Sigma_a$, and hence in S^{2n-1}, is trivial. Since the normal bundle of S^{2n-1} in $\mathbb{C}^n$ is trivial, the result follows. □

Corollary 4.6. The space Σ_a is orientable.

Proof. Since F_θ is parallelizable, it is orientable. The space Σ_a is the boundary of $\overline{F}_\theta$, so it is orientable. □

5 The Singular Homology of Σ_a

In this section, we will prove Theorem 5.5, which is a necessary and sufficient criterion for Σ_a being a topological sphere. Using Smale's generalized Poincaré conjecture, it suffices to show that Σ_a is a simply connected homology sphere. Since Σ_a is oriented and compact, we only have to determine the homology up to the middle dimension. By Poincaré duality and Alexander duality, we have

$$H_i(\Sigma_a,\mathbb{Z})=H^{2n-3-i}(\Sigma_a,\mathbb{Z})\simeq H_{i+1}(S^{2n-1}\setminus\Sigma_a,\mathbb{Z}).$$

Recall that we have a fibration $S^{2n-1}\setminus\Sigma_a\to S^1$. Notice that the action of $\pi(S^1)=\mathbb{Z}$ on $H_*(S^{2n-1}\setminus\Sigma_a,\mathbb{Z})$ is non-trivial (i.e., it is not a Serre fibration). To compute the homology of $S^{2n-1}\setminus\Sigma_a$, we need to adjust the Serre spectral sequence.

Lemma 5.1 (Wang's Sequence)**.** Given a fiber bundle $\phi : E \to S^1$ over the circle. Using the covering homotopy theorem, there is an one-parameter family of homeomorphisms

$$h_t : F_0 \to F_t$$

for $0 \le t \le 2\pi$, where h_0 is the identity. Denote $h = h_{2\pi}$, called the *characteristic homeomorphism*. There is associated an exact sequence of the form

$$\cdots \to H_{j+1}E \to H_j F_0 \xrightarrow{\mathrm{id}_* - h_*} H_j F_0 \to H_j E \to \cdots .$$

Proof. The long exact sequence of the pair (E, F_0) gives

$$\cdots \to H_{j+1}E \to H_{j+1}(E, F_0) \xrightarrow{\partial} H_j F_0 \to H_j E \to \cdots .$$

The covering homotopy $\{h_t\}$ induces a map

$$F_0 \times [0, 2\pi] \to E$$

which gives rise to an isomorphism

$$H_{j+1}(F_0 \times [0, 2\pi], F_0 \times \{0\} \cup F_0 \times \{2\pi\}) \xrightarrow{\sim} H_{j+1}(E, F_0).$$

Since $F_0 \times \{2\pi\}$ is a deformation retract of $F_0 \times [0, 2\pi]$. From the long exact sequence of the triple $(F_0 \times \{2\pi\}, F_0 \times \{0\} \cup F_0 \times \{2\pi\}, F_0 \times [0, 2\pi])$, we obtain an isomorphism

$$H_{j+1}(F_0 \times [0, 2\pi], F_0 \times \{0\} \cup F_0 \times \{2\pi\}) \xrightarrow{\sim} H_j(F_0 \times \{0\} \cup F_0 \times \{2\pi\}, F_0 \times \{2\pi\}) \simeq H_j(F_0).$$

Thus we have to understand the boundary map under the identification

$$\begin{array}{ccc} H_{j+1}(F_0 \times [0, 2\pi], F_0 \times \{0\} \cup F_0 \times \{2\pi\}) & \longrightarrow & H_j F_0 \\ \downarrow & & \vdots \\ H_{j+1}(E, F_0) & \xrightarrow[\partial]{} & H_j F_0 \end{array}$$

Given a cochain $[\phi] \in H_j F_0$. For the first row of isomorphism, the image in the left hand side is $[T_1\phi + T_1'\phi]$, where we choose the representation $T_1\phi + T_1'\phi$ to be

$$T_1\phi(t_0, \ldots, t_j) = (\phi\,(t_1 + t_0/j, \ldots, t_j + t_0/j)\,, 2\pi t_0) \in F_0 \times [0, 2\pi],$$

$$T_1'\phi(t_0, \ldots, t_j) = (\phi\,(t_1 + t_0/j, \ldots, t_j + t_0/j)\,, 2\pi(1 - t_0)) \in F_0 \times [0, 2\pi].$$

The left-hand side vertical isomorphism maps this to $[T_2\phi + T_2'\phi]$, where

$$T_2\phi(t_0, \ldots, t_j) = h_{2\pi t_0}\,(\phi\,(t_1 + t_0/j, \ldots, t_j + t_0/j)) \in E,$$

$$T_2'\phi(t_0, \ldots, t_j) = h_{2\pi(1-t_0)}\,(\phi\,(t_1 + t_0/j, \ldots, t_j + t_0/j)) \in E.$$

Finally, consider the boundary map on the lower row. The ith face of $T_2\phi$ and $T_2'\phi$ cancel up. Thus we conclude that the image of $[T_2\phi + T_2'\phi]$ is represented by

$$\partial(T_2\phi) + \partial(T_2'\phi) = \partial_0(T_2\phi) + \partial_0(T_2'\phi) = \phi - h_*\phi.$$

We complete the proof. □

Proposition 5.2. The homology group $H_i(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ vanishes for $i \neq 0, 1, n-1, n$, and the homology group $H_{n-1}(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ and $H_n(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ vanish if and only if $1 - \omega : J_a/I_a \to J_a/I_a$ is an isomorphism, where $\omega = \omega_1 \cdots \omega_n$.

Proof. As in Lemma 5.1, the family $h_t : F_0 \to F_t$ is given by

$$h_t(z_1, \dots, z_n) = (\xi_1^t z_1, \dots, \xi_n^t z_n).$$

In particular, the characteristic homeomorphism $h = h_{2\pi} : F_0 \to F_0$ is the map

$$h(z_1, \dots, z_n) = (\omega_1 z_1, \dots, \omega_n z_n).$$

Since $H_i(F_0, \mathbb{Z}) \simeq H_i(\Xi_a, \mathbb{Z})$ vanishes when $i \neq 0, n-1$, the homology $H_i(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ vanishes when $i \neq 0, 1, n-1, n$. Under the identification $H_{n-1} \simeq J_a/I_a$, the map $\mathrm{id}_* - h_*$ is the left multiplication by $1 - \omega$. Therefore, the homology group $H_{n-1}(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ and $H_n(S^{2n-1} \setminus \Sigma_a, \mathbb{Z})$ vanish if and only if $1 - \omega : J_a/I_a \to J_a/I_a$ is an isomorphism. □

Lemma 5.3. The characteristic polynomial of ω is

$$\Delta_a(t) = \prod_{0 < i_k < a_k} (t - \xi_1^{i_1} \cdots \xi_n^{i_n}).$$

Proof. Consider J_a/I_a as a tensor product

$$\bigotimes_{k=1}^{n} V_k,$$

where V_k is a $\mathbb{Z}$-module generated by $1, \omega_k, \dots, \omega_k^{a_k - 1}$. Then the automorphism ω can be consider as $\omega_1 \otimes \cdots \otimes \omega_n$. Tensor everything with $\mathbb{C}$. For each a_kth root of unity $x_k = \xi_k^{i_k}$, $0 < i_k < a_k$, the element

$$\sum_{r=0}^{a_k - 1} x_k^r \omega_k^r \in V_k \otimes \mathbb{C}$$

is an eigenvector of ω_k with eigenvalue x_k^{-1}. Therefore,

$$\prod_{k=1}^{n} \sum_{r=0}^{a_k - 1} x_k^r \omega_k^r \in J_a/I_a \otimes \mathbb{C}$$

is an eigenvector of ω with eigenvalue $\xi_1^{-i_1} \cdots \xi_n^{-i_n}$. All of these form a basis consisting of eigenvectors of ω. We conclude the desired result. □

Proposition 5.4. For $n \geq 4$, Σ_a is simply connected, hence at least $(n-3)$-connected.

Proof. Using Hurewicz's theorem, it suffices to show that $\pi_1(\Sigma_a)$ is abelian. First note that Σ_a is a deformation retract of $V(f) \setminus \{0\}$. The inclusion

$$V(f) \cap \{z_n \neq 0\} \hookrightarrow V(f) \setminus \{0\}$$

induces the surjection

$$\pi_1(V(f) \cap \{z_n \neq 0\}) \twoheadrightarrow \pi_1(V(f) \setminus \{0\}).$$

Define $\psi : V(f) \cap \{z_n \neq 0\} \to \mathbb{C}^\times$ by $z \mapsto z_n$. It is a fiber bundle with fiber $\Xi_{\hat{a}}$, where $\hat{a} = (a_1, \dots, a_{n-1})$. Indeed, we have a trivialization $\Xi_{\hat{a}} \times U \to \psi^{-1}(U)$ by

$$(z_1, \dots, z_{n-1}, s) \mapsto ((-s^{a_n})^{1/a_1} z_1, \dots, (-s^{a_n})^{1/a_{n-1}} z_{n-1}, s).$$

From Proposition 4.4 we obtain an isomorphism

$$0 = \pi_1(\Xi_{\hat{a}}) \to \pi_1(V(f) \cap \{z_n \neq 0\}) \xrightarrow{\sim} \pi_1(\mathbb{C}^\times) \to 0.$$

So $\pi_1(\Sigma_a) = \pi_1(V(f) \setminus \{0\})$ is abelian, as desired. □

Let G_a be a simple graph with n vertices, denoted by $a_1, a_2, \ldots, a_n$. Two vertices a_i, a_j are adjacent if their greatest common divisor $\gcd(a_i, a_j) > 1$.

Theorem 5.5. For $n \geq 4$, the following are equivalent:

(i) Σ_a is a topological sphere.

(ii) $\Delta_a(1) = 1$.

(iii) G_a fulfills one of the following conditions

   (a) G_a has at least two isolated points.

   (b) G_a has one isolated point and at least one connected component K with an odd number of vertices such that $(a_i, a_j) = 2$ for $a_i, a_j \in K$, $i \neq j$.

Proof. (i) $\Leftrightarrow$ (ii): By Proposition 5.4 and Proposition 5.2, Σ_a is simply connected and the homology of Σ_a with degree less than $n-2$ vanishes. For $2n-3 \geq 5$, using Smale's generalized Poincaré conjecture and Poincaré duality, Σ_a is a topological sphere if and only if the homology groups $H_{n-2}(\Sigma_a, \mathbb{Z})$, $H_{n-1}(\Sigma_a, \mathbb{Z})$ vanish. The equivalence follows immediately from Proposition 5.2 and Lemma 5.3.

(ii) $\Leftrightarrow$ (iii): It is known that the minimal polynomial of the root of unity of order d is the cyclotomic polynomial Φ_d. By Lemma 5.3, the characteristic $\Delta_a(t)$ is a product

$$\Delta_a(t) = \prod_d \Phi_d(t),$$

where d runs through the orders of $\xi_1^{i_1} \cdots \xi_n^{i_n}$, possibly several times. It is well-known that $\Phi_{p^m}(1) = p$ for every prime p and $\Phi_d(1) = 1$ if d is not a prime power. This implies that $\Delta_a(1) = 1$ if and only if for every $i = (i_1, \ldots, i_n)$ with $0 < i_k < a_k$, the order of $\xi_1^{i_1} \cdots \xi_n^{i_n}$ is not a prime power.

Let K be a component of G_a. Denote the vertices of K by $a_1, \ldots, a_r$. Let

$$\begin{aligned}\kappa(K) &= \#\{(i_1, \ldots, i_r) \mid 0 < i_k < a_k,\ \xi_1^{i_1} \cdots \xi_r^{i_r} = 1\} \\ &= \#\left\{(i_1, \ldots, i_r) \,\middle|\, 0 < i_k < a_k,\ \sum_{k=1}^{r} \frac{i_k}{a_k} \in \mathbb{Z}\right\}.\end{aligned}$$

Claim. For each component K, $\kappa(K) = 0$ if and only if K is either an isolated point, or the number of vertices of K is odd and $(a_i, a_j) = 2$ for $a_i, a_j \in K$, $i \neq j$.

Proof of Claim. ($\Leftarrow$) The case that K is an isolated point is trivial. If K satisfies the second condition, the unless $a_i = 2$ for all i, we will have $\kappa(K) = 0$. However, if all $a_i = 2$, then since $|K|$ is odd, we still have $\kappa(K) = 0$.

($\Rightarrow$) Assume that K satisfies neither two conditions, we show that $\kappa(K) > 0$. First, we show that if there is an edge $\{a_i, a_j\}$ with $(a_i, a_j) = d > 2$, then we can merge two vertices a_i, a_j into one $a_i a_j / d$ and not effecting any conditions (in fact, it becomes even better). Write $a_i' = a_i/d$, $a_j' = a_j/d$. We have

$$\frac{x}{a_i} + \frac{y}{a_j} = \frac{a_j' x + a_i' y}{d a_i' a_j'}.$$

As x, y runs over all integers, $a_j' x + a_i' y$ runs over all integers. The only question is that we can only have those x, y with $a_i \nmid x$, $a_j \nmid y$. However, if $a_j' x + a_i' y = n$, then at least one of the pairs

$$(x, y),\ (x - a_i', y + a_j'), (x - 2a_i', y + 2a_j')$$

satisfies this restriction. Therefore, $a'_j x + a'_i y$ runs through all the residue classes modulo $da'_i a'_j$ when $0 < x < a_i$, $0 < y < a_j$.

Now we reduce K to the graph such that the greatest common divisor of any two vertices is 2. If the number remaining vertices is even, we can simply choose $i_k = a_k/2$. Otherwise, let a_1 be the one that have been merged. Then we can choose $i_1 = 0$ and $i_k = a_k/2$ for other k. This complete the proof of Claim.

If there are at least two components K with $\kappa(K) = 0$, then there is no $\xi_1^{i_1} \cdots \xi_n^{i_n}$ of prime power order. Conversely, if there are less than two components with $\kappa(K) = 0$.

- If there are no component K with $\kappa(K) = 0$. Then we can choose $i = (i_1, \ldots, i_n)$ such that $\xi_1^{i_1} \cdots \xi_n^{i_n} = 1$.
- If there is exactly one component K with $\kappa(K) = 0$.
  - If K is an isolated point. WLOG, $K = \{a_1\}$. Let p be a prime divisor of a_1, then we can choose $i = (a_1/p, i_2, \ldots, i_n)$ so that $\xi_1^{i_1} \cdots \xi_n^{i_n}$ has order p.
  - If K consists of vertices $a_1, \ldots, a_r$ such that $(a_i, a_j) = 2$ for all $a_i, a_j \in K$, $i \neq j$, and r is odd. Then we can choose $i = (a_1/2, \ldots, a_r/2, i_{r+1}, \ldots, i_n)$ so that the order of $\xi_1^{i_1} \cdots \xi_n^{i_n}$ is 2.

Thus, we conclude that (ii) is equivalent to (iii). □

6 The Signature of $\overline{F}_\theta$

Let n be odd and Σ_a be a topological sphere. To determine the differential structure of $\Sigma_a = \partial \overline{F}_\theta$, we have to calculate the signature of $\overline{F}_\theta$. Recall that F_θ is diffeomorphic to Ξ_a. To calculate the intersection pairing, we have to fix the orientation.

- For each simplex $\Delta_k = \{(t_0, t_1, \ldots, t_k) \in \mathbb{R}^{k+1} \mid \sum t_i = 1\}$, we fix the standard orientation to be the one defined by the coordinate system $(t_1, \ldots, t_n)$.
- The chart $\Xi_a \cap \{z_1 \neq 0\}$ is connected, so we may assign the orientation of Ξ_a to be the one defined by the coordinate system $(\operatorname{Re} z_2, -\operatorname{Im} z_2, \ldots, \operatorname{Re} z_n, -\operatorname{Im} z_n)$ on $\Xi_a \cap \{z_1 \neq 0\}$. It is the same as the orientation defined by the coordinate system $(\operatorname{Re} z_2^{a_2}, -\operatorname{Im} z_2^{a_2}, \ldots, \operatorname{Re} z_n^{a_n}, -\operatorname{Im} z_n^{a_n})$ since $z \mapsto z^{a_i}$ is a holomorphic function.

Proposition 6.1 (Pham)**.** Under the identification $H_{n-1}(\overline{F}_\theta) \simeq H_{n-1}(\Xi_a) \simeq J_a/I_a$ and the above orientation of Ξ_a, the intersection pairing is given by

$$\langle [x], [y] \rangle = g(\bar{y}x(1-\omega_1)\cdots(1-\omega_n)), \quad x, y \in J_a,$$

where $g : J_a \to \mathbb{Z}$ is the additive homomorphism with

$$g(\omega_1^{i_1} \cdots \omega_n^{i_n}) = \begin{cases} (-1)^{(n-1)(n-2)/2} & , \ \omega_1^{i_1} \cdots \omega_n^{i_n} = 1; \\ (-1)^{(n-1)(n-2)/2+1} & , \ \omega_1^{i_1} \cdots \omega_n^{i_n} = \omega; \\ 0 & , \ \text{otherwise.} \end{cases}$$

and $y \mapsto \bar{y}$ is the automorphism of $\mathbb{Z}[\Omega_a]$ induced by $\omega_i \mapsto \omega_i^{-1}$.

Proof. Recall that the homology group $H_{n-1}(\Xi_a, \mathbb{Z})$ is generated by (Proposition 4.2)

$$e = (1-\omega_1)\cdots(1-\omega_n)\mathbf{e}.$$

The simplicial complex e can be parametrized as

$$e = \left\{ (\xi_1^{\varepsilon_1}|\alpha_1|^{1/a_1}, \ldots, \xi_n^{\varepsilon_n}|\alpha_n|^{1/a_n}) \;\middle|\; \alpha_k \in \mathbb{R},\ \varepsilon_k = \begin{cases} 1 & ,\alpha_k > 0 \\ 0 & ,\alpha_k < 0 \end{cases},\ \sum_{k=1}^{n} |\alpha_k| = 1. \right\}$$

We construct $\tilde{e}$ so that it is homotopic and transverse to e. Consider a curve in $\mathbb{C} \setminus \{0\}$

$$\begin{array}{rccc} \gamma: & \mathbb{R} & \to & \mathbb{C} \setminus \{0\} \\ & \tau & \mapsto & \gamma(\tau) = \alpha(\tau) + i\beta(\tau). \end{array}$$

such that

(i) The argument $\arg(\gamma(\tau))$ is a monotone increasing function of τ;

(ii) $\alpha(\tau) \leq 0$ for $\tau \in (-1, 1)$ and $\alpha(\tau) \geq 0$ for $\tau \notin (-1, 1)$;

(iii) $\alpha(\tau) \to \infty$ as $\tau \to \pm\infty$.

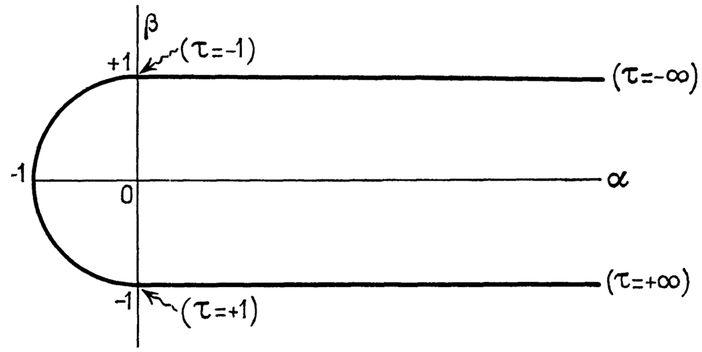

Figure 3: An example of γ

The curve in Figure 3 is an example of γ. Let $\tilde{e}$ be parametrized and defined as follows:

$$\{\tau\} = \left\{ (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \alpha(\tau_i) = 1 \right\};$$

$$\tilde{e} \begin{cases} \operatorname{Re}(z_k^{a_k}) = \alpha(\tau_k), \\ \operatorname{Im}(z_k^{a_k}) = \beta(\tau_k) - \left(\sum_{r=1}^{n} \beta(\tau_r) \right) \alpha(\tau_k); \end{cases}$$

$$\begin{array}{rcl} -\dfrac{\pi}{2a_k} \leq \arg z_k \leq \dfrac{\pi}{2a_k} & \text{for} & \tau_k \leq -1; \\ \dfrac{\pi}{2a_k} \leq \arg z_k \leq \dfrac{3\pi}{2a_k} & \text{for} & -1 \leq \tau_k \leq 1; \\ \dfrac{3\pi}{2a_k} \leq \arg z_k \leq \dfrac{5\pi}{2a_k} & \text{for} & 1 \leq \tau_k; \end{array}$$

We can divide $\tilde{e}$ into 2^n parts by the sign of $\beta(\tau_i)$ and make it into a simplicial complex. Notice that if we choose

$$\gamma_0(\tau) = \alpha_0(\tau) + i\beta_0(\tau) = \begin{cases} -\tau - 1 & ,\tau \leq -1; \\ 0 & ,-1 \leq \tau \leq 1; \\ \tau - 1 & ,\tau \geq 1, \end{cases}$$

Then we obtain the above parametrization of the union of simplices of e. By choosing some suitable homotheties, there is a homotopy from the curve γ_0 to γ. So we have a homotopy from e to $\tilde{e}$.

Now we calculate the intersection number of e and $\tilde{e}$. Notice that $\beta = \pm 1$ for $\alpha > 0$. By considering the sign of $\sum \beta(\tau_r)$, the point of $\tilde{e}$ satisfying $z_k^{a_k} \in \mathbb{R}_{\geq 0}$ must satisfies $\beta(\tau_k) = -1$ for all k or $\beta(\tau_k) = 1$ for all k. So the only intersecting simplices are $\mathbf{e}$ and $\omega_{i_1} \cdots \omega_{i_n}\mathbf{e}$, which intersect $\tilde{e}$ at

$$z^{(0)} : z_k^{(0)} = \frac{1}{n^{1/a_k}} \quad \text{and} \quad z^{(1)} : z_k^{(1)} = \frac{1}{n^{1/a_k}} \exp\left(\frac{2\pi i}{a_k}\right),$$

respectively. At the point $z^{(0)}$, $\tilde{e}$ can locally be described as

$$\operatorname{Im}(z_k^{a_k}) = 1 - n \operatorname{Re}(z_k^{a_k}).$$

So let $x_k = \operatorname{Re}(z_k^{a_k})$ be the coordinate system of $\mathbf{e}$, $y_k = \operatorname{Re}(z_k^{a_k})$ be the coordinate system of $\tilde{e}$. On Ξ_a, where $k = 2, \ldots, n$. We have

$$\begin{aligned} \frac{\partial}{\partial x_k} &= \frac{\partial}{\partial \operatorname{Re}(z_k^{a_k})} \\ \frac{\partial}{\partial y_k} &= \frac{\partial}{\partial \operatorname{Re}(z_k^{a_k})} - n\frac{\partial}{\partial \operatorname{Im}(z_k^{a_k})}. \end{aligned}$$

By considering the ordered basis

$$\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n},$$

we see that the intersection number is

$$(-1)^{(n-2)+\cdots+1} = (-1)^{(n-1)(n-2)/2}.$$

Similarly, at the point $z^{(1)}$, $\tilde{e}$ can locally be described as

$$\operatorname{Im}(z_k^{a_k}) = -1 + n \operatorname{Re}(z_k^{a_k}).$$

So let $x_k = \operatorname{Re}(z_k^{a_k})$ be the coordinate system of $\mathbf{e}$, $y_k = \operatorname{Re}(z_k^{a_k})$ be the coordinate system of $\tilde{e}$. On Ξ_a, where $k = 2, \ldots, n$. We have

$$\begin{aligned} \frac{\partial}{\partial x_k} &= \frac{\partial}{\partial \operatorname{Re}(z_k^{a_k})} \\ \frac{\partial}{\partial y_k} &= \frac{\partial}{\partial \operatorname{Re}(z_k^{a_k})} + n\frac{\partial}{\partial \operatorname{Im}(z_k^{a_k})}. \end{aligned}$$

we see that the intersection number is

$$(-1)^n \cdot (-1)^{(n-2)+\cdots+1} \cdot (-1)^{n-1} = (-1)^{(n-1)(n-2)/2+1},$$

where the factor $(-1)^n$ is given by the sign of $\omega_1 \cdots \omega_n \mathbf{e}$ in e. Therefore, the intersection indices of $\omega_1^{i_1} \cdots \omega_n^{i_n}\mathbf{e}$ and $\tilde{e}$ is

$$\langle \omega_1^{i_1} \cdots \omega_n^{i_n}\mathbf{e}, \tilde{e}\rangle = \begin{cases} (-1)^{(n-1)(n-2)/2} & , \ \omega_1^{i_1} \cdots \omega_n^{i_n} = 1; \\ (-1)^{(n-1)(n-2)/2+1} & , \ \omega_1^{i_1} \cdots \omega_n^{i_n} = \omega; \\ 0 & , \ \text{otherwise.} \end{cases}$$

We conclude that

$$\langle xe, ye\rangle = \langle x\bar{y}(1-\omega_1)\cdots(1-\omega_n)\mathbf{e}, \tilde{e}\rangle = g(x\bar{y}(1-\omega_1)\cdots(1-\omega_n)),$$

as desired. □

Theorem 6.2. Let $n \geq 5$ be odd, Σ_a be a topological sphere. Then the diffeomorphism type of Σ_a is determined by the signature $\sigma(\overline{F}_0)$, which is

$$\sigma(\overline{F}_0) = \sigma_a^+ - \sigma_a^-,$$

where

$$\sigma_a^+ = \#\left\{(j_1, \dots, j_n) \in \mathbb{Z}^n \;\middle|\; 0 < j_k < a_k,\ 0 < \sum_{k=1}^n \frac{j_k}{a_k} < 1 \mod 2\right\},$$
$$\sigma_a^- = \#\left\{(j_1, \dots, j_n) \in \mathbb{Z}^n \;\middle|\; 0 < j_k < a_k,\ -1 < \sum_{k=1}^n \frac{j_k}{a_k} < 0 \mod 2\right\}.$$

Proof. Using the same notation as in Lemma 5.3. Let

$$v_i = \prod_{k=1}^n \sum_{r=0}^{a_k-1} x_k^r \omega_k^r \quad \text{and} \quad v_j = \prod_{k=1}^n \sum_{r=0}^{a_k-1} y_k^r \omega_k^r,$$

where $x_k = \xi_k^{i_k}$ and $y_k = \xi_k^{j_k}$, be eigenvectors in $H_{n-1}(\Xi_a, \mathbb{Z}) \otimes \mathbb{C} = J_a/I_a \otimes \mathbb{C}$. By Proposition 6.1, the intersection number of v_i, v_j is

$$\begin{aligned}
\langle v_i, v_j \rangle &= g\left(v_i \bar{v}_j (1-\omega_1)\cdots(1-\omega_n)\right) \\
&= g\left(\prod_{k=1}^n \left(\sum_{r=0}^{a_k-1} x_k^r \omega_k^r\right)\left(\sum_{s=0}^{a_k-1} y_k^s \bar{\omega}_k^s\right)(1-\omega_k)\right) \\
&= g\left(\prod_{k=1}^n \left(\sum_{r=0}^{a_{k-1}} \sum_{s=0}^{a_{k-1}} x_k^r y_k^s \omega_k^r \bar{\omega}_k^s - \sum_{r=0}^{a_{k-1}} \sum_{s=0}^{a_{k-1}} x_k^r y_k^s \omega_k^{r+1} \bar{\omega}_k^s\right)\right) \\
&= (-1)^{(n-1)(n-2)/2}\left(\prod_{k=1}^n \sum_{r=0}^{a_k-1} x_k^r y_k^r - \prod_{k=1}^n \sum_{r=0}^{a_k-1} x_k^{r-1} y_k^r\right) \\
&\quad + (-1)^{(n-1)(n-2)/2+1}\left(\prod_{k=1}^n \sum_{r=0}^{a_k-1} x_k^{r+1} y_k^r - \prod_{k=1}^n \sum_{r=0}^{a_k-1} x_k^r y_k^r\right) \\
&= (-1)^{(n-1)(n-2)/2}\left(\prod_{k=1}^n (1-x_k^{-1}) \sum_{r=0}^{a_k-1} x_k^r y_k^r\right) \\
&\quad + (-1)^{(n-1)(n-2)/2+1}\left(\prod_{k=1}^n x_k(1-x_k^{-1}) \sum_{r=0}^{a_k-1} x_k^r y_k^r\right) \\
&= (-1)^{(n-1)(n-2)/2}(1 - x_1 \cdots x_n) \prod_{k=1}^n (1-x_k^{-1}) \left(\sum_{r=0}^{a_k-1} x_k^r y_k^r\right).
\end{aligned}$$

Observe that $\langle v_i, v_j \rangle \neq 0$ only if $i_k + j_k = a_k$ for every k. Therefore, $v_j + v_{a-j}$ and $i(v_j - v_{a-j})$ forms an orthogonal basis of $J_a/I_a \otimes \mathbb{R}$, and

$$\langle v_j + v_{a-j}, v_j + v_{a-j} \rangle = \langle i(v_j - v_{a-j}, i(v_j - v_{a-j}) \rangle = 2\langle v_j, v_{a-j} \rangle.$$

Compute directly, we have

$$\begin{aligned}\langle v_j, v_{a-j}\rangle &= (-1)^{(n-1)/2} a_1 \cdots a_n \left(\prod_{k=1}^{n}(1-x_k^{-1}) + \prod_{k=1}^{n}(1-x_k)\right)\\ &= 2a_1 \cdots a_n (-1)^{(n-1)/2}\operatorname{Re}\left(\prod_{k=1}^{n}(1-x_k)\right)\\ &= 2a_1 \cdots a_n (-1)^{(n-1)/2}\operatorname{Re}\left(\prod_{k=1}^{n}\left(-2ie^{\pi i j_k/a_k}\sin\pi\frac{j_k}{a_k}\right)\right)\\ &= 2a_1 \cdots a_n \operatorname{Re}\left(-\exp\left(\pi i\left(\frac{1}{2}+\sum_{k=1}^{n}\frac{j_k}{a_k}\right)\right)\prod_{k=1}^{n}2\sin\frac{\pi j_k}{a_k}\right).\end{aligned}$$

Since $\sin\frac{\pi j_k}{a_k}$ is always positive, by discussing the exponential term, the result follows. □

7 Brieskorn Exotic Spheres

From the above discussion, we conclude the following.

Example 7.1 (Brieskorn 1966)**.** For integer $n = 2m+1$, $m \geq 2$, the $(4m-1)$-spheres

$$\Sigma(\underbrace{2,\ldots,2}_{2m-1},3,6k-1) \qquad k = 1,\ldots,\frac{\sigma_m}{8}$$

represent all $\sigma_m/8$ classes of differential structure in bP_{4m}.

Proof. The graph G_a has two isolated point 3 and $6k-1$. By Theorem 5.5, Σ_a is a topological sphere which bounds a parallelizable manifold $\overline{F}_\theta$. We use Theorem 6.2 to compute the signature $\sigma(\overline{F}_\theta)$. Note that $j_1 = \cdots = j_{n-2} = 1$ and $j_{n-1} = 1$ or 2.

- For $j_{n-1} = 1$, we have

$$\sum_{k=1}^{n}\frac{j_k}{a_k} = (m-1) + \frac{5}{6} + \frac{j_k}{6k-1}.$$

  We see that it lie between $m-1$ and m if and only if $j_k = 1,\ldots,k-1$.

- For $j_{n-1} = 2$, we have

$$\sum_{k=1}^{n}\frac{j_k}{a_k} = (m-1) + \frac{7}{6} + \frac{j_k}{6k-1}.$$

  We see that it lie between m and $m+1$ if and only if $j_k = 1,\ldots,5k-1$.

Therefore, we conclude that

$$\sigma(\overline{F}_\theta) = \sigma_a^+ - \sigma_a^- = (-1)^{m-1}((k-1)-5k) + (-1)^m((5k-1)-k) = (-1)^m 8k,$$

as desired. □

References

[1] M. Kervaire and J. Milnor, *Groups of homotopy spheres: I*, Ann. of Math. Vol. 77, no. 3, 504-537, 1963.

[2] E. Brieskorn, *Beispiele zur differentialtopologie von singularitäten*, Invent. Math. 2, 1-14, 1966.

[3] Pham, F., *Formules de Picard-Lefschetz généralisées et ramificairion des intégrales*, Bull. Soc. Math. de France 93, 333-367 1965.

[4] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Stud., vol. 61, Princeton University Press, 1968.

[5] Friedrich Hirzebruch, *Singularities and exotic spheres*, Séminaire N. Bourbaki, exp. no 314, p. 13-32, 1968.

[6] Ben Knudsen, *A Brieskorn exotic sphere.* https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/2021/01/benlecture.pdf

[7] Yu-Ting, Huang, *Construction of 28 differential structures on* S^7.
http://www.math.ntu.edu.tw/~dragon/Exams/DG%20II%202021%20reports/Final%20reports%20DG%20II%20-%202021.pdf