In the following, $K=\mathbb{R}$ or $\mathbb{C}$, and every vector space is defined over K.

Definition 1. A normed space $(X,||\cdot||)$ is a Banach space if it is complete as a metric space.

Definition 2. Let X,Y be Banach spaces, $x\in X$ and $f:U(x)\to Y$ on an open neighborhood $U(x)$ of x.

(1) f is *differentiable at* x if there exists $T\in L(X,Y)$ such that

$$f(x+h)-f(x)=Th+o(\|h\|)\quad \text{as } h\to 0.$$

Denote $T=f'(x)$.

(2) f is C^1 if

$$f':U\to L(X,Y),\ x\mapsto f'(x)$$

is continuous.

Proposition 3.

(a) For $f,g:U\to Y$ differentiable at $x\in X$, we have $(cf+g)'(x)=cf'(x)+g(x)$ for all $c\in K$.

(b) For $f_i:U\to X_i$, $i=1,2$, differentiable at $x\in X$, and $(\cdot,\cdot)\in L(X_1,X_2;Y)$, we have

$$(f_1,f_2)'(x)h=(f_1'(x)h,f_2(x))+(f_1(x),f_2'(x)h).$$

(c) For $f:U\to V$ differentiable at x and $g:V\to Z$ differentiable at $f(x)$, we have

$$(g\circ f)'(x)=g'(f(x))f'(x).$$

Definition 4. If f' is differentiable at x, define the second derivative

$$f''(x):=(f')'(x)\in L(X,L(X,Y))=(X,X;Y).$$

f is C^2 if f' is C^1. Inductively we can define $f^{(k)}\in L(X,X,\dots,X;Y)$, and say f is C^k if f' is C^{k-1}.

Definition 5. Let $f\in C([a,b],Y)$. For a division $\Delta=\{a=t_0<t_1<\ldots<t_n=b\}$, let

$$I_\Delta(f):=\sum_{i=1}^{n}f(t_i^*)|t_i-t_{i-1}|\quad \text{for } t_i^*\in[t_{i-1},t_i]$$

Define the integral $\int_a^b f(t)\,dt:=\lim\limits_{\text{mesh}\Delta\to 0}I_\Delta(f)$, which is well-defined by the uniformly continuity of f.

Theorem 6 (Fundamental Theorem of Calculas)**.** Let Y be a Banach space and $f \in C([a,b], Y)$. Let $F(t) = \int_a^t f(s)\, ds$. Then $F'(t) = f(t)$.

proof: $F(t+h) - F(t) - f(t)h = \int_t^{t+h} (f(s) - f(t))\, ds$ and

$$\frac{1}{\|h\|} \left\| \int_t^{t+h} (f(s) - f(t))\, ds \right\| \leq \sup_{t' \in [t,t+h]} \|f(t') - f(t)\| \to 0 \quad \text{as } h \to 0$$

Hence $F'(t) = f(t)$. □

Corollary 6.1. If $f \in C^1([a,b], Y)$, then $f(t) - f(a) = \int_a^t f'(s)\, ds$.

Here we recall the Hahn-Banach theorem:

Theorem 7 (Hahn-Banach)**.** Let X be a vector space over K and $p : X \to \mathbb{R}$ be a seminorm. Let M be a vector subspace of X. If $f : M \to K$ is a linear functional such that $\|f(m)\| \leq p(m)$ for all $m \in M$, then there exists a linear functional $F : X \to K$ such that $F(m) = f(m)$ for all $m \in M$ and $\|F(x)\| \leq p(x)$ for all $x \in X$.

Theorem 8 (Mean Value Theorem)**.** Let X, Y be Banach spaces, U be an open subset of X, and $x, y \in U$ such that the segment $\overline{xy}$ lies in U. If $f'(c)$ exists for all $c \in \overline{xy}$, then $\|f(x) - f(y)\| \leq \|f'(c)\| \|x - y\|$ for some $c \in \overline{xy}$.

proof: Let $a = f(x) - f(y)$, and a linear functional $\lambda_1 : \operatorname{span}(a) \to \mathbb{R}$ defined by $\lambda_1(ta) := t\|a\|$. Then $|\lambda_1(u)| \leq \|u\|$. By Hahn-Banach Theorem, there exists $\lambda \in Y^*$ such that $\lambda(a) = \lambda_1(a)$ and $\lambda(u) \leq \|u\|$ for all $u \in X$, which gives $\|\lambda\| \leq 1$. On the other hand, $\|\lambda\| = \sup_{x \neq 0} \dfrac{|\lambda(x)|}{\|x\|} \geq \dfrac{|\lambda(a)|}{\|a\|} = 1$. Hence $\|\lambda\| = 1$. Consider $g : [0,1] \to \mathbb{R}$ defined by $g(t) = \lambda f(y + t(x-y))$. By mean value theorem, there is $t' \in (0,1)$ such that $g(1) - g(0) = g'(t')$. Let $c = y + t'(x-y) \in \overline{xy}$. Then

$$\begin{aligned}
\|f(x) - f(x)\| &= \lambda(f(x) - f(y)) \\
&= g(1) - g(0) \\
&= g'(t') \\
&= \lambda(f'(y + t'(x-y))(x-y)) \\
&\leq \|\lambda\| \|f'(c)\| \|x - y\| \\
&= \|f'(c)\| \|x - y\|
\end{aligned}$$

as desired. □

Theorem 9. Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k map $(k \geq 1)$ on an open neighborhood $U(x_0)$ of x_0. If $f'(x_0) : X \to Y$ is isomorphic, then f is a locally C^k-diffeomorphism at x_0.

proof: We may assume that $x_0 = 0$ and $f(0) = 0$. By replacing f by $f'(0)^{-1} f$, we may assume that $X = Y$ and $f'(0) = id_X$. There exists $r > 0$ such that $||f'(x) - id_X|| < \frac{1}{2}$ for $x \in B_r(0)$. Let $g(x) = x - f(x)$ and $g_y(x) = y + g(x)$. Since $g'_y(x) = g'(x) = id_X - f'(x)$, we have $||g'_y(x)|| < \frac{1}{2}$ for $x \in B_r(0)$. Then for $x_1, x_2 \in B_r(0)$,

$$\begin{aligned}
\|g_y(x_1) - g_y(x_2)\| &= \left\| \int_0^1 \frac{d}{dt} f(x_2 + t(x_1 - x_2))\, dt \right\| \\
&= \left\| \int_0^1 f'(x_2 + t(x_1 - x_2))(x_1 - x_2)\, dt \right\| \\
&\le \|f'(a)\| \|x_1 - x_2\| \quad \text{for some } a \text{ in the segment } \overline{x_1 x_2} \subseteq B_r(0) \\
&\le \frac{1}{2} \|x_1 - x_2\|
\end{aligned}$$

Moreover for $y \in B_{\frac{r}{2}}(0)$ and $x \in B_r(0)$, $\|g_y(x)\| < \frac{r}{2} + \|g(x) - g(0)\| < r$. By Banach fixed point theorem, there is a unique fixed $x(y)$ of $g_y(x)$, namely, $f(x(y)) = y$. Hence we have the inverse map $f^{-1} : B_{\frac{r}{2}}(0) \to B_r(0)$. For $y_1, y_2 \in B_{\frac{r}{2}}(0)$, write $f(x_i) = y_i$ for $x_i \in B_r(0)$. Then

$$\begin{aligned}
\|x_1 - x_2\| &= \|y_1 + g(x_1) - y_2 - g(x_2)\| \\
&\le \|y_1 - y_2\| + \|g(x_1) - g(x_2)\| \\
&\le \|y_1 - y_2\| + \frac{1}{2} \|x_1 - x_2\|
\end{aligned}$$

which gives $\|f^{-1}(y_1) - f^{-1}(y_2)\| \le 2\|y_1 - y_2\|$. This implies that f^{-1} is continuous.
Let $V := f^{-1}\left(B_{\frac{r}{2}}(0)\right)$, which is an open subset in $B_r(0)$. Then $f|_V : V \to B_{\frac{r}{2}}(0)$ is a homeomorphism. For $a, x \in V$, $y := f(x), b := f(a)$. Since f is C^1, we have

$$f(x) - f(a) - f'(a)(x - a) = o(x - a) \quad \text{as } x \to a.$$

That is,

$$f'(a)^{-1}(y - b) - \left(f^{-1}(y)\right) - f^{-1}(b) = f'(a)^{-1} o(x - a) \quad \text{as } y \to b$$

Since $\lim_{y \to b} \frac{o(x - a)}{\|y - b\|} = \lim_{y \to b} \frac{o(x - a)}{\|x - a\|} \frac{\|x - a\|}{\|y - b\|} = 0$, we finally have $(f^{-1})'(b) = f'(f^{-1}(b))^{-1}$. Since f^{-1} is C^0, by this equation we see that f^{-1} is C^1, and inductively, f^{-1} is C^k. □

Finally we recall the open mapping theorem: (cf. Theorem 2.11, Function Analysis, Walter Rudin)

Theorem 10. Let X be a topological vector space whose topology is induced by a complete invariant metric, and Y be a Hausdorff topological vector space. If $T : X \to Y$ is a surjective continuous linear operator, then T is an open map.

Corollary 10.1. If moreover T is bijective, then T is isomorphic.

Definition 11. Let M be a closed linear subspace of a topological vector space X. We say that M splits X if there exists a closed vector subspace N such that $X = M \oplus N$ as topological vector space.

Lemma 12 (Local Normal Forms)**.** Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k map $(k \geq 1)$ on a open neighborhood $U(x_0)$ of x_0 such that $N = \mathcal{N}(f'(x_0))$ splits X and $R = \mathcal{R}(f'(x_0))$ splits Y. Then there exists a neighborhood $W(x_0)$ of x_0 and a C^k-diffeomorphism $\varphi : U(0) \to W(x_0)$, where $U(0)$ is a neighborhood of 0 in $N \times R$, such that $f(\varphi(n,r)) = f(x_0) + r + g(n,r)$ for $(n,r) \in U(0)$ for some g satisfying $g(n,r) \in R^\perp$, $g(0,0) = 0$ and $g'(0,0) = 0$.

proof: We may assume that $x_0 = 0$ and $f(0) = 0$. By the assumption $X = N \oplus N^\perp$ and $Y = R \oplus R^\perp$, for $x = x_1 + x_2 \in X$ with $x_1 \in N, x_2 \in N^\perp$, write $f(x) = f_1(x) + f_2(x)$ with $f_1(x) \in R, f_2(x) \in R^\perp$. Note that $f(0) = 0$ and $f'(0)h = f_1'(0)h + f_2'(0)h \in R$, we have $f_1(0) = f_2(0) = 0$ and $f_2'(0) = 0$.
Consider the map $F : U(0) \to N \times R$ defined by $F(x) = (x_1, f_1(x))$. Then we have $F(0) = 0$ and $F'(0)h = (h_1, f_1'(0)h) = (h_1, f'(0)h)$, which implies $F'(0) : X \to N \times R$ is an isomorphism since the map $f'(0) : N^\perp \to R$ is. By the inverse function theorem, F is a locally C^k-diffeomorphism, that is, there is a neighborhood W of $x_0 = 0$ in X such that $F|_W : W \to U := F(W)$ is a C^k-diffeomorphism, where U is clearly a neighborhood of 0 in $N \times R$.
Let $\varphi := F|_W{}^{-1}$. Let $x = \varphi(n,r)$. We know that $n = x_1$ and $r = f_1(x)$. Then

$$f(\varphi(n,r)) = f_1(x) + f_2(x) = r + f_2(\varphi(n,r))$$

We just take $g(n,r) := f_2(\varphi(n,r)) \in R^\perp$, then $g(0,0) = 0$ and $g'(0,0) = 0$, as desired. □

Definition 13. Let X be a topological vector space.

(1) Let $p \in X$. A collection $\mathcal{P}$ of neighborhoods of p is a *local base* at p if every neighborhood of p contains a members of $\mathcal{P}$.

(2) X is *locally convex* if there exists a local base at 0 whose members are convex.

We recall another version of the Hahn-Banach theorem:

Theorem 14 (Hahn-Banach)**.** Every continuous linear functional defined on a closed vector space of a locally convex topological vector space X can be extended to a continuous linear functional on X.

Lemma 15. Let M be a closed vector subspace of a topological vector space X.

(a) If X is locally convex and $\dim M < \infty$, then M splits X.

(b) If $\operatorname{codim} M < \infty$, then M splits X.

proof:

(a) Let $\{e_1, \ldots, e_n)$ be a basis for M. For each $x \in M$, write $x = \sum_{i=1}^{n} \alpha_i(x) e_i$ for some continuous linear functional α_i on M. By Hahn-Banach theorem, each α_i extends to a continuous linear functional $\beta_i \in X^*$. Let $N = \bigcap_{i=1}^{n} \mathcal{N}(\beta_i)$. Then $X = M \oplus N$.

(b) Let $\pi : X \to {}^{X}/_{M}$ be the quotient map and let $\{e_1, \ldots, e_n)$ be a basis for ${}^{X}/_{M}$. Take $x_i \in X$ such that $\pi(x_i) = e_i$ and let $N = \operatorname{span}(x_1, \ldots, x_n)$. Then $X = M \oplus N$.

$\square$

Definition 16. Let X, Y be Banach spaces. A continuous linear operator $T : X \to Y$ is a *Fredholm operator* if both $\dim \mathcal{N}(T)$ and $\operatorname{codim} \mathcal{R}(T)$ are finite. The index of T is defined by $\operatorname{Ind}(T) = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T)$.

By Lemma 15, $\mathcal{N}(T)$ splits X and $\mathcal{R}(T)$ splits Y if T is Fredholm.

Definition 17. Let X, Y be Banach spaces and $f : X \to Y$ be a C^1 map.

(1) A point $x \in X$ is called a regular point of f if $f'(x)$ is surjective and $\mathcal{N}(f'(x))$ splits X; otherwise x is called a singular point of f.

(2) A point $y \in Y$ is called a regular value of f if $f^{-1}(y)$ contains only regular values of f; otherwise y is called a singular value.

Definition 18. Let M be a topological space.

(1) A *chart* (U, φ) in M is a pair with an open subset U in M and a homeomorphism $\varphi : U \to U_\varphi \underset{\text{open}}{\subseteq} X_\varphi$ for some Banach space X_φ, called the chart space.

(2) A *C^k-atlas* for M is a collection of charts $(U_\alpha, \varphi_\alpha)_\alpha$ such that $\bigcup_\alpha U_\alpha = M$ and any two charts are C^k-compatible, i.e., either $U_\alpha \cap U_{\alpha'} = \emptyset$ or both $\varphi_\alpha \circ \varphi_{\alpha'}^{-1}$ and $\varphi_{\alpha'} \circ \varphi_\alpha^{-1}$ are C^k.

(3) M is said to be a *C^k-Banach manifold* if M has a C^k-atlas.

Definition 19. Let $f : M \to N$ be a map between C^k-Banach manifolds.

(1) f is said to be C^k if f is C^k at each point $x \in M$ in charts.

(2) f is a C^k-diffeomorphism if f is bijective and both f and f^{-1} are C^k.

Definition 20. Let M be a C^k-manifold with $k \geq 1$, and $x \in M$.

(1) For a C^1-curve $\gamma(t)$ in M with $\gamma(t_0) = x$ for some t_0, let $\gamma_\varphi(t) = \varphi(\gamma(t))$ and $v_\varphi = \gamma'_\varphi(t_0)$. v_ϕ is called the *representative* of γ.

(2) Two C^1 curves passing through x are *equivalent at* x if the represntives are the same tangent vector at x in charts.

(3) A *tangent vector* to M at x consists of all C^1-curves equivalent at x to a fixed C^1-fixed.

(4) The *tangent space* T_xM to M at x is the set of all tangent vector to M at x.

Proposition 21. Let $f : M \to N$ be C^1 between C^1-Banach manifolds. Then there is a linear continuous map $f'(x) : T_xM \to T_{f(x)}N$ at each point $x \in M$, called the *tangent map* of f at x.

Definition 22. Let M, N be C^1-Banach manifolds and $f : X \to Y$ be a C^1 map.

(1) A point $x \in M$ is called point of f if $f'(x)$ is surjective and $\mathcal{N}(f'(x))$ splits X; otherwise x is called a singular point of f.

(2) A point $y \in N$ is called a regular value of f if $f^{-1}(y)$ contains only regular values of f; otherwise y is called a singular value.

Definition 23. Let M, N be C^k-Banach manifolds, $k \geq 1$, and $f : M \to N$ be C^k. f is called a *Fredholm operator* at x if $f'(x) : TM_x \to TN_{f(x)}$ is Fredholm.

Theorem 24 (Sard-Smale Theorem)**.** Let M, N be C^∞-Banach manifolds with M second countable. If $f : M \to N$ is C^k-Fredholm with $k > \max(\text{Ind} f'(x), 0)$ for all $x \in M$, then the set of singular values of f is meager, i.e., a countable union of nowhere dense subsets, and the set of regular values is residual, i.e., a countable intersection of open dense subsets.

To prove Sard-Smale theorem, we need some lemmas:

Lemma 25. Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k-Fredholm map on an open neighborhood $U(x_0)$ of x_0 with $k > \max(\text{Ind} f'(x_0), 0)$. Then there exists an open neighborhood $W = W(x_0)$ of x_0 such that the set of regular values of $f|_W$ is dense in Y.

proof: By definition and Lemma 15, $N := \mathcal{N}(f'(x_0))$ splits X and $R := \mathcal{R}(f'(x_0))$ splits Y. By Lemma 12, there exist a neighborhood $W = W(x_0)$ of x_0 in X and a C^k-diffeomorphism $\varphi : U(0) \to W$, where $U(0) := U_N(0) \times U_R(0)$ with $U_N(0), U_R(0)$ neighborhoods of 0 in N, R, respectively, such that $h(n, r) := f(\varphi(n, r)) = f(x_0) + r + g(n, r)$ for $(n, r) \in N \times R$ for some g satisfying $g(n, r) \in R^\perp$,

$g(0,0)=0$ and $g'(0,0)=0$. It suffices to prove the set of regular values of h is dense in Y. Let $y \in Y$ and write $y = f(x_0) + y_1 + y_2$ with $y_1 \in R$ and $y_2 \in R^{\perp}$. Consider the C^k map $\psi : U_N(0) \to R^{\perp}$ defined by $\psi(n) = g(n, y_1)$. Since now $k > \max\left(\dim N - \dim R^{\perp}, 0\right)$, by Sard's theorem, the set of regular values of ψ is dense in $R^{\perp}$. Hence it suffices to show that y is a regular value of h if y_2 is a regular value of ψ. For $h(n,r) = y$, we have $r = y_1$ and $\psi(n) = y_2$. We compute that

$$h'(n,r)(n',r') = v' + \psi'(n)(n') + g_r(n,r)(r')$$

From this we see that if $\psi'(n)$ is surjective, then so is $h'(n,r)$, as desired. □

Lemma 26. Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k-Fredholm map on an open neighborhood $U(x_0)$ of x_0 with $k \geq 1$. Then f is locally proper.

proof: It suffices to prove that $h : U(0) \to Y$ is proper. Let K be a compact set in Y. For any sequence $\{(n_m, r_m)\}_{m \in \mathbb{N}}$ in $h^{-1}(K) \subseteq U(0)$ with $y_m := h(n_m, r_m) \in K$, We may assume y_m converges to some $y \in K$ by passing to a convergent subsequence. Write $y = f(x_0) + y_1 + y_2$ with $y_1 \in R, y_2 \in R^{\perp}$. Note that $h(n_m, r_m) = f(x_0) + r_m + g(n_m, r_m)$, we see that $r_m \to y_1$ as $m \to \infty$. Now N is finite dimensional since $f'(x_0)$ is Fredholm, then there is a subseqence y_{m_k} converging to some $n \in N$. Since

$$h(n, y_1) = \lim_{k \to \infty} h(n_{m_k}, r_{m_k}) = y \in K,$$

we get that (n_{m_k}, r_{m_k}) converges to $(n, y_1) \in h^{-1}(K)$, which yields that $h^{-1}(K)$ is compact, namely, h is proper. □

Lemma 27. Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k-Fredholm map on an open neighborhood $U(x_0)$ of x_0 with $k \geq 1$. If x_0 is a regular point of f, then there exists a neighborhood of x_0 containing only regular points of f.

proof: Since $f'(x_0) : X \to Y$ is surjective, we have $R = Y$ and then $g = 0$. Now by $h(n,r) = f(x_0) + r$, we compute that $h'(n,r)(n',r') = r'$ for all $r' \in Y$, that is, $h'(n,r)$ is surjective for all $(n,r) \in U(0)$. Hence the open neighborhood $\varphi(U(0))$ of x_0 contains only regular values of f. □

Corollary 27.1. Let X, Y be Banach spaces, $x_0 \in X$ and $f : U(x_0) \to Y$ be a C^k-Fredholm map on an open neighborhood $U(x_0)$ of x_0 with $k > \max(\mathrm{Ind} f'(x_0), 0)$. Then there exists an open neighborhood $V(x_0)$ of x_0 in X such that the set of regular values of $f|_{V(x_0)}$ is open dense in Y and the set of singular values of $f|_{V(x_0)}$ is closed and nowhere dense in Y.

proof: We may take a open neighborhood $V = V(x_0)$ of x_0 in X satisfying

(i) the set of regular values of $f|_V$ is dense in Y.

(ii) $f|_V$ is proper.

By Lemma 27, the set of singular points of $f|_V$ is closed in X, hence the set of singular values is closed in Y by (ii) and nowhere dense by (i). $\square$

Now we come back to the proof of Sard-Smale theorem.

proof: By Corollary 27.1, for each point $x \in M$, we can take an open neighborhood $U(x)$ of x such that the set of singular values of $f|_{U(x)}$ is closed and nowhere dense. Now M is Lindolöf, there exists $\{x_n\}_{n\in\mathbb{N}}$ such that M is covered by $\bigcup_{n\in\mathbb{N}} U(x_n)$, and note that y is a singular value of f if and only if it is a singular value of $f|_{U(x_n)}$ for some $n \in \mathbb{N}$. Therefore the set of singular values of f is the union of the sets of singular values of $f|_{U(x_n)}$, which is meager, and then the set of regular values of f is residual. $\square$

Corollary 27.2. Let M and N be C^∞-Banach manifolds and $f : M \to N$ be proper C^k-Fredholm with $k > \max(\mathrm{Ind} f'(x), 0)$ for all $x \in M$. Then the set of regular values of f is open dense in N.

Remark 28. In the course, we apply Sard-Smale theorem to the map $\mathcal{N}^{k,p} \to \Omega^{2,+}(X, i\mathbb{R})$, where

$$\mathcal{N}^{k,p} := \left\{(A, \Phi) \in \mathcal{X}^{k,p} | D_A\Phi = 0, d^*(A - A_0) = 0, \Phi \not\equiv 0\right\}$$

is a smooth second countable Banach manifold by Proposition 8.16 in Salamon's book, and $\Omega^{2,+}(X, i\mathbb{R})$ is a Banach space.