Recall. The Γ-wall $\Omega^{2,+}_{\Gamma}(X, i\mathbb{R}) \subseteq \Omega^{2,+}(X, i\mathbb{R})$ has codim $b^+ = 1$:

$$\begin{aligned} &\Omega^{2,+}_{\Gamma}(X,i\mathbb{R}) \\ &\quad \| \\ &\{\eta \mid F_A^+ + \eta = 0 \text{ for some } A\} \quad (\Rightarrow \exists \text{ sol. } (A, \Phi = 0)) \end{aligned}$$

it is defined by $\varepsilon_\Gamma(\eta) := -\int_X \langle i\eta . w\rangle \, dVol - \pi [w] \cdot \underset{\text{det bundle}}{L_\Gamma} = 0$

w: self-dual harmonic 2-form with norm $\int |w|^2 dVol = 1$

Define $SW^{\pm}(X.\Gamma) = SW(X.\Gamma, \eta)$. $\pm \varepsilon(\eta) > 0$.

Wall crossing: compute $w(X.\Gamma) = SW^+(X.\Gamma) - SW^-(X.\Gamma)$.

Def. Let $\tilde{T} = \tilde{T}(\eta) = \{A \in \mathcal{A}(\Gamma) \mid F_A^+ + \eta = 0 . \ d^*(A - A_0) = 0\}$ $/\!/ \ H^1(X, i\mathbb{R})$

$\mathcal{G}_0(x_0) = \{u : X \to S^1 \mid d^* d \log u = 0, \ u(x_0) = 1\} \simeq H^1(X, 2\pi i \mathbb{Z})$

$\rightsquigarrow$ torus $T = \tilde{T} / \mathcal{G}_0(x_0) \simeq H^1(X, i\mathbb{R}) / H^1(X, 2\pi i\mathbb{Z})$

universal bundle $\mathcal{E} = X \times \tilde{T} \times \mathbb{C} / \mathcal{G}_0(x_0) \to X \times T$. $\left(u^*(x.A.z) := (x, u^*A, u^{-1}z) \right)$

Prop. $c_1(\mathcal{E}) = [\Omega]$, where $\Omega_{(\alpha.p)}((v.\alpha).(w.\beta)) = \frac{1}{2\pi i}(\beta(v) - \alpha(w))$

pf. Consider $X \times T \xrightarrow{f} T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$

$(x.A) \mapsto (s_1 \dots s_n, t^1 \dots t^n)$, $\quad s_j = \frac{1}{2\pi i} \log \underset{\text{basis of } \mathcal{G}_0(x_0)}{u_j}$. $A - A_0 = \sum t^j ds_j$.

$\rightsquigarrow \mathcal{E} = f^* E$, where $E = \mathbb{R}^{2n} \times \mathbb{C} / \mathbb{Z}^{2n}$, $\quad (k.l)(s.t.z) = (s+k, t+l, z e^{-2\pi i \cdot s l})$

$\rightsquigarrow c_1(\mathcal{E}) = f^* c_1(E)$, $c_1(E) = \sum ds_j \wedge dt^j$. $\square$

Thm. (Li-Liu, Ohta-Ono) If $b_1 = 2k$, then $w(X.\Gamma) = \int_T \frac{1}{k!} \left(-\frac{1}{4} \int_X \Omega^2 \wedge c_1(L_\Gamma) \right)^k$.

For each $p \in T$. consider

$$\begin{tikzcd} E_p \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X \arrow[r] & X \times T \\ x \arrow[r, mapsto] & (x, p) \end{tikzcd}$$

If $A \mapsto p_A = p$, then

$\iota_A : X \times \mathbb{C} \xrightarrow{\simeq} E_p$

$(x, z) \mapsto (x, A, z)$

is a trivialization

$\nabla_{u^*A} = u^{-1} \circ \nabla_A \circ u \rightsquigarrow$ spin^c connection ∇_p on $W_p = W \otimes E_p$.

$\rightsquigarrow$ Dirac operator $D_p : C^\infty(W^+ \otimes E_p) \to C^\infty(W^- \otimes E_p)$

Define $IND = (\ker D_p - \operatorname{coker} D_p)_{p \in T} \in K(T)$ (well-defined by Fredholm theory)

A-S index theorem (for family)

$$\Rightarrow \quad ch(IND) = \int_X ch(L_p^{1/2}) \wedge \hat{A}(X) \wedge ch(\mathcal{E}) \in H^*(T, \mathbb{Z})$$

Lemma. $c_k(-IND) = \frac{1}{k!}\left(-\frac{1}{4}\int_X \Omega^2 \wedge L_p\right)^k$

pf. Sublemma: $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is a torsion class for $\alpha_1, \alpha_2, \alpha_3 \in H^1(X, \mathbb{Z})$.

(if not, $\exists \beta$ s.t. $\int_X \beta \cup \alpha_1 \cup \alpha_2 \cup \alpha_3 > 0$. Let $w_i := \alpha_0 \cup \alpha_i + \alpha_{i+1} \cup \alpha_{i-1}$

Then $w_i^2 > 0$. $w_i \cdot w_j = 0 \Rightarrow b^+ \geq 3$ $\times$)

So $ch(\mathcal{E}) = 1 + \Omega + \frac{1}{2}\Omega^2$ $(\Omega^3 = 0)$

$ch(L_p^{1/2}) = 1 + \frac{1}{2}L_p + \frac{1}{8}L_p^2$. $\hat{A}(X) = 1 - \frac{1}{24}p_1(X)$.

$$\Rightarrow ch(IND) = \int_X \left(1 + \frac{1}{2}L_p + \frac{1}{8}L_p^2\right)\cdot\left(1 - \frac{1}{24}p_1(X)\right)\cdot\left(1 + \Omega + \frac{1}{2}\Omega^2\right)$$

$$= \frac{1}{8}(L_p^2 - \tau) + \frac{1}{4}\int_X \Omega^2 \wedge L_p$$

Write $c_t(-IND) = \prod(1 + y_j t) \Rightarrow \sum y_j = -\frac{1}{4}\int_X \Omega^2 \wedge L_p$. $\sum y_j^k = 0 \quad \forall k \geq 2$

$$\Rightarrow c_k(-IND) = \frac{1}{k!}\left(\sum y_j\right)^k$$

□

Def. We say $\eta \in \Omega^{2,+}_\Gamma(X, i\mathbb{R})$ is <u>regular</u> if

(a) $\forall (A, \Phi) \in \widetilde{\hat{M}}^X(X, \Gamma, \eta)$, $\operatorname{coker} D_{A.\Phi} = H^0(X, i\mathbb{R})$

(b) $\forall A \in \tilde{T}(\eta)$, $0 \neq \Phi \in \ker D_A$, $H^1(X, i\mathbb{R}) \twoheadrightarrow \operatorname{coker} D_A$
$\alpha \longmapsto \pi_A(\Gamma(\alpha)\Phi)$

(c) $\forall A \in \tilde{T}(\eta)$, $0 \neq \Phi \in \ker P_A$. $\exists \varphi \in C^\infty(W^+)$ s.t.
$\pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) \neq 0$. $D_A\varphi \in \Gamma(H^1(X, i\mathbb{R}))\Phi$.

SW-eq. $\begin{cases} d^*(A - A_0) = 0 \\ F_A^+ + \eta + itw = \sigma^+((\Phi\Phi^*)_0) \\ D_A\Phi = 0 \end{cases}$

$$D_{A.\Phi}\begin{pmatrix}\alpha \\ \varphi\end{pmatrix} = \begin{pmatrix} d^*\alpha \\ d^+\alpha - \sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0) \\ D_A\varphi + \Gamma(\alpha)\Phi \end{pmatrix}$$

$\pi_A : L^2(W^-) \longrightarrow \operatorname{coker} P_A$

$\pi^+ : \Omega^{2,+}(i\mathbb{R}) \longrightarrow H^{2,+}(i\mathbb{R})$

are orthogonal projections

Prop. (i) $\{\eta \text{ regular}\} \subseteq \Omega_p^{2,+}(X. i\mathbb{R})$
open. dense
wrt C^∞-top.

(ii) $\forall \eta_0$ reg., $p>0$, $\exists \varepsilon>0$ s.t. $\forall \eta \in \Omega^{2,+} \setminus \Omega_p^{2,+}$ with $\|\eta-\eta_0\|_{L^p} < \varepsilon$

$\Rightarrow$ η is reg. ($M^\chi(X,\Gamma,\eta)$ is smooth with expected dim $\frac{1}{4}(L_p^2 - 2\chi - 3\tau)$)

pf of wall-crossing formula assuming this prop.

Choose $\eta \in \Omega_p^{2,+}$ reg. Consider $M^\chi(\eta_t) = M^\chi(X,\Gamma,\eta_t)$, $\eta_t = \eta + it\omega$. $|t| \leq \varepsilon$.

Then $\operatorname{coker} D_{A.\Phi} = H^0(X, i\mathbb{R})$ $\forall (A.\Phi) \in M^\chi(\eta_t)$ (ε small)

$$\partial M^\chi(\{\eta_t\}) = M^\chi(\eta_\varepsilon) - M^\chi(\eta_{-\varepsilon}) = M(\eta_\varepsilon) - M(\eta_{-\varepsilon})$$

$\uparrow$

dim = index $D_A + b_1 - 1 \geq 1$, but non-compact at $t=0$.

Take $\delta>0$ small s.t. $(A.\Phi) \in \tilde{\mathcal{M}}(\eta_\varepsilon) \cup \tilde{\mathcal{M}}(\eta_{-\varepsilon}) \Rightarrow \|\Phi\|_{L^2}^2 > \delta$. (need $\mathcal{M}(\eta_{\pm\varepsilon})$ cpt)

Define $\tilde{\mathcal{M}}_\delta(\{\eta_t\}) = \{(A.\Phi.t) \mid (A.\Phi) \in \tilde{\mathcal{M}}(\eta_t),\ \|\Phi\|_{L^2}^2 \geq \delta\} \longrightarrow \underset{\substack{\| \\ M_\delta(\{\eta_t\})}}{\tilde{M}_\delta(\{\eta_t\})/\mathcal{G}_0}$

Lemma. For $\delta>0$ small. $\tilde{M}(\{\eta_t\}) \pitchfork \{(A.\Phi.t) \mid \|\Phi\|_{L^2}^2 = \delta\}$

pf. Tang. of $\tilde{M}$ at $(A.\Phi.t)$ is $\left\{(\alpha.\varphi.\tau) \mid D_{A.\Phi}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ -i\tau\omega\\ 0\end{pmatrix} \ (*)\right\}$

Tang of $\{\|\Phi\|_{L^2}^2 = \delta\}$ at $(A.\Phi.t)$ is $\{(\alpha.\varphi.\tau) \mid \langle\varphi.\Phi\rangle = 0\}$

" $\pitchfork$ " $\Leftrightarrow$ $\exists (\alpha.\varphi.\tau)$ with $\langle\varphi.\Phi\rangle \neq 0$.

Consider $\Pi^+ D_{A.\Phi} \Pi$. $\Pi^+\begin{pmatrix}\xi\\ \gamma\\ \psi\end{pmatrix} = \begin{pmatrix}\xi\\ \gamma - \pi^+\gamma\\ \psi\end{pmatrix}$, $\Pi\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}\alpha\\ \varphi\end{pmatrix} - \frac{\langle\Phi.\varphi\rangle}{\|\Phi\|^2}\begin{pmatrix}0\\ \Phi\end{pmatrix}$

(ortho comp of Φ)

If $\operatorname{coker}(\Pi^+ D_{A.\Phi} \Pi) = H^0(X.i\mathbb{R}) \oplus H^{2,+}(X, i\mathbb{R}) = \operatorname{coker}(\Pi^+ D_{A.\Phi})$,

then $\dim\{(*), \langle\varphi.\Phi\rangle = 0\} = \dim\{(*)\} - \underset{\overset{\|}{1}}{\underline{\operatorname{ind}\Pi}}$ $\rightsquigarrow$ DONE.

$$\Pi\, D^{*}_{A,\Phi}\, \Pi^{+} \begin{pmatrix}\xi\\ \gamma\\ \psi\end{pmatrix} = \underbrace{D^{*}_{A,\Phi}\, \Pi^{+} \begin{pmatrix}\xi\\ \gamma\\ \psi\end{pmatrix}}_{\begin{gathered}\shortparallel\\ \begin{pmatrix} * \\ D_A^{*}\psi - \frac{1}{2}(\sigma^{+})^{-1}(\gamma-\pi^{+}\gamma)\Phi \end{pmatrix}\end{gathered}} + \frac{1}{2}\,\Pi \begin{pmatrix} 0 \\ (\sigma^{+})^{-1}(\gamma-\pi^{+}\gamma)\,\Phi \end{pmatrix}$$

(

formal adjoint

$$\mathrm{End}(W^{+})^{0} \xrightarrow[\sim]{\sigma^{+}} \Lambda^{2,+}T_X^{*} \otimes \mathbb{C}$$

$$\langle D_A^{*}\psi, \Phi\rangle - \langle \psi, D_A\Phi\rangle = 0$$

estimate: $\|(\xi,\gamma,\psi)\|_{1,\lambda} := \|d\xi\|^2_{L^2} + \lambda^2\|\pi^{+}\gamma\|^2_{L^2} + \|d^{*}\gamma\|^2_{L^2} + \|\psi\|^2_{W^{1,2}}$

$$\lesssim \|\underbrace{D^{*}_{A,\Phi}(\xi,\gamma,\psi)}_{\begin{gathered}\shortparallel\\ (\alpha,\varphi)\end{gathered}}\|_{0,\lambda} := \frac{1}{\lambda^2}\|\pi(\alpha)\|^2_{L^2} + \|\alpha-\pi\alpha\|^2_{L^2} + \|\varphi\|^2_{L^2}$$

$(\pi: \Omega^1(X, i\mathbb{R}) \to H^1(X, i\mathbb{R}))$

where $\lambda = \|\Phi\|_{L^2} = \sqrt{\delta}$ small.

Indeed, $$\|(\alpha,\varphi)\|^2_{0,\lambda} = \frac{1}{\lambda^2}\|\pi(i\langle\psi, i\Gamma(-)\Phi\rangle)\|^2$$
$$+ \|d\xi + d^{*}\gamma + (\mathrm{id}-\pi)(i\langle\psi, i\Gamma(-)\Phi\rangle)\|^2$$
$$+ \|D_A^{*}\psi - \frac{1}{2}(\sigma^{+})^{-1}(\gamma)\Phi\|^2$$

(using $(a+b)^2 \geq \frac{a^2}{2} - b^2$ and $\|\mathrm{id}-\pi\| \leq \|\mathrm{id}\|$)

$$\geq \underline{\|\pi(i\langle\psi, i\Gamma(-)\lambda^{-1}\Phi\rangle)\|^2}$$
$$+ \frac{1}{2}\|d\xi + d^{*}\gamma\|^2 - \|i\langle\psi, i\Gamma(-)\Phi\rangle\|^2$$
$$+ \underline{\frac{1}{2}\|D_A^{*}\psi - \frac{1}{2}(\sigma^{+})^{-1}(\pi^{+}\gamma)\Phi\|^2} - \frac{1}{4}\|(\sigma^{+})^{-1}(\gamma-\pi^{+}\gamma)\Phi\|^2 \quad (\star)$$

η reg. $\Rightarrow$ $$P_{A,\Phi}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix} \pi^{+}\sigma^{+}((\varphi\Phi^{*} + \Phi\varphi^{*})_0) \\ D_A\varphi + \Gamma(\alpha)\Phi \end{pmatrix}$$ is surj. $\forall \Phi$.

$$P_{A,\Phi}: H^1(i\mathbb{R}) \oplus W^{1,2}(W^{+}) \to H^{2,+}(i\mathbb{R}) \oplus L^2(W^{-})$$

$$\rightsquigarrow \|\underbrace{P^{*}_{A,\Phi}(\gamma,\psi)}_{\begin{gathered}\shortparallel\\ (\pi(i\langle\psi, i\Gamma(-)\Phi\rangle),\ D_A^{*}\psi - \frac{1}{2}(\sigma^{+})^{-1}(\gamma)\Phi)\end{gathered}}\|_{L^2} \gtrsim \|(\gamma,\psi)\|_{W^{1,2}}$$

small

$$\text{So } (\star) \geq \frac{1}{2}\underline{\|P^{*}_{A,\lambda^{-1}\Phi}(\lambda\pi^{+}\gamma,\psi)\|^2} + \frac{1}{2}\|d\xi + d^{*}\gamma\|^2 - \left(\|\psi\|^2_{L^4} + \frac{1}{4}\|\gamma-\pi^{+}\gamma\|^2_{L^4}\right)\overset{\downarrow}{\|\Phi\|_{L^4}}$$

$$\gtrsim \lambda^2\|\pi^{+}\gamma\|^2 + \|\psi\|^2_{W^{1,2}} + \|d\xi\|^2 + \|d^{*}\gamma\|^2 - \underset{\uparrow}{c}\left(\|\psi\|^2_{W^{1,2}} + \|d^{*}\gamma\|^2\right)\|\Phi\|_{L^4}$$

Sobolev embedding.

$$\gtrsim \|(\xi,\gamma,\psi)\|_{1,\lambda} \quad \text{for } \Phi \text{ small.}$$

Then $\| \Pi D^*_{A.\Phi} \Pi^+(\xi.\gamma.\psi) - D^*_{A.\Phi} \Pi^+(\xi.\gamma.\psi) \|_{0.\lambda}$

Sobler ↓

$$\lesssim \| (\sigma^+)^{-1}(\gamma - \pi^+\gamma)\Phi \|^2_{L^2} \leq \|\gamma - \pi^+\gamma\|^2_{L^4} \|\Phi\|^2_{L^4} \lesssim \|d^*\gamma\|^2_{L^2} \|\Phi\|^2_{L^4}$$

$$\lesssim \| \Pi^+(\xi.w.\psi)\|_{1.\lambda} \|\Phi\|^2_{L^4} \lesssim \| D^*_{A.\Phi} \Pi^+(\xi.\gamma.\psi)\|_{0.\lambda} \cdot \|\Phi\|^2_{L^4} .$$

$\rightsquigarrow \quad \| \Pi^+(\xi, w.\psi)\|_{1.\lambda} \lesssim \| \Pi D^*_{A.\Phi} \Pi^+(\xi.\gamma.\psi)\|_{0.\lambda}$ $\square$.

So $M_\delta(\{\eta_t\})$ is a cpt mfd with boundary $M(\eta_\varepsilon) - M(\eta_{-\varepsilon}) - M_\delta(\eta)$.

$$M_\delta(\eta) = \{(A, \Phi, t) \in \mathfrak{M}(\{\eta_t\}) \mid \|\Phi\|^2_{L^2} = \delta\} / \mathcal{G}_0$$

Lemma. $M_\delta(\eta)$ is cobordant to $M^0_\delta(\eta) = \{(A,\Phi) \mid A \in \tilde{T}, D_A\Phi = 0, \|\Phi\|^2_{L^2} = \delta\} / \mathcal{G}_0$

pf. $t = \langle \underbrace{F_A^+ + \eta}_{\parallel} + itw , iw \rangle = \langle \sigma^+((\Phi\Phi^*)_0), iw \rangle$

$F_A^+ - F_{A_0}^+ \in \operatorname{Im} d^+$ $(\Omega^{2,+} = H^{2,+} \oplus \operatorname{Im} d^+$ by Hodge theory$)$

$$\text{So SW-eq.} \Leftrightarrow \begin{cases} d^*(A - A_0) = 0 \\ F_A^+ + \eta = (id - \pi^+)\sigma^+((\Phi\Phi^*)_0) \\ D_A\Phi = 0 \end{cases}$$

$$M^0_\delta(\eta) \sim M_\delta(\eta) \text{ is now given by } M^s_\delta(\eta) := \text{sol. of } \begin{cases} d^*(A - A_0) = 0 \\ F^A + \eta = s\,(id - \pi^+)\sigma^+((\Phi\Phi^*)_0) \\ D_A\Phi = 0 \\ \|\Phi\|^2 = \delta \end{cases}$$

$\square$

Now, $w(X.\Gamma) = \int_{M(\eta_\varepsilon) - M(\eta_{-\varepsilon})} c_1(L)^d = \int_{M_\delta(\eta)} c_1(L)^d = \int_{M^0_\delta(\eta)} c_1(L)^d$, $d = \operatorname{ind} D_A + b_1 - 2$

Lemma. Let $D : E \to F$ be a complex Fredholm operator / cpt $2n$-dim mfd M.

Suppose $D|_{SE} : SE \to F$ is transversal to the zero section.

↖ sphere bundle

Then $\int_{M(D)} c_1(L)^d = \int_M c_n(-\operatorname{IND}(D))$, L : tautological line bundle

where $M(D) = \{(x.\xi) \in SE \mid D_x\xi = 0\} / S^1 \subseteq \mathbb{P}E$, $d = \operatorname{ind} D_x + n - 1 \geq 0$

↑ L

pf. Take $E_1 \subseteq E$ fin. codim. s.t. $D|_{E_1}$ is inj. $\rightsquigarrow$ $E = E_0 \oplus E_1$

$F_1 = DE_1 \subseteq F$ also fin codim $\rightsquigarrow$ $F = F_0 \oplus F_1$

$$\rightsquigarrow D = \begin{pmatrix} D_{00} & D_{01} \\ 0 & D_{11} \end{pmatrix}$$

After a perturbation, may assume $D_{00}|_{SE_0} \pitchfork 0|_{SE_0}$ in F_0

$$\rightsquigarrow \underset{D_{00}+D_{11} : E \to F}{\underset{\shortparallel}{D'}}|_{SE} \pitchfork 0|_{SE} \text{ in } F$$

Take $K_t : E \to F$ s.t. $K_{t,x}$ is a cpt operator $\forall t,x$, and

$K_0 = 0$, $K_1 = D_{01} : E_1 \to F_0$

since D_{11} is bij

$\rightsquigarrow$ cobordism $M(D) \sim M(D') \sim M(D_{00})$

So may assume E, F are fin. dim bundles.

Choose $\underset{\mathrm{rk}E - \mathrm{rk}F + n - 1}{\underset{\shortparallel}{d}}$ sections $\varphi_1 \cdots \varphi_d \in C^\infty(E^\vee)$ s.t. $D' = D \oplus \varphi_1 \oplus \cdots \oplus \varphi_d : E \to F \oplus \mathbb{C}^d$ has reg. crossings.

$\rightsquigarrow$ $\underset{s_0, s_1, \dots, s_N}{\underset{\shortparallel}{D''}} : \underset{\mathbb{C}^{N+1}}{\underset{\wr\shortparallel}{E \oplus E'}} \to F \oplus \mathbb{C}^d \oplus E'$, may assume $s_1 \dots s_N$ are lin indep

$\rightsquigarrow F' = \langle s_1, \dots, s_N \rangle \subseteq F \oplus \mathbb{C}^d \oplus E'$

$$\rightsquigarrow \int_{M(D)} c_1(L)^d = \# M(D'') = \#(\bar{s}_0 \pitchfork F/F') = \int_M c_n(F \oplus \mathbb{C}^d \oplus E'/F')$$

$$= \int_M c_n(F \ominus E) = \int_M c_n(-IND(D)) \qquad \square$$

So $w(X,\Gamma) = \int_{M^0_\delta(\eta)} c_1(L)^d = \int_T c_k(-IND) = \int_T \frac{1}{k!}\left(-\frac{1}{4}\int_X \Omega^2 \wedge L_p\right)^k$ ■

Def. A ruled surface is a cpt smooth mfd $X \to \underset{\substack{\uparrow \\ \text{Riemann surface}}}{\Sigma}$ with $\mathbb{CP}^1$ fiber.

A symplectic 4-fold X is minimal if $\nexists\, S^2 \underset{\substack{\uparrow \\ \text{symplectic sub.}}}{\subset} X$ with self intersection $(i_*[S^2])^2 = -1$.

Thm. (Taubes) Let $E \to X$ be a cpx l.b. with $SW^+(X, \Gamma_E) \neq 0$.

Then $\exists\, C \underset{\substack{\uparrow \\ \text{symplectic sub.}}}{\subseteq} X$ represents $c_1(E)$.

Every conn. comp. $C_i \subseteq C$ satisfies $K \cdot C_i \leq g(C_i) - 1 \leq C_i^2$. $(*)$

Sketch of pf of $(*)$:

Adjunction formula $\Rightarrow$ $2g(C_i) - 2 = C_i^2 + K \cdot C_i$

$\dim \mathcal{M}(X, [C_i]) = C_i^2 - K \cdot C_i \geq 0$

$$\Rightarrow C_i^2 = \frac{(C_i^2 + K\cdot C_i) + (C_i^2 - K \cdot C_i)}{2} \geq \frac{2g(C_i) - 2}{2} \geq \frac{(C_i^2 + K \cdot C_i) - (C_i^2 - K\cdot C_i)}{2} = K \cdot C_i$$

Cor. Let $E, E' \to X$ be cpx lb.'s with $SW^+(X, \Gamma_E), SW^+(X, \Gamma_{E'}) \neq 0$.

Then $E \cdot E' \geq 0$. (So, if $b^+ \geq 2$, then $K^2 \geq 0$)

pf. Write $c_1(E) = \sum [C_i]$, $c_1(E') = \sum [C_j']$. with $\begin{cases} g(C_i) - 1 \leq C_i^2 \\ g(C_j') - 1 \leq (C_j')^2 \end{cases}$

X min. $\Rightarrow$ $C_i^2 \geq 0$. $(C_j')^2 \geq 0$.

So $C_i \cdot C_j' \geq 0$ $\forall i, j$. $\Rightarrow$ $E \cdot E' = \sum_{i,j} C_i \cdot C_j' \geq 0$ $\square$

Thm. Let X be a minimal symplectic 4-fold with $K^2 < 0$. Then X is a ruled surface.
($\Rightarrow b^+ = 1$ by Cor)

Thm. (McDuff) If $\exists\, S^2 \simeq C \subseteq X$ with self-intersection number $[C]^2 \geq 0$,

then X is rational or ruled.

Lemma. If $\exists$ l.b. $E \to X$ s.t. $SW^+(X, \Gamma_E) \neq 0$, $E^2 + K \cdot E < 0$.

then X is rational or ruled.

pf. By Taube's thm, $c_1(E) = \sum_i [C_i]$.

Adjunction $\Rightarrow$ $2g(C_i) - 2 = \underset{\substack{\Pi \\ E \cdot C_i}}{C_i^2} + K \cdot C_i$

$$\Rightarrow \sum_i (2g(C_i) - 2) = (E+K) \cdot E < 0 \Rightarrow C_i \simeq S^2 \text{ for some } i.$$

$K \cdot C_i < 0$, X minimal $\Rightarrow$ $C_i^2 \geq 0$ $\Rightarrow$ X rat. or ruled by the thm. $\square$

Since $K^2 < 0$, this shows that $SW^+(X, \Gamma_K) = SW^-(X, \Gamma_{can}) = 0$.

For any $e \in H^2(X, \mathbb{Z})$, let $w(e) = \int_T \frac{1}{k!}\left(\int_X \Omega^2 \wedge (K - 2e)\right)^k$, $k = \frac{b_1}{2}$.

$= w(X, \Gamma_E)$ if $c_1(E) = e$

$\rightsquigarrow$ $w(0) = SW^+(X, \Gamma_{can}) = 1$.

a line ($a.b \in \mathcal{H} \Rightarrow a^2, b^2 . ab = 0 \Rightarrow a // b$ since $b^+ = -1$)

Consider the set $\mathcal{H} := H^1(X, \mathbb{Z}) \cup H^1(X, \mathbb{Z}) \subseteq H^2(X, \mathbb{Z})$,

which contains non-torsion elements (by $w(0) \neq 0$) when $k \neq 0$.

Claim. $\exists\ a \in H^2(X, \mathbb{Z})$ s.t.

(i) $a^2 = 0$, $K \cdot a < 0$, (ii) $\underset{\substack{\shortparallel \\ (pK - qa)^2 - K \cdot (pK - qa) = (p^2 - p)K^2 + q(2p-1)(-K \cdot a)}}{\underline{\dim M(X, \Gamma_{pK-qa})}} \geq 0 \Rightarrow w(pK - qa) \neq 0$.

If $k = 0$, then (ii) is always true and (i) is easy.

If $k > 0$, take $\underset{\substack{\shortparallel \\ 0}}{a} \in \mathcal{H}$ $\Rightarrow$ $\Omega^2 \wedge a = 0$, $a^2 = 0$ by the sublemma above

Then $w(pK - qa) = \int_T \frac{1}{k!}\left(\underbrace{\int_X \Omega^2 \wedge (1-2p)K}_{\Downarrow}\right)^k = (1-2p)^k \underset{\substack{\shortparallel \\ 1}}{w(0)} \neq 0$.

$K \cdot a \neq 0 \Rightarrow$ may assume $K \cdot a < 0$

Now, $w(e) = SW^+(X,\Gamma_e) + (-1)^k SW^+(X,\Gamma_{K-e}) \neq 0 \Rightarrow SW^+(X,\Gamma_e) \neq 0$

or $SW^+(X,\Gamma_{K-e}) \neq 0$

Goal: find p.q s.t. $SW^+(X,\Gamma_{\underset{pK-qa}{\underset{\shortparallel}{e}}}) \neq 0$, $e^2+K\cdot e<0$ ($\Rightarrow$ DONE by lemma.)

Assume $\nexists$ such $(p.q)$.

$e = a \Rightarrow \dim M = -K\cdot a > 0 \Rightarrow w(a) \neq 0$, $a^2+K\cdot a<0$

$\Rightarrow SW^+(X,\Gamma_{K-a}) \neq 0 \Rightarrow (K-a)^2 + K\cdot(K-a) = 2K^2 - 3K\cdot a \geq 0$

Let $s = -K^2 > 0$, $t = -K\cdot a > 0$, $\lambda = \frac{s}{t} \leq \frac{3}{2}$

The condition $e^2 + K\cdot e < 0$, $\dim M = e^2 - K\cdot e \geq 0$

$$\Leftrightarrow \begin{cases} -sp^2 + 2tpq - sp + tq < 0 \\ -sp^2 + 2tpq + sp - tq \geq 0 \end{cases} \Leftrightarrow \underset{f(p)}{\underset{\shortparallel}{\frac{2p+1}{p^2+p}}} < \frac{\lambda}{q} \leq \underset{f(p-1)}{\underset{\shortparallel}{\frac{2p-1}{p^2-p}}} \quad (\infty \text{ if } p=1)$$

Since $f(p)$ is strictly dec. (in p), $\forall q \geq 1$. $\exists!\, p \geq 2$ s.t. the condition holds.

$\Rightarrow SW^+(X,\Gamma_{K-e}) \neq 0 \Rightarrow \underset{(p-1)s-(p+q-1)t}{\underset{\shortparallel}{(K-a)\cdot(K-e)}} \geq 0$ by Cor.

$\Rightarrow \lambda \geq \frac{p-1}{p+q-1} \Rightarrow \frac{2p-1}{p^2-p} \geq \frac{p+q-1}{(p-1)q} \Rightarrow 2pq - q \geq p^2 + pq - p$

$\Rightarrow (p-q)(1-p) \geq 0 \Rightarrow q \geq p$

But $p \geq 2 \Rightarrow \frac{5}{3} \leq \frac{2p+1}{p+1} < \frac{\lambda}{q} p \leq \frac{3}{2}\cdot\frac{p}{q} \Rightarrow \frac{p}{q} > \frac{10}{9}$ $\times$.

Thm. For X minimal symplectic, TFAE:

(i) X admits a positive scalar curvature metric

(ii) X admits a symplectic structure ω with $K\cdot[\omega]<0$

(iii) X is rational or ruled.

(iii) $\Rightarrow$ (i), (ii) : $X=\mathbb{CP}^2$: take Fubini-Study metric

X ruled ⇝ take standard metric on $\left\{\begin{array}{l}\mathbb{F}_n\\ \mathbb{C}\times\mathbb{CP}^1/\Lambda\\ \mathbb{H}\times\mathbb{CP}^1/\Gamma \quad (\text{with } \mathbb{CP}^1 \text{ small})\end{array}\right.$

Thm. Suppose (X,ω) is a symplectic 4-fold with $b^+=1$.

Then $SW^+(X,\Gamma_{can})=1$, and $\forall E$, $SW^+(X,\Gamma_E)\neq 0 \Rightarrow E\cdot[\omega]\geq 0$

("$=$" $\Leftrightarrow$ $E=\mathbb{C}$)

Sketch of pf. Suppose $SW^+(X,E)\neq 0$

Then $\varepsilon_{\Gamma_E}(\underbrace{\pi\lambda\omega}_{\eta}) = \lambda\int_X \omega\wedge\omega + \pi(K-2E)\cdot[\omega] > 0 \Rightarrow M(X,\Gamma,\eta)\neq\emptyset$

for λ large.

$\Rightarrow \exists$ sol (B,φ_0,φ_2) of
$$\left\{\begin{array}{l}\bar\partial_B\varphi_0+\bar\partial_B^*\varphi_2=0\\ 2F_B^{0,2}=\bar\varphi_0\varphi_2\\ 4i(F_B)_\omega=4\pi\lambda+|\varphi_2|^2-|\varphi_0|^2\end{array}\right.$$
$\Leftrightarrow D_A\Phi=0$

$\Leftrightarrow F_{A_{can}}+F_B+\eta=\sigma^+((\Phi\Phi^*)_0)$

$A=A_{can}+B\in\mathcal{A}(\Gamma_E)$

$\Phi=\varphi_0+\varphi_2\in\Omega^{0,0}(E)\oplus\Omega^{0,2}(E)$

So $0=\|\bar\partial_B\varphi_0+\bar\partial_B^*\varphi_2\|^2=\|\bar\partial_B\varphi_0\|^2+\|\bar\partial_B^*\varphi_2\|^2+2\langle\underbrace{\bar\partial_B^2\varphi_0}_{F_B^{0,2}\varphi_0-\frac{1}{4}(\partial_B\varphi_0)\circ N_J},\varphi_2\rangle$

$=2\|\bar\partial_B\varphi_0\|^2+\|\bar\varphi_0\varphi_2\|^2-\frac{1}{2}\langle(\partial_B\varphi_0)\circ N_J,\varphi_2\rangle$

$2\|\bar\partial_B\varphi_0\|^2=\langle\underbrace{\bar\partial_B^*\bar\partial_B\varphi_0}_{d_B^*d_B\varphi_0-2i(F_B)_\omega\varphi_0},\varphi_0\rangle=\|d_B\varphi_0\|^2-8\pi^2\lambda\cdot E\cdot[\omega]+2\pi\lambda\|\varphi_2\|^2$

$+\frac{2}{\lambda}\int_X(4\pi\lambda-|\varphi_0|^2)^2-\frac{1}{2}\|\bar\varphi_0\varphi_2\|^2.$

$\frac{1}{2}\langle(\partial_B\varphi_0)\circ N_J,\varphi_2\rangle\leq\varepsilon\|d_B\varphi_0\|^2+\frac{c}{\varepsilon}\|\varphi_2\|^2$

take λ large

⇝ $(1-\varepsilon)\|d_B\varphi_0\|^2+(2\pi\lambda-\frac{c}{\varepsilon})\|\varphi_2\|^2+(\text{positive})\leq 8\pi^2\lambda\cdot E\cdot[\omega]$ ⇝ $E\cdot[\omega]\geq 0$

(i) ⇒ (ii) : positive scalar curvature ⇒ $\mathcal{M}(X,\Gamma_{can},\eta) = \emptyset$ for small η.

$SW^+(X,\Gamma_{can}) \neq 0 \Rightarrow b^+ = 1$ and $\mathcal{E}_\Gamma(0) = \pi\cdot[\omega]\cdot K < 0 \Rightarrow \underset{\underset{H^2(X)_{\text{free}}}{\cap}}{\bar{K}} \neq 0$.

If $K^2 < 0$, then X is ruled ⇒ (iii) ⇒ (ii)

If $K^2 \geq 0$, then $\bar{K} = \underset{\underset{\text{any metric } g.}{\uparrow}}{(K\cdot[\omega_g])}[\omega_g] + \underset{\underset{H^{2,-}(X)}{\cap}}{K^-} \Rightarrow K\cdot[\omega_g] \neq 0$ (otherwise $K^2 = (K^-)^2 < 0$)

(ii) ⇒ (iii) : May assume $K^2 \geq 0$.

$\|$

$2\chi + 3\tau = 4 - 4b^1 + 2b^+ + 2b^- + 3b^+ - 3b^- = 9 - 4b^1 - b^-$.

$\rightsquigarrow b^1 = 0$ or $b^1 = 2$, $b^- = 0, 1$.

Case 1. $b^1 = 0$, Q_X odd

$\Rightarrow \exists\, \alpha, \beta_1 \dots \beta_m \in H^2 \quad Q_X = \begin{pmatrix} 1 & \\ & -I_m \end{pmatrix} \quad m \leq 9$.

$K = \lambda\alpha + \sum \mu_i\beta_i$, $[\omega] \overset{\overset{\text{scaling}}{\downarrow}}{=} \alpha + \sum \varepsilon_i\beta_i$

(⇒ λ, μ_i odd by adjunction) (⇒ $\sum \varepsilon_i^2 < 1$)

$K^2 = \lambda^2 - \sum\mu_i^2 = 9 - m \Rightarrow \lambda^2 \geq 9$

If $\lambda > 0$, then $K\cdot[\omega] = \lambda - \sum\mu_i\varepsilon_i < 0 \Rightarrow \lambda^2 < (\sum\mu_i\varepsilon_i)^2$

$\leq (\sum\mu_i^2)\cdot(\sum\varepsilon_i^2)$

$< \sum\mu_i^2 = \lambda^2 + m - 9$ $\quad \times$

So $\lambda \leq -3$.

Take $c_1(E) = \alpha \Rightarrow SW^-(X,\Gamma_\alpha) = \pm SW^+(X,\Gamma_{K-\alpha}) = 0$

$\underline{(K-\alpha)\cdot[\omega] < 0}$ (1)

$\dim \mathcal{M}(\Gamma_\alpha) = \underline{\alpha^2 - K\cdot\alpha > 0}$ (2) $\Rightarrow SW^+(X,\Gamma_\alpha) = 1 \overset{\text{Lemma}}{\Rightarrow} X$ rat. or ruled.

$k = \frac{b_1}{2} = 0$ $\quad 1 + \lambda = \underline{\alpha^2 + K\cdot\alpha < 0}$ (3)

Case 2. $b^1 = 0$, Q_X even ⇒ $Q_X = H$ or $H \oplus E_8$

$Q_X = H = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \Rightarrow K = \lambda\alpha + \mu\beta$, $[\omega] = \alpha + \varepsilon\beta$. $\quad \lambda, \mu < 0$ even, $\varepsilon > 0$ ⇒ (1), (2), (3)

$Q_X = H \oplus E_8 \Rightarrow K = \lambda\alpha + \mu\beta + \sum \nu_i\gamma_i$. $\quad \lambda, \mu, \nu_i$ even $\quad K^2 = 0 \Rightarrow \lambda\mu > 0$.

$[\omega] = \alpha + \varepsilon\beta + \sum \delta_j\gamma_j$ $\quad [\omega]^2 > 0 \Rightarrow \varepsilon > 0$.

$$\left(\sum \nu_i \delta_j \langle \gamma_i . \gamma_j \rangle\right)^2 \leq (K - \lambda\alpha - \mu\beta)^2 \cdot ([w] - \alpha - \varepsilon\beta)^2$$

$$= (2\lambda\mu) \cdot (2\varepsilon) \leq (\varepsilon|\lambda| + |\mu|)^2$$

So $K \cdot [w] = \lambda\varepsilon + \mu + \sum \nu_i \delta_j \langle \gamma_i . \gamma_j \rangle < 0 \Rightarrow \lambda . \mu < 0$.

$\Rightarrow (K-\alpha)\cdot[w] = K\cdot[w] < 0$, (1) $\quad \alpha^2 - K\cdot\alpha = -\mu > 0$. (2) $\quad \alpha^2 + K\cdot\alpha = \mu < 0$ (3)

Case 3. $b' = 2$. Q_X odd

$K\cdot[w] < 0 \Rightarrow SW^-(X, \Gamma_{can}) = SW^+(X, \Gamma_k) = 0 \Rightarrow w \neq 0$.

$\Rightarrow \exists \underset{\neq 0}{\gamma} \in H^2$ with $\gamma^2 = 0 \Rightarrow b^- > 0 \Rightarrow Q_X = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$K = \lambda\alpha + \lambda\beta \qquad\qquad \lambda < 0$

$\Rightarrow$

$[w] = \alpha + \varepsilon\beta$ may assume $|\varepsilon| < 1$

$\because w(0) = 1$, $\alpha + \beta \notin \mathcal{H} = H' \cup H' \Rightarrow e = \alpha - \beta \in \mathcal{H}$

$\Rightarrow w(e) = \int_T \int_X \Omega^2 \wedge (k - 2e) = w(0) = 1$

$k = \frac{b_1}{2} = 1$

Also. (1) $(K - e)\cdot[w] = (1-\varepsilon)\lambda - (1+\varepsilon) < 0$

(2) $e^2 - K\cdot e = -2\lambda > 0$

(3) $e^2 + K\cdot e = 2\lambda < 0$

Case 4. $b' = 2$, Q_X even $\Rightarrow Q_X = H = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$(K^2 = 0)$

$K = \lambda\alpha \qquad\qquad \lambda < 0$

$\Rightarrow$

$[w] = \alpha + \varepsilon\beta \qquad\qquad \varepsilon > 0$

$w(0) = 1 \Rightarrow \alpha \notin \mathcal{H} \Rightarrow \beta \in \mathcal{H} \Rightarrow w(\beta) = w(0) = 1$

Also. (1) $(K - \beta)\cdot[w] = \lambda\varepsilon - 1 < 0$

(2) $\beta^2 - K\cdot\beta = -\lambda > 0$

(3) $\beta^2 + K\cdot\beta = \lambda < 0$

$\square$