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Date April 9, 1998

TOPOLOGY OF BIRATIONAL MANIFOLDS AND APPLICATIONS TO DEGENERATIONS

A thesis presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the subject of

Mathematics

Harvard University
Cambridge, Massachusetts

March 1998

Contents

Acknowledgement

I am very grateful to my advisor, Shing-Tung Yau, for his steady encouragement during my five years of graduate study. His strong enthusiasm and wide scope in mathematics has a great impact on me. The characteristic style and the atmosphere that he has created in his "Student Seminar" gave me many wonderful memories.

I have benefitted from conversation with Bo Cui, Bing Cheng, A. Johan de Jong, Wee Teck Gan, Mark Gross, Brendan Hassett and Kang Zuo. I also received invaluable friendship from Mutao Wang at one decisive point of my graduate study. I would like to take this opportunity to thank all of them.

I am especially indebted to Ching-Li Chai and Wee Teck Gan for sharing their insights on the problems that I have studied and presented here. Several inaccuracies were pointed out and useful comments were provided by them. I am also grateful to Raoul Bott, Hélène Esnault, Benedict H. Gross, Joe Harris, Barry Mazur, Wilfried Schmid, Eckart Viehweg and Jing Yu for their interest in my work. The extremely comfortable environment provided by the Mathematics Department has been determinantal for my study.

Last but not the least, I want to thank my mother Feng Cheng and my wife Hui-Wen Lin for their tremendous love and constant support. Without their tolerance and devotion, I could have never gone this far.

Abstract

The main theme in this thesis is to prove the invariance of Betti numbers of smooth complex projective varieties under certain birational correspondences and to discuss its applications to degeneration problems of smooth minimal models.

There are two parts of it. In the first part, it is shown that if $f\colon X \dashrightarrow X'$ is a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci, then X and X' have the same Betti numbers. In particular, birational smooth minimal models have the same Betti numbers.

The main idea is to use the Weil conjecture. To proceed, we first observe that the whole problem is reduced to the p-adic case, and then use Weil's formula to identify the number of rational points with certain p-adic integral. The next key point is to show that the canonical bundles become equivalent after pulled back to a common resolution of the given birational map. Putting this information into the p-adic integrals of both varieties shows that they have the same Jacobian factor in the change of variable formula, hence settles the theorem.

In the second part, it is shown that for a degeneration of three dimensional smooth minimal models acquiring nontrivial terminal singularities, the punctured family can not be completed into a smooth projective family. Since there are examples such that the monodromy is trivial in the C^∞ sense, this gives a negative answer to the so called "filling in" problem in dimension three.

The proof makes use of various results developed in the Mori theory. The key lemma in the first part and the main result mentioned above also play a very important role here. This degeneration problem is motivated by the study of the Weil-Petersson metric on the moduli spaces of Calabi-Yau manifolds. In fact, we propose the equivalence between incomplete boundary points and degenerations of Calabi-Yau manifolds acquiring at most canonical singularities.

This thesis was written under the supervision of Professor Shing-Tung Yau.

Introduction

I would like to describe personal reflections of my past five years of graduate study, to recall how those problems I was dealing with came to my mind and to explain how they were solved. But overall, I need to first say something about minimal models, which is the main subject that attracted me for most of the time. And perhaps, I hope, that I finally got some feeling of it.

The concept of minimal models goes back to the Italian algebraic geometers. It plays a decesive role in the classification theory of algebraic surfaces and is also fundamental in many applications. However, its range of applications are not extended to higher dimensions until S. Mori's fundamental work on the structure of rational curves appeared in the late 70's.

During the last two decades, the minimal model theory has been extensively developed by S. Mori, M. Reid, Y. Kawamata, E. Viehweg, V. Shokurov, J. Kollár and many others. It becomes clear that it forms an important reduction step in the study of higher dimensional algebraic geometry. One of their most significant achievement is that many important conjectures in dimension three were thus solved.

What is a minimal model? It is a birational model with numerically effective canonical divisors and with at most terminal singularities (perhaps with some factoriality assumption). This could make sense only when one glances at Mori's cone theorem, and its extension by Kawamata, Shokurov and Kollár to the singular case. Basically, it says that if the canonical bundle is not nef, then the variety admits further contractions. There are serious problems to continue this process due to the wild singularities that one may encounter after contractions. This was finally resolved by Mori in 1988 by proving the existence of flips, hence settled the existence of minimal models in dimension three.

So far, the existence problem is completely open in higher dimensions, but even worse, the minimal model is not unique except in dimension two. It is then important to see what kind of invariants are shared by those birational minimal models. More generally, we would like to know how certain topological invariants change under certain elementary birational transformations. The first main result in this thesis provides the answer for the Betti numbers:

Theorem A. *Let $f\colon X \dashrightarrow X'$ be a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci, then X and X' have the same Betti numbers. In particular, birational smooth minimal models have the same Betti numbers.*

This type of problem was first studied by Kollár in the case of threefolds about ten years ago, by refining Kawamata's result on three dimensional flops. His method is basically geometrical. In fact he obtained a complete understanding of birational maps between three dimensional minimal models — they are composed by a sequence of flops and he has a clear local picture of flops. Namely he established, among other things, the invariance of singularities, cohomologies and intersection cohomologies under flops in dimension three (cf. 5.1). In this way, Theorem A only generalizes the "Betti number" statement to arbitrary dimensions, and is still under the very restricted smoothness assumption.

However, there are some interesting immedeate consequences of Theorem A. One of them is that the exceptional loci of the given birational map also share the same Betti numbers (Corollary 4.5). This is obtained by applying the Mayer-Vietoris argument to the birational correspondence we may construct via H. Hironaka's theorem on the resolution of singularities, and then make use of Theorem A. In fact, in all the examples known to the author, the exceptional loci are actually birational to each other componentwise! But we have no proof of this.

The proof of Theorem A is based on some general considerations in birational geometry and Grothendieck-Deligne's solution to the Weil conjecture [D1, D2]. The bridge to connect these two is the theory of p-adic integrals. Essentially, all of the algebro-geometric results we need were well developed in the 80's. And all the arithmetic results we need were done even earlier. Moreover, instead of the details, we even just need the statements existed in the literatures! It seems that all we need to do is to put them together and to see what happens. However, this is the step that people seemed to ignore. The real intention of this research is an attemption to combine these two theory together. From this point of view, we seem to have a very good start.

I would like to say some words about the development on this problem. In fact, even the idea to use the Weil conjecture via p-adic integrals to compute cohomologies is not new. It has to be dated back to the 70's to the works of G. Harder and M.S. Narasimhan [HN], although it was used there in a somewhat different way. p-adic integrals were also studied extensively in the context of Igusa-Weil local zeta functions by J. Denef and F. Loeser since late 80's [Ig, DF1]. Recently this approach was taken up again by Batyrev, and he first established Theorem A in the special case of projective Calabi-Yau manifolds. In his case, essentially no minimal model theory needs to be involved. At that time, an even more striking result to me appeared, that was D. Huybrechts' stronger statement about Hyper-Kähler manifolds [Hu] (cf. 5.2).

These results were made famous, at least to me, because it was used by Beauville in explaining Yau-Zaslow's formula on the number of rational curves on K3 surfaces. Although only hearing this development oversea, from the previous experience in the minimal model theory, notably the abundance conjecture, I then soon convinced myself that the same result must hold true for general minimal models. I cooked out the first version of Theorem A in October 1997 under the further assumption that the canonical bundle is semi-ample (with a help from C.-L. Chai). It is an argument based on p-adic integrals and birational correspondences. At about the same time, Batyrev's proof in the Calabi-Yau case appeared on the network [Ba] where his "measure theoretic" argument came to my mind.

By extending these developments further, I then realized that our original argument based on birational correspondences in fact works equally well without the semi-ampleness assumption. The key point is that the assumption can even be localized to the exceptional loci. This leads to the concept of "K-partial ordering", which is introduced in §1 and is closely related to interesting geometric situations arising from the minimal model theory. The applicability of the Weil conjecture is largely clarified in terms of this notion (cf. Proposition 2.16 and Theorem 3.1). Moreover, this approach also provides a natural setting in the singular case.

I have tried to develop this, together with the p-adic measure, as far as possible so that it could fit the need of the minimal model theory. In fact, an easy but very interesting fact observed here is that the integral points of a p-adic variety has finite p-adic measure if and only if it has at most terminal singularities (Proposition 2.12). This give me the belief that p-adic integrals fit naturally into the framework of minimal model theory. But due to technical reasons, I have restricted myself to the smooth case when I state and prove Theorem A. (See however 2.17 and 5.3 for more about the singular case.)

I have to point out at least two aspects that Theorem A is still unsatisfactory, the torsion elements are not considered and no natural maps between cohomologies has been even mentioned. Although there is one obvious candidate for this map, the cohomology correspondence induced from the birational correspondence, it is not clear how to show directly that it induces isomorphisms. In fact, there is no strong evidence why this should be true.

In the simplest cases, we can show that smooth minimal models minimize $H^2(X, \mathbf{Z})$ compatible with the Hodge structure among birational smooth projective varieties. And in the singular case, at least we know that minimal models minimize the group of Weil divisors among birational projective varieties with at most terminal singularities. The proof is elementary (does not use the Weil conjecture) and is contained in §4 together with some related results. In fact, it is simply another application of the notion of K-partial ordering. Nevertheless, it is worth pointing out that in stating both theorems, what we have in mind is that there should be a "minimal cohomology theory" among birational varieties. Moreover, it should be realized exactly by the minimal models.

* * * * *

Turn to its applications — a basic question in the analysis of boundary point of moduli spaces is the problem about degenerations. There are many powerful tools developed in this area, from the traditional Picard-Lefschetz theory to the modern theory entitled with the name "variations of Hodge structures". However, there are some questions seem to be beyond the scope of Hodge theory. One is the so called "filling in problem".

This is concerned about a degenerating family of smooth projective varieties over the disk such that the punctured family is smoothly equivalent to a trivial product. The question is whether this punctured family can be completed into a smooth analytic family. Negative answer to this question is well known in the curve theory, however, it is mainly due to the presence of nontrivial fundamental groups. So it is natural to consider only simply connected varieties. In this setup, V. Kulikov's classification theorem on semi-stable degenerations of K3 surfaces [Ku] (in the late 70's) provided the first important class of examples that the filling in problem has a positive answer.

In the 80's, R. Friedman [F1] and J. Morgan [Mo] had also studies these kind of questions. A negative answer has thus been obtained by them for certain degenerating families of surfaces of general type. From this, they also constructed negative examples for dimensions at least four. But at that moment, people did not know how to answer this question for a given specific family with finite order monodromy, even for the simplest examples – even dimensional nodal degenerations studied in the Picard-Lefschetz theory. The nonfilliability of this was finally proved by C. Voisin in 1990 [Vo].

In his survey paper on Calabi-Yau threefolds [F4], Friedman remarked that for families of quintic hypersurfaces acqriring an A_2 singularity, the monodromy has finite order inside the mapping class group. He also expected that the filling in problem has a negative answer for any finite base change. This question caught my interest for three reasons. One, the fiber dimension is three, which belongs to the unknown zone of the existing list of examples. Two, the singularity is so simple. And more importantly, it is Calabi-Yau, a "natural candidate" for K3 surfaces in three dimensios, and we already know a positive answer for K3's (sounds like a paradox)!

The second main result of this thesis is to provide a general theorem on terminal degenerations, which in particular answers the filling in problem in negative.

Theorem B. *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a Gorenstein 3-fold $\mathcal{X}_0$ with nontrivial terminal singularities and with $K_{\mathcal{X}_0}$ nef. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$ with $\mathcal{X}_t \cong \mathcal{X}'_t$ for $t \neq 0$.*

From this point of view, the above mentioned paradox is simply that there are no terminal singularities in dimension two! Also not a surprise, the proof uses many technical results in the three dimensional minimal model theory. Friedman's study on simultaneous resolution of threefold double point [F3] is also fundamental to the proof. And notably, Theorem A is used in an essential step. However, we need to make use of its strong form obtained by Kollár mentioned above, because from our Theorem A, we don't know whether the smoothness is preserved between birational **Q**-factorial minimal models. Nevertheless, we still expect that further investigation will lead to interesting applications of Theorem A in higher dimensional geometry.

In fact, Theorem B was obtained in 1995, two years before the proof of Theorem A was found. The most exciting thing to me is that in both theorems, the most technical step (to me) is the same! This is what I called the "Key Lemma" in §1. I spent several months in obtaining this lemma when I tried to prove Theorem B. At the end, I found out that a weaker form of it was already in the literature, namely Kollár's paper [Ko]! The remaining step for me is just to generalize it and fortunately this could be done without too much difficulity.

Theorem B is closely related to the study of the Weil-Petersson geometry of Calabi-Yau moduli spaces. This is the original problem that Professor Yau gave me. My original motivation to prove Theorem B is to provide "essential" metric incomplete boundary point of the moduli space of Calabi-Yau threefolds. §9 is devoted to this aspect. Needless to say, all of these are somehow related to the study of "Mirror Symmetry" phenomenon.

This article is concluded with certain speculations related to E. Viehweg's program on the quasi-projectivity of certain moduli spaces and with a question on finite distance degenerations of Calabi-Yau manifolds. The central object in this circle of ideas is an understanding of canonical singularities — as has been introduced to us by M. Reid more than twenty years ago. In fact, it is this concept, together with Mori's cone theorem and the Kawamata-Viehweg vanishing theorem, that gave the way of the whole development of the minimal model theory started in the early 80's!

Chapter One — Birational Invariants

§1 Birational Geometry

We begin with some standard definitions. For a complete treatment of minimal model theory, the reader should consult [KMM].

Let X be an n dimensional complex normal $\mathbf{Q}$-Gorenstein variety. That is, the canonical divisor K_X is $\mathbf{Q}$-Cartier. Recall that X has (at most) terminal (resp. canonical, resp. log-terminal) singularities if there is a resolution $\phi : Y \to X$ such that in the canonical bundle relation

$$K_Y =_{\mathbf{Q}} \phi^* K_X + \sum a_i E_i, \tag{1.1}$$

we have that $a_i > 0$ (resp. $a_i \geq 0$, resp. $a_i > -1$) for all i. Here, the E_i's vary among the prime components of all the exceptional divisors. Although (1.1) holds only up to $\mathbf{Q}$-linear equivalence, the divisor $\sum a_i E_i \in Z_{n-1} \otimes \mathbf{Q}$ is uniquely determined. Moreover, the condition on a_i's is readily seen to be independent of the chosen resolution. It is also elementary to see that smooth points are all terminal.

Let Z be a proper subvariety of X. A $\mathbf{Q}$-Cartier divisor D is called numerically effective (nef) along Z if $D.C := \deg_{\tilde{C}}(f^* D) \geq 0$ for all effective curves $C \subset Z$, where $f\colon \tilde{C} \to C$ is the normalization of C. And D is simply called nef if $Z = X$. A projective variety X is called a minimal model if X is terminal and K_X is nef.

Two normal varieties X and X' are birational if they have isomorphic function fields $K(X) \cong K(X')$ (over $\mathbf{C}$). Geometrically, this means that there is a rational map $f\colon X \dashrightarrow X'$ such that f^{-1} is also rational. The exceptional loci of f are defined to be the smallest subvarieties $Z \subset X$ and $Z' \subset X'$ such that f induces an isomorphism $X - Z \cong X' - Z'$.

Among the class of birational $\mathbf{Q}$-Gorenstein varieties, We have the notion of **K-partial ordering** (where the "K" is for canonical divisors):

Definition 1.2. For two $\mathbf{Q}$-Gorenstein varieties X and X', we say that $X \leq_K X'$ (resp. $X <_K X'$) if there is a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$ with Y smooth, such that $\phi^* K_X \leq_{\mathbf{Q}} \phi'^* K_{X'}$ (resp. "$<_{\mathbf{Q}}$"). Moreover, "$X \leq_K X'$" plus "$X \geq_K X'$" implies that "$X =_K X'$", ie. $\phi^* K_X =_{\mathbf{Q}} \phi'^* K_{X'}$. In this case, we say that X and X' are K-equivalent.

The well-definedness of this notion follows from the canonical bundle relations

$$K_Y =_{\mathbf{Q}} \phi^* K_X + E =_{\mathbf{Q}} \phi'^* K_{X'} + E', \tag{1.3}$$

since we know that $X \le_K X'$ if and only if $E \ge E'$. In the terminal case, this means that ϕ has more exceptional divisors than ϕ' (so heuristically, X is "smaller" than X').

Here is the typical geometric situation that we can compare their K-partial order:

Key Lemma 1.4. *Let $f\colon X \dashrightarrow X'$ be a birational map between two varieties with canonical singularities. Suppose that the exceptional locus $Z \subset X$ is proper and that K_X is nef along Z, then $X \le_K X'$. Moreover, if X' is terminal, then Z has codimension at least two.*

Proof. Let $\phi : Y \to X$ and $\phi' : Y \to X'$ be a good common resolution of singularities of f so that the union of the exceptional set of ϕ and ϕ' is a normal crossing divisor of Y. This can be done by considering $\bar{\Gamma}_f \subset X \times X'$, the closure of the graph of f, blowing up the exceptional set of $\bar{\Gamma}_f \to X$ and $\bar{\Gamma}_f \to X'$ and then taking Y to be a Hironaka (embedded) resolution [Hi].

Consider the canonical bundle relations:

$$\begin{aligned} K_Y &=_{\mathbf{Q}} \phi^* K_X + E \equiv \phi^* K_X + F + G \\ &=_{\mathbf{Q}} \phi'^* K_{X'} + E' \equiv \phi'^* K_{X'} + F' + G'. \end{aligned} \tag{1.5}$$

Here F and F' denote the sum of divisors (with coefficients ≥ 0) which are both ϕ and ϕ' exceptional. G (resp. G') denotes the part which is ϕ exceptional but not ϕ' exceptional (resp. ϕ' but not ϕ exceptional). Notice that $\phi(G') \subset Z$.

To proceed, we write

$$\phi'^* K_{X'} =_{\mathbf{Q}} \phi^* K_X + G + (F - F' - G'). \tag{1.6}$$

It is enough to prove that $F - F' - G' \ge 0$, because this implies that $F - F' \ge 0$ and $G' = 0$, and so $E \ge E'$.

By taking a generic hyperplane section H of Y $n-2$ times, the problem is reduced to a problem on surfaces. Namely

$$H^{n-2}.\phi'^* K_{X'} =_{\mathbf{Q}} H^{n-2}.\phi^* K_X + \zeta + (\xi - \xi' - \zeta'), \tag{1.7}$$

where $\xi = H^{n-2}.F$ and $\zeta = H^{n-2}.G$ etc. If $\xi - \xi' - \zeta'$ is not effective, write it as $H^{n-2}.(A - B) = a - b$ with A and B effective. Then by taking the intersection of (1.7) with b, we get

$$B.H^{n-2}.\phi'^* K_{X'} =_{\mathbf{Q}} B.H^{n-2}.\phi^* K_X + b.\zeta + b.a - b^2. \tag{1.8}$$

The left hand side is always zero since B is ϕ' exceptional. Moreover, if $B \subset F'$ then $B.H^{n-2}.\phi^* K_X = 0$ too. If $B \subset G'$ then the curve $\phi(B.H^{n-2}) \subset \phi(G') \subset Z$ is inside the exceptional locus. So the first three terms in the right hand side are non-negative since K_X is nef along Z and a, b and ζ are different components. However, since b is a nontrivial combination of ϕ' exceptional curves in H^{n-2}, we have from the Hodge index theorem for surfaces that $b^2 < 0$, a contradiction! Hence $F - F' - G' \geq 0$.

For the second statement, from the construction of Y, we know that all components of the exceptional sets, denoted by $\operatorname{Exc}\phi$ and $\operatorname{Exc}\phi'$ respectively, are divisors. If X' is assumed to be terminal, then all ϕ' exceptional divisors occur as components of E'. So $G' = 0$ implies that $\operatorname{Exc}\phi' \subset \operatorname{Exc}\phi$. With this understood, from

$$X - \phi(\operatorname{Exc}\phi) \cong Y - \operatorname{Exc}\phi \cong X' - \phi'(\operatorname{Exc}\phi) \subset X' - \phi'(\operatorname{Exc}\phi'), \tag{1.9}$$

we conclude that $Z \subset \phi(\operatorname{Exc}\phi)$ is of codimension at least two. Q.E.D.

Corollary 1.10. *Let $f\colon X \dashrightarrow X'$ be a birational map between two varieties with at most canonical singularities such that K_X (resp. $K_{X'}$) is nef along the exceptional locus $Z \subset X$ (resp. $Z' \subset X'$), then $X =_K X'$. Moreover, f extends to an isomorphism in codimension one if X and X' are terminal. This applies, in particular, if both X and X' are minimal models.*

Variant 1.11. Instead of assuming that the exceptional locus in X is proper, one can generalize Key Lemma 1.4 to the relative case, namely f is a S-birational map and that $X \to S$ and $X' \to S$ are proper S-schemes. The proof is identical to the one given above by changing notation.

Remark 1.12. This type of argument is familiar in the minimal model theory. Notably, in analyzing the log-flip diagram (eg. [KMM; 5-1-11]) or more specially, the flops. Key Lemma 1.4 implies that if X' is a flip of X, then $X \geq_K X'$ (in fact, more is true: $X >_K X'$). Corollary 1.10 implies that flop induces K-equivalence. Since flip/flop will not be used in any essential way in this paper, we will refer the interested reader to [KMM] for the definitions. The proof given above is inspired by Kollár's treatment of flops in [K1].

§2 The Weil Conjecture and p-adic Integrals

To prove Theorem A, we will show that X and X' have the same number of rational points over certain finite fields when a suitable good reduction is taken. That is, we prove that they have the same "zeta function". The theorem will then follow from the statement of the Weil conjecture.

2.1. The reduction procedure. This is standard in algebraic geometry and in number theory: as long as we perform only a finite number of "algebraic constructions" in the complex case, e.g. consider morphisms, since all the objects involved can by defined by a finite number of polynomials, we can take $S \subset \mathbf{C}$ a finitely generated subring over $\mathbf{Z}$ so that everything is defined over S. S has the property that the residue field S/m of any maximal ideal $m \subset S$ is finite.

If we start with "smooth objects", general reduction theory then says that for an infinite number of "good primes" (in fact, Zariski dense in $\mathrm{Spec}\,(S)$), we may get good reductions so that everything is defined smoothly over the finite residue field $\mathbf{F}_q$ with $q = p^r$ for some prime number p. We may also assume that this reduction has a lifting such that everything is defined smoothly over R, the maximal compact subring of a p-adic local field $\mathbf{K}$, i.e. a finite extension field of $\mathbf{Q}_p$, with residue field $\mathbf{F}_q$.

More precisely, let F be the quotient field of S. Based on the fact (and others) that $\mathbf{Q}_p$ has infinite transcendence degree, the "embedding theorem" (see for example [Ca; p.82]) says that for an infinite number of p's, there is an embedding of fields $i : F \to \mathbf{Q}_p$ such that $i(S) \subset \mathbf{Z}_p$. Moreover, i may be chosen so that a prescribed finite subset of S, say the coefficients of those defining polynomials, is mapped into the set of p-adic units. This embedding then gives the desired lifting.

Let P be the unique maximal ideal of R (so $R/P \cong \mathbf{F}_q$). We denote by $\bar{X}$, $\bar{U}, \ldots$ those objects constructed from X, $U \ldots$ via reductions mod P. That is, objects lie over the point $\mathrm{Spec}\, R/P \to \mathrm{Spec}\, R$ — they are defined over $\mathbf{F}_q$. We also denote the reduction map by $\pi : X(R) \to \bar{X}(\mathbf{F}_q)$ etc.

2.2. The Weil conjecture. Let $\bar{X}$ be a variety defined over a finite field $\mathbf{F}_q$. After fixing an algebraic closure, the Weil zeta function of $\bar{X}$ is defined by

$$Z(\bar{X}, t) := \exp\left(\sum\nolimits_{k \geq 1} |\bar{X}(\mathbf{F}_{q^k})| \frac{t^k}{k} \right). \tag{2.3}$$

In 1949, Weil conjectured several nice properties of this zeta function for smooth projective varieties and expalined how some of these would follow once a suitable cohomology theory exists [W1]. This lead Grothendieck to his creation of étale cohomology theory.

More precisely, Grothendieck proved a "Lefschetz fixed point formula" in a very general context (eg. constructible sheaves over seperated schems of finite type ...) [D2], which in particular implies that the zeta function is a rational function:

$$Z(\bar{X}, t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots P_{2n}(t)}, \tag{2.4}$$

where $P_j(t)$ is a polynomial with integer coefficients such that $P_j(0) = 1$ and $\deg P_j(t) = h^j$, the j-th Betti number of compactly supported ℓ-adic étale cohomologies (for a prime $\ell \neq p$). Moreover, when $\bar{X}$ comes from a good reduction of a smooth complex projective variety X in the sense described in (2.1), h^j coincides with the j-th Betti number of the singular cohomologies of $X(\mathbf{C})$.

Deligne [D1] completed the proof of the Weil conjecture by proving the important "Riemann Hypothesis" that all roots of $P_j(t)$ have absolute value $q^{-j/2}$. In particular, the complete information about the $\mathbf{F}_{q^k}$-rational points determines the h^j's and all the roots.

2.5. Counting points via p-adic integrals. How do we count $\bar{X}(\mathbf{F}_q)$? If $\bar{X}$ comes from the good reduction of a smooth R-scheme, we will see that such a counting can be achieved by using p-adic integrals (cf. Theorem 2.8). We will first recall some elementary aspects of the p-adic integral over $\mathbf{K}$-analytic manifolds and over R-schemes.

Consider the Haar measure on the locally compact field $\mathbf{K}$, normalized so that the compact open "disk" R has volume 1:

$$\int_R |dz| = 1. \tag{2.6}$$

We may extend this to the multivariable case and define the p-adic integral of any regular n form $\Psi = \psi(z_1, \cdots, z_n) dz_1 \wedge \cdots \wedge dz_n$ by

$$\int_{R^n} |\Psi| := \int_{R^n} |\psi(z)||dz_1 \wedge \cdots \wedge dz_n|. \tag{2.7}$$

Here $|a| := q^{-\nu_p(\mathrm{N}_{\mathbf{K}/\mathbf{Q}_p}(a))}$ is the usual p-adic norm.

We may define an integral slightly more general than (2.7): suppose that Ψ is a r-pluricanonical form such that in local analytic coordinates we have

$\Psi = \psi(z_1, \cdots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes r}$. We define the integration of a "r-th root of $|\Psi|$" by

$$(2.7') \qquad \int_{R^n} |\Psi|^{1/r} := \int_{R^n} |\psi(z)|^{1/r} |dz_1 \wedge \cdots \wedge dz_n|.$$

This is independent of the choice of coordinates, as can be checked easily by the same method as in [W2; p.14]. So we can extend the definition to (not necessarily complete) **K**-analytic manifolds with Ψ a (possibly meromorphic) pluricanonical form. Certainly then the integral defined may not be finite.

The key property we need is the following (slightly more general form of a) formula of Weil [W2; 2.2.5]. We briefly sketch its proof.

Theorem 2.8. *Let U be a smooth R-scheme and Ω a nowhere zero r-pluricanonical form on U, then*

$$\int_{U(R)} |\Omega|^{1/r} = \frac{|\bar{U}(\mathbf{F}_q)|}{q^n}.$$

Proof. The proof given by Weil in [W2] goes through without difficulties — one first observes that the reduction map $\pi: U(R) \to \bar{U}(\mathbf{F}_q)$ induces an isomorphism between $\pi^{-1}(\bar{t})$ and PR^n for any $\bar{t} \in \bar{U}(\mathbf{F}_q)$ (Hensel's lemma) such that there is a function ψ with $|\psi(z)| = 1$ and

$$(2.9) \qquad \Omega = \psi(z) \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes r}$$

in the **K**-analytic chart PR^n. This implies that $\int_{\pi^{-1}(\bar{t})} |\Omega|^{1/r} = 1/q^n$ for any $\bar{t} \in \bar{U}(\mathbf{F}_q)$. Summing over $\bar{t}$ then gives the result. Q.E.D.

The right hand side of (2.8) shows that the integral is independent of the choice of the form Ω. One may also see this by observing that any two such forms differ by a nowhere vanishing function on U (over R) which takes values in the units on all R-points. This allows one to define a canonical p-adic measure on the R-points of smooth R-schemes by "gluing" the local integrals. We will define it in the singular case with the hope that it may be useful for later development.

2.10. Canonical measure on Q-Gorenstein R-schemes. We will only consider those R-schemes, eg. X, that come from complex **Q**-Gorenstein varieties as in (2.1). Let $r \in \mathbf{N}$ such that rK_X is Cartier (locally free). We may assume that we have a R-resolution of singularities $\phi: Y \to X$, which is a projective R-morphism, so that the reduced part of the exceptional set is a simple normal crossing R-variety. We will define a measure on $X(R)$ such that the measurable sets are exactly the compact open subsets in the **K**-analytic topology.

Let U_i's be a Zariski open cover of X such that $rK_X|_{U_i}$ is actually free. Then for a compact open subset $S \subset U_i(R) \subset X(R)$, we define its measure by

$$m_X(S) \equiv \int_S |\Omega_i|^{1/r} := \int_{\phi^{-1}(S)} |\phi^*\Omega_i|^{1/r}, \tag{2.11}$$

where Ω_i is an arbitrary generator of $rK_X|_{U_i}$. Notice that the properness of ϕ implies that $\phi^{-1}(S) \subset Y(R)$. This allows us to operate the integral entirely on R-points.

For general compact open $S \subset X(R)$, we may break S into disjoint pieces S_j so that S_j is contained in some $U_i(R)$ (in fact, S_j may be chosen to lie entirely in a fiber of the reduction map π), and then define $m_X(S) = \sum_i m_X(S_i)$. Notice that $m_X(S)$ is again independent of the choice of U_i, Ω_i and Y.

The following proposition explains the possible connection between the canonical measure and the minimal model theory:

Proposition 2.12. *For a* **Q***-Gorenstein R-variety X, $X(R)$ has finite measure if and only if X has at most log-terminal singularities.*

Proof. Consider the canonical bundle relation for $\phi: Y \to X$

$$rK_Y = \phi^* rK_X + \sum_i e_i E_i \tag{2.13}$$

with rK_X being Cartier and $e_i \in \mathbf{Z}$. To determine the finiteness of $m_X(X(R))$, we only need to consider those R-points on the exceptional fibers. Locally, $\operatorname{div} \phi^*\Omega = \sum_i e_i E_i$ for a generator Ω of rK_X. So the integral is a product of one dimensional integrals of the form

$$I_i := \int_R |z^{e_i}\, dz^{\otimes r}|^{1/r} = \int_R |z|^{e_i/r}\, |dz|. \tag{2.14}$$

If this is finite, then

$$I_i = \int_{PR} |z|^{e_i/r}\, |dz| + (q-1)\frac{1}{q} = q^{-(e_i/r+1)} I_i + \frac{q-1}{q}. \tag{2.15}$$

Since $I_i > 0$, this makes sense only if $q^{e_i/r+1} > 1$. That is, $e_i/r > -1$, which is exactly the definition of log-terminal singularities. Q.E.D.

Since the measure is defined Zariski-locally via p-adic integrals, for smooth X, we have from Weil's formula (2.8) that:

Corollary 2.16. *Let X be an n-dimensional smooth R-variety with finite residue field $\mathbf{F}_q$, then*

$$m_X(X(R)) = \frac{|\bar{X}(\mathbf{F}_q)|}{q^n}.$$

Remark 2.17. If X is singular, $m_X((X(R))$ is a weighted counting of the rational points. By definition, if $\phi: Y \to X$ is a crepant R-morphism, ie. $K_Y =_{\mathbf{Q}} \phi^* K_X$, then $m_X((X(R)) = m_Y((Y(R))$. In particular, $m_X((X(R))$ counts the rational points of $\bar{Y}$ if Y is smooth! This applies to many interesting "pure canonical" singularities and to terminal singularities having small resolutions. However, further investigation on the precise "geometric meaning" of this weighted counting is still needed for the general case (cf. 5.3).

§3 The Proof of Theorem A

We will in fact prove a result which connects the notion of K-partial ordering and the canonical measure. This will largely clarify the role played by the Weil conjecture.

Theorem 3.1. *Let X and X' be two birational log-terminal R-varieties. Then $m_X(X(R)) \leq m_{X'}(X'(R))$ if $X \leq_K X'$. In particular, K-equivalence implies measure equivalence.*

Proof. Consider as before, a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$ over R with Y a smooth R-variety. Let $r \in \mathbf{N}$ be such that both rK_X and $rK_{X'}$ are Cartier. Then $X \leq_K X'$ if and only if in the canonical bundle relations $rK_Y = \phi^* rK_X + E = \phi'^* rK_{X'} + E'$, we have $E \geq E'$.

From the properness of ϕ and ϕ', we have that $\phi^{-1}(X(R)) = Y(R) = \phi'^{-1}(X'(R))$. So from the definition of the measure (2.11), it suffices to show that for any compact open subset $T \subset Y(R)$ with $\pi(T)$ a single point $\bar{y} \in \bar{Y}(\mathbf{F}_q)$, we have

$$\int_T |\phi^*\Omega|^{1/r} \leq \int_T |\phi'^*\Omega'|^{1/r}. \tag{3.2}$$

Here Ω is an arbitrary local generator of rK_X on a Zariski open set U where rK_X is actually free and such that $\bar{\phi}(\bar{y}) \in \bar{U}$ (and with similar conditions for Ω').

Clearly, (3.2) can fail to be an equality only if $\bar{y} \in \bar{E} \cup \bar{E}'$. However, in this case $E \geq E'$ says that the order of $\phi^*\Omega$ is no less than that of $\phi^*\Omega$. (3.2) then follows from the definition of the p-adic integral (2.7′) (see also (2.15)). Q.E.D.

If X and X' are smooth, combining this with (2.16) gives

Corollary 3.3. *Let X and X' be two birational smooth R-schemes. Then $|\bar{X}(\mathbf{F}_q)| \leq |\bar{X}'(\mathbf{F}_q)|$ if $X \leq_K X'$.*

With this done, by working on cyclotomic extensions of $\mathbf{K}$, the same proof shows that $|\bar{X}(\mathbf{F}_{q^k})| \leq |\bar{X}'(\mathbf{F}_{q^k})|$ for all $k \in \mathbf{N}$. In particular, $Z(\bar{X}, t) \leq Z(\bar{X}', t)$ for all $t > 0$. The same is true for all the derivatives, but it is not clear how to make use of these. The simplest application is given by:

Corollary 3.4. *Let X and X' be two birational complex smooth varieties. They have the same Euler number for the compactly supported cohomologies if $X =_K X'$.*

Proof. Apply the reduction procedure (2.1) to reduce this to the p-adic case. The statement then follows from Grothendieck's Lefschetz fixed point formula (2.4) and the above comparison of zeta functions. Q.E.D.

So far we have not used Deligne's theorem on the "Riemann Hypothesis". To use it, we need to impose the projective assumption.

Theorem 3.5. *Let X and X' be two birational smooth projective R-schemes. If $X =_K X'$ then $m_X(X(R)) = m_{X'}(X'(R))$. This is equivalent to $Z(\bar{X}, t) = Z(\bar{X}', t)$. In particular, they have the same "Betti numbers" by the Weil conjecture.*

Now we may come back to our original geometric situation:

Theorem A. *Let $f\colon X \cdot\cdot\to X'$ be a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci, then X and X' have the same Betti numbers. In particular, birational smooth minimal models have the same Betti numbers.*

Proof. By Corollary 1.10, X and X' are K-equivalent. So Theorem A simply follows from the reduction procedure (2.1) and Theorem 3.5. Q.E.D.

Remark 3.6. In the preliminary version of this paper (dated October 1997), Theorem A was stated with the assumption that the canonical bundle is semi-ample, that is, rK_X is generated by global sections for some $r \in \mathbf{N}$. The proof proceeds by cutting out the pluri-canonical divisors and applying p-adic integrals to the birational correspondence, where the notion of K-equivalence is essential for this step to work.

By using Weil's formula (2.8), the proof is then concluded by induction on dimensions. In this approach, the usage of integration of a r-th root of the absolute value of a pluricanonical form was suggested to the author by C.-L. Chai in order to deal with the case that $r > 1$. Happily enough, as the author realized later, the semi-ample assumption can be removed once we observed that the problem can even be localized to the exceptional loci.

Remark 3.7. The equivalence of zeta functions is a stronger statement than the equivalence of Betti numbers. Moreover, we have in fact established the equivalence of zeta functions for a dense set of primes. From the theory of motives, this suggests that we may in fact have the equivalence of Hodge structures. Further investigation in this should be interesting and important.

Question 3.8. Is Theorem A true for Kähler manifolds?

§4 Miscellaneous Results

Now we come back to the complex number field and begin with an elementary observation:

Lemma 4.1. *If the exceptional loci of a birational map $f\colon X \cdots\to X'$ between two smooth projective varieties have codimension at least two then for $i \le 2$ we have $\pi_i(X) \cong \pi_i(X')$ and $H^i(X,\mathbf{Z}) \cong H^i(X',\mathbf{Z})$ which is compatible with the rational Hodge structures.*

Proof. The real codimension four statement plus the transversality argument shows that $\pi_i(X) \cong \pi_i(X')$, $H_i(X,\mathbf{Z}) \cong H_i(X',\mathbf{Z})$ and $H^i(X,\mathbf{Z}) \cong H^i(X',\mathbf{Z})$ canonically for $i \le 2$. Moreover, by Hartog's extension we know that the Hodge groups $H^0(\Omega^i)$ are all birational invariants among smooth varieties. The orthogonality of Hodge filtrations then shows that $H^i(X,\mathbf{Q})$ and $H^i(X',\mathbf{Q})$ share the same rational Hodge structures for $i \le 2$. Q.E.D.

A slightly deeper result is given by

Proposition 4.2. *If the exceptional loci $Z \subset X$ and $Z' \subset X'$ of a birational map f between two smooth varieties have codimension at least two, then $h^i(X) - h^i(Z) = h^i(X') - h^i(Z')$.*

Proof. Construct a birational correpondence $X \leftarrow Y \rightarrow X$ as in §1 and denote the exceptional divisor of $\phi\colon Y \to X$ (resp. $\phi'\colon Y \to X'$) by E (resp. E'). Since Hironaka's resolution process only blows up smooth centers inside the singular set of the graph of f, the isomorphism $X - Z \cong X' - Z'$ implies that $\phi(E \cup E') \subset Z$ and $\phi'(E \cup E') \subset Z'$, hence that $E_{\rm red} = E'_{\rm red}$, $Z = \phi(E)$ and $Z' = \phi'(E')$.

Consider an open cover $\{V, W\}$ of X by letting $V := X - Z$ and $W \supset Z$ be a deformation retract neighborhood. Let $\tilde V := \phi^{-1}(V)$ and $\tilde W := \phi^{-1}(W) \supset E$ be the corresponding open cover of Y. Then we have the following commutative diagram of integral cohomologies

$$(4.3)\qquad \begin{array}{ccccccc} H^{i-1}(\tilde V \cap \tilde W) & \to & H^i(Y) & \to & H^i(\tilde V) \oplus H^i(E) & \to & H^i(\tilde V \cap \tilde W) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^{i-1}(V \cap W) & \to & H^i(X) & \to & H^i(V) \oplus H^i(Z) & \to & H^i(V \cap W) \end{array}$$

It is a general fact that $\phi^*\colon H^i(X) \to H^i(Y)$ is injective (by the projection formula, that ϕ is proper of degree one implies that $\phi_! \circ \phi^*(a) = a$ for all $a \in H^i(X)$). Since $\tilde V \cong V$ and $\tilde V \cap \tilde W \cong V \cap W$, simple diagram chasing shows that $H^i(Z) \to H^i(E)$ is also injective. We may then break (4.3) into short exact sequences

$$(4.4)\qquad 0 \to \phi^* H^i(X) \to H^i(Y) \to H^i(E)/\phi^* H^i(Z) \to 0.$$

Similarly, we have for $\phi' : Y \to X'$:

$$(4.4')\qquad 0 \to \phi'^* H^i(X') \to H^i(Y) \to H^i(E')/\phi'^* H^i(Z') \to 0.$$

Since $E_{\rm red} = E'_{\rm red}$, the proposition follows immedeately. Q.E.D.

Combining this with Theorem A gives

Corollary 4.5. *Let $f : X \dashrightarrow X'$ be a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci, then the exceptional loci also have the same Betti numbers. In particular, this applies to birational smooth minimal models.*

Remark 4.6. The proof of Theorem A in fact also shows that $\bar{Z}$ and $\bar{Z}'$ have the number of $\mathbf{F}_q$-rational points. This is simply because $|\bar{X}(\mathbf{F}_q)| = |\bar{X}'(\mathbf{F}_q)|$ and $\bar{X} - \bar{Z} \cong \bar{X}' - \bar{Z}'$. In particular, if Z and Z' are smooth then they have the same Betti numbers. Although this argument apparently only works for smooth Z and Z', which is very restricted, it is more than just a special case of (4.5) — since it carries certain nontrivial arithmetic information.

We are now in a position to show that minimal models are really minimal in the sense of cohomologies:

Theorem 4.7. *Smooth minimal models minimize $H^2(X, \mathbf{Z})$ compatible with the Hodge structure among birational smooth projective varieties. In the singular case, the minimal models minimize the group of Weil divisors among birational projective varieties with at most terminal singularities.*

Proof. Let $f : X \dashrightarrow X'$ be a birational map between two n dimensional smooth projective varieties where only X is assumed to be minimal. In the notation of §1, Key Lemma 1.4 says that $E \geq E'$. So we obtain canonical morphisms $H^i(E) \to H^i(E')$ induced from $E' \subset E$. Since $Z := \phi(E)$ and $Z' := \phi'(E')$ are of codimension at least two, $H^{2n-2}(Z) = 0 = H^{2n-2}(Z')$. By comparing (4.4) and (4.4′) via the surjective map $H^{2n-2}(E) \to H^{2n-2}(E')$, we obtain a canonical embedding:

$$(4.8)\qquad \phi^* H^{2n-2}(X, \mathbf{Z}) \subset \phi'^* H^{2n-2}(X', \mathbf{Z}).$$

which respects the Hodge structures. This induces an injective map

$$(4.9)\qquad \phi'_! \circ \phi^* : H^{2n-2}(X, \mathbf{Z}) \to H^{2n-2}(X', \mathbf{Z}),$$

which by the projection formula is easily seen to be independent of the choices of Y, hence canonical. Poincaré duality then concludes the first statement of 4.7.

For the second statement, we may simply copy the above proof by replacing (4.4) with the similar formula for the Weil divisors. Q.E.D.

One can also interpret this result in terms of the Picard group if the terminal varieties considered are assumed to be factorial or $\mathbf{Q}$-factorial.

§5 Further Comments

We conclude this chapter with two historical remarks and three technical remarks:

5.1. Birational geometry. A version of Key Lemma 1.4, or rather the Corollary 1.10, was used before by Kollár in his study of three dimensional flops. In fact, he proved that three dimensional birational $\mathbf{Q}$-factorial minimal models all share the same singularities, singular cohomologies and intersection cohomologies with pure Hodge structures (via deep results due to Saito). See [K1] for the details.

More recently, the author used a relative version of (1.10), namely variant 1.11, to study degenerations of minimal projective threefolds [W; §4] and obtained a negative answer to the so called "filling-in problem" in dimension three. This result is now included in chapter two with some refinement of the original proof.

5.2. Previous results. After Kollár's result on threefolds, the problem on the equivalence of Betti numbers seemed to be ignored for a while until recently when Batyrev treated the case of projective Calabi-Yau manifolds [Ba].

In the special case of projective hyper-Kähler manifolds, Theorem A has also been proved recently by Huybrechts [Hu] using quite different methods. In fact, he proved more — these manifolds are all inseparable points in the moduli space (hence are diffeomorphic and share the same Hodge structures)!

This problem on general minimal models, to the best of the author's knowledge, has not been studied until the present work. In our case, the homotopy types will generally be different. In fact, it is well known that for a single elementary transform of threefolds, although the singular cohomologies are canonically identified, the cup product must change. However, inspired by Kollár's result and Remark 3.7, we still expect that the (non-polarized) Hodge structures will turn out to be the same.

5.3. Singular case. In order to generalize Theorem A to the singular case, our approach works equally well in the log-terminal case, with the only problem being that we need a good interpretation like Weil's formula (2.8) for the precise meaning of the weighted counting, which is the key to relate p-adic integrals to the Weil conjecture.

Since a suitable version of the Weil conjecture for singular varieties has already been proved by Deligne in [BBD] in terms of the intersection cohomologies introduced by Goresky and MacPherson [GM], this problem is thus reduced to the calculation of local Lefschetz numbers.

More precisely, one needs to evaluate the p-adic integrals over a singular point and to reconstruct the "constructible complexes of sheaves" which it may correspond to. If luckily enough, it is the intersection cohomology complexes, then we may get our conclusion again via Deligne's theorem. A detailed discussion on this will be continued in a subsequent paper.

5.4. Minimal cohomology. For Theorem 4.7, it is likely that a similar argument works for proving that terminal minimal models also minimize the second intersection cohomology groups and that they all share the same pure Hodge structures. The important injectivity of $\phi^* : IH^i(X) \to IH^i(Y)$ needed to conclude (4.4) is now a consequence of the so called "decomposition theorem" of projective morphisms. ([BBD] again!)

An interesting question arises: is the Picard number (or the second Betti number) of a non-minimal model always strictly bigger than the one attained by the minimal models?

Mazur raised the following question: can one extract the expected "minimal cohomology piece" directly from any smooth model without refering to the minimal models?

5.5. Recent development. We first notice that the proof of Theorem A can be formally seperated into three parts:

1. Geometric situations lead to the conclusion of K-equivalence. This is done Theorem 1.4, or Corollary 1.10. In particular, this applies to birational minimal models.
2. A reasonable integration/measure theory attached to a variety. Here we deal with p-adic integrals, or equivalently, the number of rational points in the case of smooth varieties. Theorem 3.1 shows that K-equivalence implies measure equivalent. In the notation used there, E and E' are exactly the Jacobian factor occuring in the changing of variables formula from X and X' to Y respectively.
3. Topological/geometrical interpretation of the integral. In our case, this corresponds to Grothendieck-Deligne's solution to the Weil conjecture.

We can then formulate a meta theorem via the above steps by considering more general integrals.

Recently, based on an idea of Kontsevich, Denef and Loeser [DL2] has constructed a motivic integration on the space of arcs of an algebraic variety, which generalizes the p-adic integral. Using this new integration theory in step 2 and Deligne's theorem on the existence of functorial mixed Hodge structures on compactly supported cohomologies of algebraic varieties in step 3, Theorem A can be strengthened to the statement that X and X' also have the same Hodge numbers. Moreover, the usage of motivic integration allows much better understanding of the exceptional loci. However, like the case of p-adic integrals, the topological meaning of the full measure in the singular case is still not well understood.

After the present work was completed, their preprint [DL2] and then the preprint version of this chapter became avaliable in the network. Afterwards, the above implication was also observed and pointed out to the author by Loeser. Since their construction of motivic integration is quite delicate, we will not try to say anything about it here. The interested reader is referred to [DL2] for the details of this wonderful theory.

Chapter Two — Filling in Problem in Dimension Three

§6 Degenerations with Trivial Monodromy

6.1. Degenerations and monodromies. We are interested in the case of a degeneration $\mathcal{X} \to \Delta$ of polarized Kähler n-folds. By this we mean that $\mathcal{X}$ is a Kähler $(n+1)$-fold and $\mathcal{X} \to \Delta$ is a proper flat holomorphic map with the general fiber $\mathcal{X}_t$, $t \neq 0$, a smooth Kähler n-fold. Notice that the resulting family over the punctured disk has a polarization (a locally constant Kähler class) induced from the Kähler form on $\mathcal{X}$.

In general, $\mathcal{X} \to \Delta$ is called a degeneration of certain type if $\mathcal{X}_0$ has only singularities of that type. And by "$\mathcal{X} \to \Delta$ is a smoothing of $\mathcal{X}_0$", we will mean that $\mathcal{X} \to \Delta$ is a proper flat family with smooth $\mathcal{X}_t$ for $t \neq 0$ but without assuming the complex space $\mathcal{X}$ to be smooth. A degeneration $\mathcal{X} \to \Delta$ is called semi-stable if $\mathcal{X}_0$ is a reduced divisor with normal crossings in $\mathcal{X}$. By a theorem of Mumford, every degeneration has a semi-stable reduction by a sequence of blow-ups and base-changes.

The diffeomorphism type of the punctured family $\mathcal{X}^\times \to \Delta^\times$ depends only on its restriction to a circle. Fix a reference point $t \neq 0$ in the circle, by using local trivializations along the circle, one obtains a diffeomorphism $T : \mathcal{X}_t \to \mathcal{X}_t$ up to isotopies. That is, T is an element in the mapping class group of $\mathcal{X}_t$. We will call T "the monodromy" of the given degeneration.

In the cohomology level, a generator of $\pi_1(\Delta^\times) \cong \mathbf{Z}$ induces the so called Picard-Lefschetz transformation – the monodromy T acting on $H^m_{\mathbf{Z}}$, which is known to be quasi-unipotent. Under the semi-stable asssumption, T will be unipotent and we will consider the associated nilpotent operator $N := \log T$ acting on $H^m_{\mathbf{Q}}$. The quasi-unipotent statement is also known to be true for any abstract polarized Variation of Hodge Structures [Sc]. In the following, we will usually assume that T is unipotent by allowing a base change implicitly.

6.2. Guiding examples — a preliminary discussion. There exists smoothable Calabi-Yau 3-folds with canonical singularities such that the smoothing comes from a birational contraction of a smooth family over the disk, which induces isomorphisms outside the puncture. These examples are due to Wilson [Wi] in his deep study of the jumping phenomenon of Kähler cones. More precisely, his proposition 4.4 says that the "type III primitive contraction" with the exceptional divisor a quasi-ruled surface over an elliptic curve provides such an example.

In the surface case, these correspond to smoothings of K3 surfaces with RDP's. By Kulikov's classification theorem [Ku] they are birational to smooth families possibly after a base change. We will call this knid of degenerations "trivial" since they do not degenerate at all for certain polarizations.

If the monodromy of a degeneration $\mathcal{X} \to \Delta$ is not of finite order, the degeneration is clearly "nontrivial" in the above sense. We will however interested in the extremal case, namely degenerations with trivial monodromy. The above examples are of trivial monodromy and are in fact "projectively trivial" possibly after a base change. By this we simply mean that the punctured family can be filled in smoothly in the projective category.

Is there any degeneration with C^∞ trivial monodromy but can not be filled in smoothly? As we have already mentioned in the introduction, examples already occurs for curves. However, they are due to the presence of the nontrivial fundamental groups. Simply connected examples were found and studied by Friedman and Morgan in the 80's. They obtained examples for surfaces of general type and used them to construct examples for dimensions bigger than or equal to four.

6.3. Picard-Lefschetz theory. We start by recalling the cohomological form of the classical Picard-Lefschetz theorem:

Theorem 6.4. *For a nodal degeneration of smooth n-folds, the monodromy operator T acting on cohomologies is trivial except possibly in the middle dimensional cohomology. In the middle dimensional case, we have that*

I. $(T^2 - I)^2 = 0$ *if n is odd, and that*

II. $T^2 = I$ *if n is even.*

The standard proof is to write down the explicit reflection formula of T in terms of the "vanishing cycles". However, even to see whether T is of finite order in the cohomology level (in the odd case), one needs to know whether the vanishing cycles represent nontrivial homology classes. Clearly, this is not just a local problem of the singular points. For example, nodal degenerations of odd dimensional quadrics have trivial monodromy on cohomology, since the middle cohomology is trivial! (This was pointed out to the author by J. de Jong.) But this seems to be not the case for general varieties.

In the case that n is even, more is known. Morgan [Mo] proved that the monodromy actually has finite order. That is, after a finite base change, the punctured family is a C^∞ product. A nice result proved by Voisin [Vo] says that they are however not filliable by smooth manifolds in the cohomologically Kähler category.

6.5. Three dimensional case. Explicit calculations done by Candelas *et al.* [COGP] shows that there are nodel degenerations of Calabi-Yau 3-folds such that the monodromy is not of finite order. A theoretic proof of this statement turns out to be delicate (even for Calabi-Yau 3-folds). We will give a sketch of it by showing the existence of nontrivial vanishing cycles, following a suggestion by Mark Gross.

Let us assume that our threefolds are all simply conected. First of all, a nodal threefold $\mathcal{X}_0$ always admits (not necessarily projective) small resolutions $X \to \mathcal{X}_0$ with smooth rational curves $X \supset C_i \to p_i \in \mathcal{X}_0$ contracted to ODP's. In the case of Calabi-Yau threefolds (Gorenstein threefolds with trivial canonical bundle and with $h^1(\Omega) = 0$), the existence of global smoothing $\mathcal{X} \to \Delta$ of $\mathcal{X}_0$ forces that there are nontrivial relations of $[C_i] \in H_2(X)$ by Friedman's result [F3, F4]. That is, the canonical map $e : \bigoplus_i \mathbf{Z}[C_i] \to H_2(X, \mathbf{Z})$ has nontrivial kernel dimension $s > 0$. Consider the resulting surgery diagram:

$$\begin{array}{c} X \\ \downarrow \\ \mathcal{X}_0 \subset \mathcal{X} \supset \mathcal{X}_t \end{array} \tag{6.6}$$

It has the following local description: let $V_i \ni p_i$ be a contrctible neighborhood of an ODP, $V_i' \subset \mathcal{X}_t$ be the smoothing of V_i and $U_i \subset X$ be the inverse image of V_i. Then

I. U_i is a deformation retract neighborhood C_i and so has the homotopy type of $S^2 \sim D^4 \times S^2$.
II. V_i' has the homotopy type of $S^3 \times D^3$. Where the sections $\sigma_i \sim S^3$ are the so called vanishing cycles.
III. The surgery from X to $\mathcal{X}_t$ is induced from $\partial(D^4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times D^3)$.

Let us assume that there are k ODP's.

An immedeate consequence of (6.6) is the Euler number formula:

$$\chi(X) - k\chi(\mathbf{P}^1) = \chi(\mathcal{X}_0) - k\chi(\text{pt}) = \chi(\mathcal{X}_t) - k\chi(S^3). \tag{6.7}$$

Let W be the "common open set" of X, $\mathcal{X}_o$ and $\mathcal{X}_t$ away from all points p_i's such that W and V_i's cover $\mathcal{X}_t$ etc. A portion of the Mayer-Vietoris sequence of the covering $\{W, V_i'\}$ of $\mathcal{X}_t$ gives

$$0 \to H_3(W) \to H_3(\mathcal{X}_t) \to \bigoplus_i \mathbf{Z}[C_i] \to H_2(X) \to H_2(\mathcal{X}_t) \to 0. \tag{6.8}$$

Hence that $b_2(X) = b_2(\mathcal{X}_t) + (k - s)$.

Take into account of $b_2(\mathcal{X}_0) = b_2(\mathcal{X}_t)$ and $b_4(\mathcal{X}_0) = b_4(X)$ (which also follows from suitable Mayer-Vietoris sequences), simple manipulations with (6.7) shows that $b_3(\mathcal{X}_t) = b_3(\mathcal{X}_0) + s$. Comparing with the (Mayer-Vietoris) sequence defining the vanishing cycles:

$$\bigoplus_i \mathbf{Z}[\sigma_i] \to H_3(\mathcal{X}_t) \to H_3(\mathcal{X}_0) \to 0, \tag{6.9}$$

we conclude that $s > 0$ is the dimension of the sapce of vanishing cycles. Q.E.D.

6.10. Filling in problem in dimension three. In [F4], Friedman remarked that a degeneration of quintic hypersurfaces in $\mathbf{P}^4$ acquiring an isolated A_2 singularity (locally of the form: $x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$) actually has $N = 0$ (due to Clemens). Moreover, by Morgan's result [Mo], the monodromy has finite order in the mapping class group! He asked that whether this punctured family can be filled in smoothly in any finite base change. (He expected that the answer in NO.) If not, this will be the first known simply connected example in dimension three.

The main goal of this chapter is to prove a general theorem about the non-filliability of degenerations of three dimansional smooth minimal models acquiring nontrivial terminal singularities. In particular, we obtain in (8.7) a negative answer to Friedman's queation (as he has expected).

§7 Two Key Lemmas

We start with the following important fact that we have established in §1. Namely the Key Lemma 1.4, or more precisely, Corollary 1.10.

Lemma 7.1. *Let $\mathcal{X} \to \Delta$ and $\mathcal{X}' \to \Delta$ be two projective families with smooth general fiber $\mathcal{X}_t \cong \mathcal{X}'_t$ for $t \neq 0$. Assume that*
I. *$\mathcal{X}$ and $\mathcal{X}'$ have at most terminal singularities,*
II. *$K_{\mathcal{X}}$ (resp. $K_{\mathcal{X}'}$) is nef on the central fiber and*
III. *the map which identifies the general fibers is bimeromorphic, then this map extends to an isomorphism in codimension one. In particular, $\mathcal{X}_0$ and $\mathcal{X}'_0$ are birational to each other.*

The only point we need to be careful is that in general "the map" which identifies the general fibers may not be bimeromorphic! In fact, this map is not well defined if the general fibers have continuous automorphisms. In our case, the nefness will implies that there is no holomorphic vector fields on the general fiber. However, then we still need a further base change to get a bimeromorphic map which extends the prescribed identification on general fibers. The reader is refered to Freidman's paper [F2] for more details. It is however clear, ss we will see in the next section, this point does not affect our proof of Theorem B and our solution toward the filling in problem.

The next lemma follows from a special case of the Shokurov-Kollár connectedness theorem [K2; Theorem 17.4]. Which is in turn a consequence of the Kawamata-Viehweg vanishing theorem [KMM]. We present the proof for the sake of completeness.

Lemma 7.2. *The total space of a small smoothing of Gorenstein canonical singularities has at most Gorenstein terminal singularities.*

Proof. Let $\mathcal{X} \supset X$ be a smoothing of a complex space X with canonical singularities. Take a log resolution $f : \mathcal{X}' \to \mathcal{X}$ of the pair $(\mathcal{X}, X)$, that is, the union of the proper transform X' of X in $\mathcal{X}'$ and the exceptional divisors E_i of f form a normal crossing divisors. The restriction $g := f|_{X'} : X' \to X$ is then a resolution of singularities of X. It is an elementary fact from commutative algebra that the total space of a small smoothing of Gorenstein singularities is also Gorenstein, so we may write

$$K_{\mathcal{X}'} = f^* K_{\mathcal{X}} + \sum a_i E_i, \qquad a_i \in \mathbf{Z}. \tag{7.3}$$

If $f^* X = X' + \sum \ell_i E_i$ then $\ell_i \in \mathbf{N}$ since the singular set of $\mathcal{X}$ is contained in the singular set of X. So

$$K_{\mathcal{X}'} + X' = f^*(K_{\mathcal{X}} + X) + \sum (a_i - \ell_i)E_i. \tag{7.4}$$

By the adjunction formula, we have

$$K_{X'} = g^*K_X + A|_{X'} - F|_{X'}, \tag{7.5}$$

where A (resp. F) is the exceptional part with $a_i - \ell_i \geq 0$ (resp. ≤ -1). By the assumption on X, this shows that $F|_{X'} = 0$. The lemma then follows if we can show that "$F \neq 0$ implies $F \cap X' \neq 0$". Because for the remaining part A, we have that $a_i \geq \ell_i \geq 1$. That is, $\mathcal{X}$ has only Gorenstein terminal singularities.

We first rewrite (7.4) into the form

$$K_{\mathcal{X}'} + \big(- f^*(K_{\mathcal{X}} + X)\big) = A - (F + X'), \tag{7.6}$$

and we claim that $F + X'$ is connected in any neighborhood of a fiber of f. From (7.6), as $-f^*(K_{\mathcal{X}} + X)$ is f-big and f-nef trivially, we may apply the Kawamata-Viehweg vanishing theorem [KMM; 1-2-3] to get:

$$R^1 f_* \mathcal{O}_{\mathcal{X}'}(A - (F + X')) = 0. \tag{7.7}$$

This leads to the following exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}'}(A - (F + X')) \to \mathcal{O}_{\mathcal{X}'}(A) \to \mathcal{O}_{F+X'}(A) \to 0. \tag{7.8}$$

Localize (7.8) to a fiber $f^{-1}(x)$ with $x \in X$ shows that $F + X'|_{f^{-1}(x)}$ must be connected — since the quotient of the cyclic module $\mathcal{O}_{\mathcal{X}'}(A)$ can not have two cyclic modules as direct summands. This complete the proof because the connectedness property implies thaf $F \cap X' \neq 0$. Q.E.D.

§8 The Proof of Theorem B

In this section, by using several results of Reid, Kawamata and Kollár in the theory of 3-fold birational geometry along with Friedman's result on the simultaneous resolution of 3-fold double points, a negative answer to the "filling problem" as stated at the end of §1 is given for any projective smoothing of a terminal Gorenstein 3-fold with numerical effective canonical bundle even if the monodromy is C^∞ trivial! As a consequence, any smoothable terminal Calabi-Yau 3-fold provides nontrivial examples. Here is the main theorem:

Theorem B. *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a Gorenstein 3-fold $\mathcal{X}_0$ with nontrivial terminal singularities and with $K_{\mathcal{X}_0}$ nef. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$ with $\mathcal{X}_t \cong \mathcal{X}'_t$ for $t \neq 0$.*

Proof. Assume that such a smooth family $\mathcal{X}' \to \Delta$ exists. We will check the conditions needed in Lemma 7.1. I is satisfied by Lemma 7.2 since terminal singularities are by definition canonical, and II is clearly satisfied since

$$K_{\mathcal{X}}\,|_{\mathcal{X}_0} = K_{\mathcal{X}_0}, \tag{8.1}$$

which is nef.

Since all conditions in Lemma 7.1 are satisfied, we know that $\mathcal{X}_0$ is birational to $\mathcal{X}'_0$. We will show that this is impossible.

If $\mathcal{X}_0$ is **Q**-factorial then $\mathcal{X}_0$ and $\mathcal{X}'_0$ are birationally equivalent minimal models. Recall that a minimal model is a normal variety which is **Q**-factorial, terminal and has nef canonical class. By Kollár's theorem on flops [K1], they are related by a sequence of flops. But a flop does not change the singularities in the terminal case, so we get a contradiction.

If $\mathcal{X}_0$ is not **Q**-factorial, a theorem of Reid-Kawamata (see eg. [K3, (6.7.4)]) says that we still have a projective small morphism $X \to \mathcal{X}_0$ from a (**Q**-factorial) minimal model X to $\mathcal{X}_0$. X is birational to $\mathcal{X}_0$ and so is birational to $\mathcal{X}'_0$. As before, this implies that X is smooth and it is related to $\mathcal{X}'_0$ by a sequence of flops. By Kollár's result again [K1], X and $\mathcal{X}'_0$ have the same integral homologies and hence have the same homologies as the general fiber $\mathcal{X}_t$ in $\mathcal{X}$. Here we may also apply our Theorem A since in the later argument we will only make use of the rank of the homology groups.

Consider the following "small contraction/smoothing" diagram:

$$\begin{array}{c} X \\ \downarrow \\ \mathcal{X}_0 \subset \mathcal{X} \supset \mathcal{X}_t \end{array} \tag{8.2}$$

If $\mathcal{X}_0$ has only ODP singularities, (8.2) is nothing but a "surgery diagram" appeared in the Picard-Lefschetz theory (6.5). The explicit formula (6.7) (or (6.8)) which relates the homologies of X and $\mathcal{X}_t$ shows in particular that they can not be the same. We will need (6.7) (or (6.8)) in a generalized form suitable for our purpose. The proof is identically the same.

Lemma 8.3. *Given a diagram as above in the C^∞ category such that near each singular point of $\mathcal{X}_0$ it is a "small contraction-smoothing" diagram of a germ of ODP. Let C_i be the rational curves contracted to those ODP's and let $e : \bigoplus_i \mathbf{Z}[C_i] \to H_2(X, \mathbf{Z})$ be the map which associates to each C_i its class in X, then $H_2(\mathcal{X}_t) = \operatorname{coker} e$.*

So, $H_2(\mathcal{X}_t) \cong H_2(X)$ means the image of e is zero, which is impossible because X is projective. This is the desired contradiction in the case when $\mathcal{X}_0$ has only ODP's as singular points.

In the general case, since the singularities are Gorenstein, by Reid's classification they are exactly isolated cDV singular points, that is, one parameter deformation of surface RDP's. By Friedman's result [F1], if $p \in V$ is a germ of an isolated cDV point and $C \subset U$ is the corresponding germ of the exceptional set (which is a curve) contracted to p, then the versal deformation spaces $\mathrm{Def}(p, V)$ and $\mathrm{Def}(C, U)$ are both smooth and there is an inclusion map of complex spaces $\mathrm{Def}(C, U) \to \mathrm{Def}(p, V)$. Moreover, one can deform the complex structure of a small neighborhood of C so that in this new complex structure, C decomposes into several $\mathbf{P}^1$'s and the contraction map deforms to a nontrivial contraction of these $\mathbf{P}^1$'s down to ODP's, while keeping a neighborhood of these ODP's to remain in the versal deformations of the germ $p \in V$.

We can preform this analytic process for all C's and p's simultaneously in each corresponding small neighborhoods and then patch them together smoothly. As a result, we obtain a deformed diagram which satisfies the conditions stated in lemma 3.3:

$$\begin{array}{c} \widetilde{X} \\ \downarrow \\ \widetilde{\mathcal{X}}_0 \subset \widetilde{\mathcal{X}} \supset \widetilde{\mathcal{X}}_t \end{array} \tag{8.4}$$

By our construction, $\widetilde{X}$ is diffeomorphic to X and $\widetilde{\mathcal{X}}_t$ is diffeomorphic to $\mathcal{X}_t$ for $t \neq 0$. The later is true because $\mathrm{Def}(p, V)$ is smooth and the constructiuon is local. Now we have again,

$$H_2(\widetilde{\mathcal{X}}_t) \cong H_2(\mathcal{X}_t) \cong H_2(X) \cong H_2(\widetilde{X}). \tag{8.5}$$

This implies that the image of e is zero. Since the original exceptional curve has nontrivial homology class, at least one deformed rational curve has nontrivial homology class. This leads to the desired contradiction again and we are done. Q.E.D.

A Calabi-Yau variety is by definition a normal projective variety which is Gorenstein and has trivial canonical (Cartier) divisors. Usually we also impose the condition that $h^1(\mathcal{O}) = 0$ to distinguish them from abelian varieties.

In the case of Calabi-Yau 3-folds with at most canonical singularities, $h^1(\mathcal{O}) = 0$ implies $h^2(\mathcal{O}) = 0$ by the Grothendieck-Serre duality theorem for Gorenstein varieties. Hence any smoothing $\mathcal{X} \to \Delta$ must be projective by the semi-continuity of $h^2(\mathcal{O}_{\mathcal{X}_t})$, and in fact $\mathcal{X}_t$ must still be Calabi-Yau. So we conclude the following:

Theorem 8.6. *Let $\mathcal{X} \to \Delta$ be a smoothing of a Calabi-Yau 3-fold with nontrivial terminal singularities. Then $\mathcal{X} \to \Delta$ is not birational to a smooth family with identical general fibers.*

8.7. Negative answer to the filling in problem. Why Theorem B (or Theorem 8.6) answers the filling in problem? Notice that the assumptions we made in these theorems are all invariant under base changes. If for some base change the punctured family can be filled in smoothly and projectively, then a further base change also has this property. We may arrange the base change so that the identification map of the general fibers becomes bimeromorphic. Then we may apply Theorem B or Theorem 8.6 to obtain the desired contradiction. Q.E.D.

§9 Weil-Petersson Geometry of Calabi-Yau Moduli

The classical Weil-Petersson metric on the Teichmüller space of compact Riemann surfaces is a Kähler metric which is complete only in the case of elliptic curves [Wo]. It has a natural generalization to the deformation spaces of higher dimansional polarized Kähler-Einstein manifolds. It is still Kähler. Moreover, in the case of abelian varieties and K3 surfaces, the Weil-Petersson metric turns out to be equal to the Bergman metric of the Hermitian symmetric period domain, hence is in fact "complete" Kähler-Einstein [Sc].

The completeness is an important property for differential geometric reason. Motivated by the above examples, one may naively think that the completeness of the Weil-Petersson metric still holds true for general Calabi-Yau manifolds. However, explicit calculation done by physicists (eg. Candelas *et al.* [CGH] for some special nodal degenerations of Calabi-Yau 3-folds) indicated that this may not always be the case.

Naturally, we need to clarify what do we actually mean that the metric is complete or incomplete. This depends on how we define the "moduli space", which is already very interesting in the case of K3 surfaces. We will gradually explain what is our understanding of this problem. And it would then become clear that the Weil-Petersson metric is in general incomplete if one sticks on "moduli" of smooth varieties.

9.1. The Weil-Petersson metric. For a given family of polarized Kähler manifolds $\mathcal{X} \to S$ with Kähler metrics $g(s)$ on $\mathcal{X}_s$, one can define a possibly degenerate hermitian metric G on S as follows: at $s \in S$ with fiber $X = \mathcal{X}_s$, we consider the Kodaira-Spencer map $\rho : T_{S,s} \to H^1(X, T_X) \cong \mathbf{H}^{0,1}_{\bar\partial}(T_X)$ into harmonic forms with respect to $g(s)$; so for v, $w \in T_s(S)$, we may define

$$G(v, w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)}. \tag{9.2}$$

When $\mathcal{X} \to S$ is a polarized Kähler-Einstein family and ρ is injective, $G_{WP} := G$ is called the Weil-Petersson metric on S.

When X is a Calabi-Yau manifold, we have Yau's solution to Calabi's conjecture [Ya] that X has an unique Ricci flat metric in each Kähler class and the Bogomolov-Tian-Todorov theorem that the Kuranishi space of X is unobstructed [Ti, To].

Let $\mathcal{X} \to S$ be a maximal subfamily of the Kuranishi family with a fixed polarization class $[\omega]$, then ρ is clearly injective. Let $g(s)$ be the unique Ricci flat metric in the given polarization. Using the fact that the global holomorphic n-form $\Omega(s)$ is flat with respect to $g(s)$, it was shown in [Ti, To] that

$$G_{WP}(v,w) = \frac{Q(C(i(v)\Omega), \overline{i(w)\Omega})}{Q(C\Omega, \bar{\Omega})}, \tag{9.3}$$

where $H^1(X, T_X) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \cong H^{n-1,1}$ via the interior product $v \mapsto i(v)\Omega$ is the well-known isomorphism. The tangent space T_S is mapped to $P^{n-1,1}$ isomorphically and hence leads to the fact that the n-th flag period map is an local embedding. So the Weil-Petersson metric is induced from the Hodge metric on the n-th piece of the horizontal tangent bundle. For convienence, let's write $\widetilde{Q} = \sqrt{-1}^n Q$ $(= Q(\cdot, \bar{\cdot})$ on $H^{n,0} = P^{n,0})$. Tian observed that $\widetilde{Q}$ is a Kähler potential of G_{WP}, that is,

$$\omega_{WP} = \frac{\sqrt{-1}}{2}\mathrm{Ric}_{\widetilde{Q}}(H^{n,0}) = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\widetilde{Q}, \tag{9.4}$$

where ω_{WP} denotes the fundamental real 2-form of G_{WP} (this formula shows in particular that ω_{WP} is independent of the polarization). The proof is essentially part of Griffiths' curvature calculation [Gr], hence is purely Hodge theoretic. So we can extend the definition of G_{WP} to polarized VHS over S with $h^{n,0} = 1$ by (9.4), although it is only semi-positive. Since it makes sense to talk about geodesics and distances, we will still call it the Weil-Petersson metric.

Clearly, our aim is to characterize all finite distance degenerations and then to describe the possible picture of the completion. We get strong evidence that it is closely related to the minimal model program in birational geometry. However, the results we can rigorously proved so far are not enough to answer the full question. We do formulate a conjecture in §10 to complete our discussion here.

The result is this section are mostly exercises in Hodge theory. We will recall what we need. Details can be found in [Gr, GS, Cl, Sc].

9.5. Schmid's theory on limiting MHS. Let D be the period domain for certain polarized Hodge structures and let $\check{D}$ be its compact dual. For a polarized VHS $\phi : \Delta^\times \to \langle T\rangle\backslash D$; the map ϕ lifts to the upper half plane $\Phi : \mathbf{H} \to D$ with the coordinates $t \in \Delta^\times$ and $z \in \mathbf{H}$ related by $t = e^{2\pi\sqrt{-1}z}$. Set

$$A(z) = e^{-zN}\Phi(z) : \mathbf{H} \to \check{D}, \tag{9.6}$$

(instead of D). Since $A(z+1) = A(z)$, A descends to a function $\alpha(t)$ on $\Delta^\times$. The very first part of Schmid's "nilpotent orbit theorem" says that $\alpha(t)$

extends holomorphically over $t = 0$. The special value $F_\infty := \alpha(0)$ is called the limiting filtration and is in general outside D. However, the nilpotent operator N uniquely defines a "monodromy weight filtration" on V: $0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m-1} \subset W_{2m} = V$ such that $N(W_k) \subset W_{k-2}$ and induces an isomorphism

$$(9.7) \qquad N^\ell : G^W_{m+\ell} \cong G^W_{m-\ell},$$

where $G^W_k := W_k/W_{k-1}$ is the graded piece. These two filtrations F^p_∞ and W_k together define a "polarized mixed Hodge structure" on V in the following sense: the induced Hodge filtration

$$(9.8) \qquad F^p_\infty G^W_k := F^p_\infty \cap W_k / F^p_\infty \cap W_{k-1}, \quad p = 0, \ldots, m$$

defines a (pure) Hodge structure of weight k on G^W_k. The operator N acts on them as a morphism of MHS's of type $(-1,-1)$. That is, $N(F^p_\infty G^W_k) \subset F^{p-1}_\infty G^W_{k-2}$. Moreover, for $\ell \geq 0$, the primitive part $P^W_{m+\ell} := \ker N^{\ell+1} \subset G^W_{m+\ell}$ is polarized by $Q(\cdot, N^\ell \bar{\cdot})$.

When ϕ comes from geometric situations, namely the period map of a degeneration $\mathcal{X} \to \Delta$, by adding together with the non-primitive part, the total cohomology $H^m(\mathcal{X}_t, \mathbf{C})$ still admits non-polarized MHS.

We now give the basic criterion for finite Weil-Petersson distance in the case of one parameter degenerations of polarized Hodge structures $\phi : \Delta^\times \to \langle T \rangle \backslash D$ with $h^{n,0} = 1$:

Theorem 9.9. *The center of a degeneration of polarized Hodge structures of weight n with $F^n \cong \mathbf{C}$ has finite Weil-Petersson distance if and only if $NF^n_\infty = 0$.*

Proof. Let $\Phi : \mathbf{H} \to D$ be the lifting. To start the computation, all we need is a good choice of a holomorphic section Ω of $H^{n,0}$. Let $p^n : D \to \mathbf{P}(V)$ be the projection to the F^n part. we have $\Phi^n(z) = (e^{zN}\alpha(t))^n = e^{zN}\alpha^n(t)$. Here $*^n := p^n(*) \in \mathbf{P}(V)$ means the n-th flag. Near $t = 0$, we can consider a vector (local homogeneous coordinates) representation $\mathbf{a}$ of α^n in V. Then $\mathbf{a}(t) = a_0 + a_1 t + \cdots$ is holomorphic in t. We have orrespondingly

$$(9.10) \qquad \mathbf{A}(z) = a_0 + a_1 e^{2\pi\sqrt{-1}z} + a_2 e^{4\pi\sqrt{-1}z} + \cdots.$$

The crucial point here is that the function $e^{2\pi\sqrt{-1}z} = e^{2\pi\sqrt{-1}x}e^{-2\pi y}$ has the property that all the partial derivatives in x and y decay to 0 exponentially as $y \to \infty$, with rate of decay independent of x. For ease of notation, let h be the function class satisfying the above property and $\mathbf{h}$ the corresponding function class with values in V.

Now let $\Omega(z) = e^{zN}\mathbf{A}(z)$. This is the desired section because vector representations correspond to sections of the tautological line budle of $\mathbf{P}^n$ which pull back to $H^{n,0}$ by Φ. So the Kähler form ω_{WP} of the induced Weil-Petersson metric G_{WP} on $\mathbf{H}$ is given by

$$\omega_{WP} = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\widetilde{Q}(e^{zN}\mathbf{A}(z), e^{\bar{z}N}\overline{\mathbf{A}(z)}). \tag{9.11}$$

Since we are in one complex variable, write $G_{WP} = G|dz|^2$, then $G = -(1/4)\triangle\log\widetilde{Q}$. We have $Q(Tu,Tv) = Q(u,v)$, it follows easily that $Q(Nu,v) = -Q(u,Nv)$ and $Q(e^{zN}u,v) = Q(u,e^{-zN}v)$. Since $\mathbf{A} = a_0 + \mathbf{h}$, we have

$$\begin{aligned} \widetilde{Q}(e^{zN}\mathbf{A}, e^{\bar{z}N}\bar{\mathbf{A}}) &= \widetilde{Q}(e^{zN}a_0, e^{\bar{z}N}\bar{a_0}) + h \\ &= \widetilde{Q}(e^{2\sqrt{-1}yN}a_0, \bar{a_0}) + h = p(y) + h, \end{aligned} \tag{9.12}$$

where $p(y)$ is a polynomial in y with

$$d = \deg p(y) = \max\{\,\ell \mid N^\ell\alpha_0 \neq 0\,\}. \tag{9.13}$$

This a consequence of the polarization condition for the mixed Hodge structure (9.5) and the fact that $a_0 \in G_{n+d}$. So

$$\begin{aligned} 4G &= \frac{(p'+h)^2 - (p+h)(p''+h)}{(p+h)^2} = \frac{(p'^2 - pp'') + h}{p^2 + h} \\ &\sim \frac{p'^2 - pp''}{p^2} + h \sim \frac{d^2 - d(d-1)}{y^2} + h = \frac{d}{y^2} + h. \end{aligned} \tag{9.14}$$

Here we have used the fact that $p^{-2}h \in h$. Obviously, if $NF^n_\infty = 0$ then $d = 0$ and $G = h$, so $\int_t^\infty \sqrt{G}\,|dz| < \infty$ for some curve (e.g. $x = c$). When $NF^n_\infty \neq 0$ we have $d \geq 1$ and for y large enough we can make $h < 1/y^3$ uniformly in x, then clearly $\int_t^\infty \sqrt{G}\,|dz| \sim 2\log y\,|_t^\infty = \infty$ for any path with $y \to \infty$. Q.E.D.

Return to the geometric situation, namely the semi-stable degeneration of polarized Calabi-Yau manifolds. As a simple application of the Clemens-Schmid exact sequence [Cl], we have

Theorem 9.15. *The central fiber X has finite Weil-Petersson distance if and only if some irreducible component $X_i \subset X$ has $H^{n,0} \neq 0$. This is equivalent to that there is exact one component with $h^{n,0} = 1$.*

Proof. By the results of Schmid in (9.5), F_∞ and N defines a MHS on $H^n(\mathcal{X}_t)$ for a reference fiber $\mathcal{X}_t$ with $t \neq 0$. It follows from (9.7) that $(\ker N) \cap F^n_\infty \equiv G^W_n F^n_\infty$. So $NF^n_\infty = 0$ if and only if $F^n_\infty = G^W_n F^n_\infty$.

Recall that the "geometric genus formula" [Cl] says that

$$h^{n,0}(\mathcal{X}_t) \geq \sum\nolimits_i h^{n,0}(X_i), \tag{9.16}$$

and the RHS corresponds to all the invariant cycles in F^n_∞, that is, $(\ker N) \cap F^n_\infty$. Since the LHS of (9.16) corresponds to F^n_∞, the eqality holds if and only if $F^n_\infty = (\ker N) \cap F^n_\infty = G^W_n F^n_\infty$, that is, if and only if $NF^n_\infty = 0$.

In our case, Theorem 9.9 says that finite distance is equivalent to $NF^n_\infty = 0$. Since $h^{n,0}(\mathcal{X}_t) = 1$, this is equivalent to that there exist some (and so at most one) component with $h^{n,0} \neq 0$ (and so in fact it must be 1). The proof is now complete. Q.E.D.

As a corollary, we deduce the following theorem which we believe to be very close to the final answer of the completion problem:

Theorem 9.17. *Let X be a Calabi-Yau varieties which admits a smoothing to Calabi-Yau manifolds. If X has only canonical singularities then X has finite Weil-Petersson distance along the base.*

Proof. For any resolution $f : \widetilde{X} \to X$, we have as in the above that $H^{n,0}(\widetilde{X}, \mathbf{C}) = \Gamma(\widetilde{X}, K_{\widetilde{X}}) = \Gamma(\widetilde{X}, \sum e_i E_i)$ (notice that e_i's are integers). Since E_i's are exceptional, it follows easily that $H^{n,0}(\widetilde{X}, \mathbf{C}) \neq 0$ precisely when X has at most canonical singularities.

Now let $\mathcal{X} \to \Delta$ be a smoothing of X. Take a semi-stable reduction of it, then there is a component in the central fiber of the semi-stable reduction which corresponds to the proper transform of X. Then it has $h^{n,0} = 1$. Now apply Theorem 9.15 and notice that finite distance in a special smoothing implies finite distance in the whole smoothing component. Q.E.D.

Example 9.18. According to [Re], hypersurface singularities of monomial type $\sum_i x^{d_i} = 0$ is canonical if and only if $\sum_i 1/d_i > 1$. In the three dimensional case, the finiteness of the Weil-Petersson distance with singularities of this type were known to Candelas *et al.* [CGH] via direct calculations. Theorem 9.17 seems to indicate that canonical singularities may also play significant role in certain physics problems.

§10 Speculations

Now we may put everything together. In the case of Calabi-Yau 3-folds, Wilson's example shows that finite distance degenerations could be "trivial" if the singular set is a smooth elliptic curve, which is **canonical** but not terminal. But our Theorem B shows that all **terminal** degenerations gives nontrivial finite distance points, hence shows that in general the Weil-Petersson metric is not complete. This phenomenon does not occur in the case of K3 surfaces (two dimension Calabi-Yau manifolds) because there are no two dimensional terminal singulariteis. Moreover, Theorem 9.17 shows that it is very likely that the completion could be achieved by considering all smoothable Calabi-Yau 3-folds with at most canonical singularities.

In [Vi], Viehweg proved a general theorem on the quasi-projectivity of moduli spaces of polarized manifolds. In fact he mentioned that his approach works equally well for normal varieties with at most canonical singularities once the "locally closedness" of the moduli functor can be proved. Very recently, Kawamata announced a proof that deformations of canonical singularities are again canonical [Ka]. This implies the required locally closedness and hence complete Viehweg's program. (Compare with our Lemma 7.2.)

From the point of view of the Weil-Petersson geometry, this amounts to say that the completion of the Weil-Petersson metric is the enlarged quasi-projective moduli corresponding to Viehweg's program including canonical singularities. In fact, our original motivation to study the Weil-Petersson metric (or more generally, the Hodge metric) is to hope to give a purely differential geometric approach to the quasi-projectivity problem.

This motivates the following question:

Question 10.1. Is the converse of Theorem 9.17 true? More precisely, if a degeneration of Calabi-Yau manifolds has finite Weil-Petersson distance, is that true this degeneration is birational to another degeneration such that the central fiber is an irreducible Calabi-Yau variety with only canonical singularities?

This is obviously the most important step toward the completion program. In the following, we will discribe two heuristic reasons why we think this question may has an affirmative answer.

10.2. Via minimal model conjecture. We have used the geometric genus inequality to obtain Theorem 9.16. But in fact N. Nakayama has proved in [Na] that if the minimal model conjectures are all true — including the abundance conjecture, then one also has the plurigenus inequality:

$$P_m(\mathcal{X}_t) \geq \sum_i P_m(X_i), \tag{10.3}$$

for all $m \in \mathbf{N}$. Where $P_m(X) = h^0(X, K_X^m)$. In our case, $P_m(\mathcal{X}_t) = 1$ for all m and if X_i is the unique component of $\mathcal{X}_0$ with $h^0(K) = 1$, then $P_m(X_i) \geq 1$ for all m. This implies that $P_m(X_i) = 1$ and $P_m(X_j) = 0$ for $j \neq i$. In dimension three, a theorem of Miyaoka then says that these X_j's must be uniruled. (A simplified proof given by Shepherd-Barron is in [K2].) In general, this is still conjectured to be true.

One may then try to prove that the ruling is in fact an extremal rays and then contract it via the Kawamata-Shokurov contraction theorem [KMM]. Since X_j are uniruled, the whole X_j will be contracted. The contraction theorem gaurantees that the contracted space still has terminal singularities. If one can keep on contracting all X_j with $j \neq i$, then the resulting family $\widetilde{\mathcal{X}} \to \Delta$ will has central fiber an irreducible Calabi-Yau variety with canonical singularities.

This process is basically the same as what Kulikov did in his classification theorem of semi-stable degeneration of K3 durfaces, where the idea of log-flips first appeared. This means that our heuristic reason is exactly the first "hard part" of extending minimal model theory to higher dimensions.

10.4. Via Hausdorff convergence. Around the early 90's, works done by H. Nakajima, Z.Gao and M. Anderson had demonstrated that the Hausdorff limit of real Einstein four manifolds with diameter upper bound and volume lower bound is again Einstein with at worst quotient singularities (cf. [An]). In the Kähler category, this gives a differential geometric point of view of Kulikov's type I degenerations of K3 surfaces.

We first notice the following fact: a generic hyperplane section of canonical singularities is again canonical. This implies that the generic point of codimension two stratum of canonical singularities is nothing but the RDP — the SU(2) quotient singularities.

If we are given degenerations $\mathcal{X} \to \Delta$ of polarized Calabi-Yau manifolds, We may imagine that $\mathcal{X}_t$ are fibered by families of complex surface slices. In the case of finite distance degenerations, as $t \to 0$, the central fiber can be viewed as the Hausdorff limit of $\mathcal{X}_t$. The volume bound is trivial since the it is in fact a const. The diameter bound is implied by the finiteness of the Weil-Petersson distance. This is not hard to see from the definition of the Weil-Petersson metric (as the variation of the Kähler-Einstein structures).

If we believe that the Ricci curvature bound can be "preserved" for generic surface slices, then by applying the known result mentioned above, we conclude that the generic point of codimension two singularities of $\mathcal{X}_0$ is canonical — a positive evidence to our question. However, it is also obvious that it is extremely difficult to fully answer our question in this way even if the above argument indeed works. This is due to the fact that one has no idea of how to deal with the singularities of higher codimensions.

A workable example is given by Calabi-Yau 3-folds with K3 fiberation structures. Wilson's theorem on the invariance of Kahler cone under deformations [Wi] indicates the possibility that the K3 fibration structure is stable under deformations of complex structures. This will then give the surface slices we want. Needless to say, a lot of analysis needs to be done to justify the above argument.

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