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κ -Noncollapsing Estimates Along The Ricci Flow

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k-Noncollapsing Estimates Along The Ricci Flow

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沿著瑞奇流的κ-noncollapsing估計

摘要

在這篇文章裡我們描述了兩種由Perelman提出建立沿著瑞奇流的κ-noncollapsing 定理的方法。第一種方法是使用Perelman entropy。第二種方法是利用Perelman's reduced volume的單調性來建立。Reduced volume是對non-collapsing定理更局部的看法，因此我們學習Perelman的証明中關於龐加萊猜想裡ancient κ-noncollapsing的解時(這種解不必是緊緻因此不被總體的量所控制)，第二個方法是重要的。我們的論述主要是依據Cao-Zhu [6]，關於Perelman's W functional我們參考O. Rothaus [3]給予更詳細的說明。

κ-Noncollapsing Estimates Along The Ricci Flow

Abstract

In this paper we report on the two methods pioneered by G. Perelman [1] to establish his κ-noncollapsing thm of the Ricci flow. The first method uses the Perelman entropy. The second proof uses the monotonicity of the Perelman's reduced volume. The second proof is important, because the reduced volume is a more localized quantity in its definition and so one can in fact establish local versions of the non-collapsing theorem which turn out to be important when we study ancient κ-noncollapsing solutions in Perelman's proof of the Poincaré conjecture. Such solutions need not be compact and so cannot be controlled by global quantities (such as the Perelman entropy). Our treatment follows closely the cuticle by Cao-Zhu [6], with some more details on Perelman's $\mathcal{W}$ functional by O. Rothaus [3].

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κ-NONCOLLAPSING ESTIMATES ALONG THE RICCI FLOW

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ABSTRACT. In this paper we report on the two methods pioneered by G. Perelman [1] to establish his κ-noncollapsing thm of the Ricci flow. The first method uses the Perelman entropy. The second proof uses the monotonicity of the Perelman's reduced volume. The second proof is important, because the reduced volume is a more localized quantity in its definition and so one can in fact establish local versions of the non-collapsing theorem which turn out to be important when we study ancient κ-noncollapsing solutions in Perelman's proof of the Poincaré conjecture. Such solutions need not be compact and so cannot be controlled by global quantities (such as the Perelman entropy). Our treatment follows closely the cuticle by Cao-Zhu [6], with some more details on Perelman's $\mathcal{W}$ functional by O. Rothaus [3].

1. Introduction

1.1. **Contents of this note.** Consider a complete Riemannian manifold M of dimension $n \geq 3$ with the Riemannian metric g_{ij}. Let $g = g(t)$ be a smooth solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric$$

on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with a given initial metric $g(0) = g_0$.

In Section 2.1-2.3, a new monotonic quantity, namely the reduced volume $\tilde{V}$, is introduced. It is defined in terms of so-called $\mathcal{L}$-geodesics. Let (p, t_0) be a fixed spacetime point. Define the backward time by $\tau = t_0 - t$. Given a curve $\gamma(\tau)$ in M defined on $0 \leq \tau \leq \bar{\tau}$ (i.e. going backward in real time) with $\gamma(0) = p$, its $\mathcal{L}$-length is defined to be

$$\mathcal{L}(\gamma) = \int_0^{\bar{\tau}} \sqrt{\tau}(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R(\gamma(\tau), t_0 - \tau))d\tau.$$

Let $L(q, \bar{\tau})$ be the infimum of $\mathcal{L}(\gamma)$ over curves γ with $\gamma(0) = p$ and $\gamma(\bar{\tau}) = q$. Put

$$\ell(q, \bar{\tau}) = \frac{L(q, \bar{\tau})}{2\sqrt{\bar{\tau}}}.$$

The reduced volume is defined by

$$\tilde{V}(\bar{\tau}) = \int_M (4\pi\bar{\tau})^{-\frac{n}{2}} e^{-\ell(q,\bar{\tau})} dV.$$

The remarkable fact is that if g is a Ricci flow solution then $\tilde{V}$ is nonincreasing in $\bar{\tau}$, i.e. nondecreasing in real time t. The proof of monotonicity uses a subtle cancelation between the $\bar{\tau}$-derivative of $\ell(\gamma(\bar{\tau}), \bar{\tau})$ along an $\mathcal{L}$-geodesic and the Jacobian of the so-called $\mathcal{L}$-exponential map.

In Section 2.3, a modified "entropy" functional $\mathcal{W}(g, f, \tau)$ is introduced. It is nondecreasing in t provided that g is a Ricci flow solution, $\tau = t_0 - t$ and $(4\pi\tau)^{\frac{n}{2}} e^{-f}$ satisfies the conjugate heat equation.

In Section 3.1, the entropy functional $\mathcal{W}$ is used to prove a no local collapsing theorem. The statement is that if g is a given Ricci flow on a finite time interval $[0, T)$ then for any (scale) ρ, there is a number $\kappa > 0$ so that if $B_t(x, r)$ is a time-t ball with radius r less than ρ, then

$$|Rm| \leq \frac{1}{r^2} \quad \Rightarrow \quad Vol(B_t(x, r)) \geq \kappa r^n \quad \text{on } B_t(x, r).$$

The method of proof is to show that if $r^{-n}Vol(B_t(x,r))$ is very small than the evaluation of $\mathcal{W}$ at time t is very negative, which contradicts the monotonicity of $\mathcal{W}$.

In Section 3.2 we will use a cut-off argument to extend the no local collapsing theorem to any complete solution with bounded curvature. In some sense, the second no local collapsing theorem gives a good relative estimate of the volume element for the Ricci flow.

1.2. **Historical remarks.** Historically, in the 1980's, it was Richard Hamilton who initiated the program of using the Ricci flow to solve the Poincaré conjecture as well as the hyperbolicity conjecture of three dimensional manifolds. His idea is to do surgeries on the manifold when the curvature tends to blow-up in some region during the Ricci flow. In order to perform surgeries Hamilton needs to classify the neighborhood of the blow-up region. He used a standard parabolic scaling of the Ricci flow to perform the blow-up analysis and he also proved a convergence theorem of the rescalled region when a "Little Loop Lemma" holds. Unfortunately his proof of the Little Loop Lemma turns out to be incorrect and it becomes the first main obstacle to carry out Hamilton's program.

In [1], G. Perelman made a breakthrough in Hamilton's program. Among many other things, Perelman formulated and proved the No Local Collapsing Theorem which in particular implies the Little Loop Lemma as a corollary. The $\mathcal{L}$-geodesic, reduced length ℓ as well as the reduced volume $\tilde{V}$ are all due to Perelman. Since Perelman's paper was written in a rather dense manner, it is highly desirable to have more transparent proofs, with more details filled in, of the results proved or announced in [1]. Since then several nice articles had appeared aiming at understanding Perelman's argument.

Our purpose here is simply to understand Perelman's No Local Collapsing Theorems, both the compact and non-compact cases. Our treatment follows closely the cuticle by Cao-Zhu [6], with some more details on Perelman's $\mathcal{W}$ functional by using results in O. Rothaus [3]. It is the author's hope that this note will be helpful as a supplementary reading for people who wants to read Perelman's work.

2. Perelman's Reduce Volume

2.1. **The $\mathcal{L}$-geodesics.** We write the Ricci flow in the backward version

$$\frac{\partial g_{ij}}{\partial \tau} = 2R_{ij}$$

on a manifold M with $\tau = \tau(t)$ satisfying $\frac{d\tau}{dt} = -1$. We always assume that either M is compact or $g_{ij}(\tau)$ are complete and have uniformly bounded curvature. The $\mathcal{L}$-length of a (smooth) space curve $\gamma : [\tau_1, \tau_2] \to M$ is defined by

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2) d\tau,$$

where the scalar curvature $R(\gamma(\tau))$ and the norm $|\dot{\gamma}(\tau)|$ are evaluated using the metric at time $t = t_0 - \tau$. Here $\tau_1 > 0$.

To derive the $\mathcal{L}$-geodesic equation, as in the standard Riemannian geometry we consider an 1-parameter family of curves $\gamma_s : [\tau_1, \tau_2] \to M$, parametrized by $s \in (-\epsilon, \epsilon)$. Equivalently, we have a map $\widetilde{\gamma}(s, \tau)$ with $s \in (-\epsilon, \epsilon)$ and $\tau \in [\tau_1, \tau_2]$. Putting $X = \frac{\partial \widetilde{\gamma}}{\partial \tau}$ and $Y = \frac{\partial \widetilde{\gamma}}{\partial s}$, we have $[X, Y] = 0$. This implies that $\nabla_X Y = \nabla_Y X$. Writing δ_Y as shorthand for $\frac{d}{ds}|_{s=0}$, and restricting to the curve $\gamma(\tau) = \widetilde{\gamma}(0, \tau)$. We have $(\delta_Y Y)(\tau) = Y(\tau)$ and $(\delta_Y X)(\tau) = (\nabla_X Y)(\tau)$. Then

$$\begin{aligned}
&\delta_Y(\mathcal{L}) \\
&= \int_{\tau_1}^{\tau_2} \sqrt{\tau}(\langle \nabla R, Y\rangle + 2\langle X, \nabla_Y X\rangle) d\tau \\
&= \int_{\tau_1}^{\tau_2} \sqrt{\tau}(\langle \nabla R, Y\rangle + 2\langle X, \nabla_X Y\rangle) d\tau \\
&= \int_{\tau_1}^{\tau_2} \sqrt{\tau}(\langle \nabla R, Y\rangle + 2\frac{d}{d\tau}\langle X, Y\rangle - 2\langle \nabla_X X, Y\rangle - 4Ric(X, Y)) d\tau \\
&= 2\sqrt{\tau}\langle X, Y\rangle\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \sqrt{\tau}\langle Y, \nabla R - 2\nabla_X X - 4Ric(\cdot, X) - \frac{1}{\tau}X\rangle d\tau.
\end{aligned}$$

Hence the $\mathcal{L}$-geodesic equation is

$$\nabla_X X - \frac{1}{2}\nabla R + \frac{1}{2\tau}X + 2Ric(X, \cdot) = 0$$

where the 1-form $Ric(X, \cdot)$ has been identified with the corresponding dual vector field.

Give any $p, q \in M$ and $\tau_2 > \tau_1 > 0$ there exists a $\mathcal{L}$-shortest geodesic $\gamma : [\tau_1, \tau_2] \to M$ such that $\gamma(\tau_1) = p, \gamma(\tau_2) = q$ and satisfies the $\mathcal{L}$-geodesic equation. Multiplying $\sqrt{\tau}$ to the- $\mathcal{L}$-geodesic equation, we get

$$\nabla_X(\sqrt{\tau}X) = \frac{\sqrt{\tau}}{2}\nabla R - 2\sqrt{\tau}Ric(X,\cdot) \quad \text{on } [\tau_1,\tau_2].$$

That is,

$$\frac{d}{d\tau}(\sqrt{\tau}X) = \frac{\sqrt{\tau}}{2}\nabla R - 2Ric(\sqrt{\tau}X,\cdot) \quad \text{on } [\tau_1,\tau_2]. \tag{1}$$

Thus, if a continuous curve defined on $[\tau_1,\tau_2]$ satisfying the $\mathcal{L}$-geodesic equation (1) for any subinterval $0<\tau_1<\tau<\tau_2$, then $v = \lim_{\tau\to 0^+}\sqrt{\tau}X(\tau)$ exists. This allows us to extend the definition of the $\mathcal{L}$-length to include the case $\tau_1 = 0$ for all those (continuous) curves $\gamma : [0,\tau_2] \to M$ which are smooth on $(0,\tau_2]$ and have limits $\lim_{\tau\to 0+}\sqrt{\tau}\dot{\gamma}(\tau)$.

This means that for a fixed $p \in M$, by taking $\tau_1 = 0$ and $\gamma(0) = p$, the vector $v = \lim_{\tau\to 0}\sqrt{\tau}X(\tau)$ is well-defined in T_PM. The $\mathcal{L}$-exponential map $\mathcal{L}\exp_\tau : T_PM \to M$ sends v to $\gamma(\tau)$.

2.2. **Perelman's reduced volume.** The function $L(q,\bar{\tau})$ is the infimum of the $\mathcal{L}$-length among curves γ with $\gamma(0) = p$ and $\gamma(\bar{\tau}) = q$.

When we perform standard variational calculations of the function L, we can get the following results (see [2], 18-22):

$$\frac{dL(\gamma(\bar{\tau}),\bar{\tau})}{d\bar{\tau}} = \sqrt{\bar{\tau}}(R(\gamma(\tau))) + |X(\bar{\tau})|^2) \tag{2}$$

and

$$\bar{\tau}^{\frac{3}{2}}(R(\gamma(\bar{\tau}) + |X(\bar{\tau})|^2) = -K + \frac{1}{2}L(q,\bar{\tau}), \tag{3}$$

where

$$K = \int_0^{\bar{\tau}} \tau^{\frac{3}{2}} H(X(\tau))d\tau$$

and

$$H(X) = -R_\tau - \frac{1}{\tau} - 2\langle\nabla R, X\rangle + 2Ric(X,X).$$

Also

$$L_{\bar{\tau}}(q,\bar{\tau}) = 2\sqrt{\bar{\tau}}R(q) - \frac{1}{2\bar{\tau}}L(q,\bar{\tau}) + \frac{1}{\bar{\tau}}K, \tag{4}$$

$$\Delta L \le \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}}\int_0^{\bar{\tau}}\tau^{\frac{3}{2}}H(X)d\tau = \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}}K \tag{5}$$

and

$$\bar{L}_{\bar{\tau}} + \Delta\bar{L} \le 2n \quad \text{where} \quad \bar{L}(q,\tau) = 2\sqrt{\tau}L(q,\tau). \tag{6}$$

Moreover,

$$\frac{d|Y|^2}{d\tau}\Big|_{\tau=\bar{\tau}} \leq \frac{1}{\bar{\tau}} - \frac{1}{\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}} \sqrt{\tau} H(X, \tilde{Y}) d\tau. \tag{7}$$

Defining the reduced length by

$$\ell(q,\tau) = \frac{L(q,\tau)}{2\sqrt{\tau}}$$

and the reduced volume by

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\ell(q,\tau)} dq.$$

The goal is to show that $\tilde{V}(\tau)$ is nonincreasing in τ, i.e. nondecreasing in t. To do this one uses the $\mathcal{L}$-exponential map to write $\tilde{V}(\tau)$ as an integral over T_pM :

$$\tilde{V}(\tau) = \int_{T_pM} (4\pi\tau)^{-\frac{n}{2}} e^{-\ell(\mathcal{L}\exp_\tau(v),\tau)} \mathcal{J}(v,\tau)\chi_\tau(v) dv,$$

where $\mathcal{J}(v,\tau) = \det d(\mathcal{L}_{\exp_\tau})_v$ is the Jacobian factor in the change of variables and χ_τ is a cutoff function related to the $\mathcal{L}$-cut locus of p.

To show that $\tilde{V}(\tau)$ is nonincreasing in τ, it suffices to show that

$$\tau^{-\frac{n}{2}} e^{\ell(\mathcal{L}\exp_\tau(v),\tau)} \mathcal{J}(v,\tau)$$

is nonincreasing in τ, or equivalently that

$$-\frac{n}{2}\log(\tau) - \ell(\mathcal{L}\exp_\tau(v),\tau) + \log \mathcal{J}(v,\tau)$$

is nonincreasing in τ. Hence it is necessary to compute

$$\frac{d\ell(\mathcal{L}\exp_\tau(v),\tau)}{d\tau} \quad \text{and} \quad \frac{d\mathcal{J}(v,\tau)}{d\tau}.$$

The fact that $\tilde{V}(\tau)$ is nonincreasing in τ is then used to show that the Ricci flow solution cannot collapse near p.

Theorem 2.1 (Monotonicity of Perelman's reduced volume)**.** *Let g_{ij} be a family of complete metrics evolved by the Ricci flow $\frac{\partial}{\partial\tau}g_{ij} = 2R_{ij}$ on a manifold M with bounded curvature. Fix a point p in M and let $\ell(q,\tau)$ be the reduced distance from $(p,0)$. Then Perelman's reduced volume*

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\ell(q,\tau)} dV_\tau(q)$$

is nonincreasing in τ.

Proof. Fix $p \in M$. From the discussion above, we can write

$$\tilde{V}(\tau) = \int_{T_pM} (4\pi\tau)^{-\frac{n}{2}} e^{-\ell(\mathcal{L}\exp_\tau(v),\tau)} \mathcal{J}(v,\tau)\chi_\tau(v)dv,$$

where $\mathcal{J}(v,\tau) = \det\ d(\mathcal{L}_{\exp_\tau})_v$ is the Jacobian factor in the change of variable and χ_τ is a cutoff function related to the $\mathcal{L}$-cut locus of p.

We first show that for each v, the expression

$$-\frac{n}{2}\log(\tau) - \ell(\mathcal{L}\exp_\tau(v),\tau) + \log \mathcal{J}(v,\tau)$$

is nonincreasing in τ. Let γ be the $\mathcal{L}$-geodesic with initial vector $v \in T_pM$. From (2) and (3),

$$\text{(8)}\quad \frac{d\ell(\gamma(\tau),\tau)}{d\tau}\Big|_{\tau=\bar{\tau}} = -\frac{1}{2\bar{\tau}}\ell(\gamma(\bar{\tau}) + \frac{1}{2}(R(\gamma(\bar{\tau}) + |X(\bar{\tau})|^2) = -\frac{1}{2}\bar{\tau}^{-\frac{3}{2}}K.$$

Next, let $\{Y_i\}_{i=1}^n$ be a basis for the Jacobi fields along γ that vanish at $\tau = 0$. We can write

$$\log \mathcal{J}(v,\tau)^2 = \log\ \det((d(\mathcal{L}\exp_\tau)_v)^* d(\mathcal{L}\exp_\tau)_v = \log\ \det(S(\tau)) + const.,$$

where S is the matrix

$$S_{ij}(\tau) = \langle Y_i(\tau), Y_j(\tau)\rangle.$$

Then

$$\frac{d\log \mathcal{J}(v,\tau)}{d\tau} = \frac{1}{2}\mathrm{Tr}\Big(S^{-1}\frac{dS}{d\tau}\Big).$$

To compute the derivative at $\tau = \bar{\tau}$, we can choose a basis so that $S(\bar{\tau}) = I_n$, i.e. $\langle Y_i(\bar{\tau}), Y_j(\bar{\tau})\rangle = \delta_{ij}$ then using (7) and the same method as in ([2], 21),

$$\text{(9)}\qquad \frac{dln\mathcal{J}(v,\tau)}{d\tau}\Big|_{\tau=\bar{\tau}} = \frac{1}{2}\sum_{i=1}^{n} \frac{d|Y_i|^2}{d\tau}\Big|_{\tau=\bar{\tau}} \leq \frac{n}{2\bar{\tau}} - \frac{1}{2}\bar{\tau}^{-\frac{3}{2}}K.$$

From (8) and (9), we deduce that

$$\tau - \frac{n}{2} e^{-\ell(\mathcal{L}\exp_\tau(v),\tau)}\mathcal{J}(v,\tau)$$

is nonincreasing in τ. Finally, if $\tau \leq \tau'$ then $\Omega_{\tau'} \subset \Omega_\tau$, so $\chi_\tau(v)$ is nonincreasing in τ. Hence $\tilde{V}(\tau)$ is nonincreasing in τ. $\square$

2.3. **Perelman's $\mathcal{W}$ functional.**

Definition 2.2. Perelman's $\mathcal{W}$ functional is defined by

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dv$$

where g_{ij} is a *Riemannian* metric, f is a smooth function on M, and τ is a positive scalar parameter. The functional $\mathcal{W}$ is invariant under simultaneous scaling of τ and g_{ij} (or equivalently the parabolic scaling), and invariant under diffeomorphism. Namely, for any positive number a and any diffeomorphism φ

$$\mathcal{W}(a\varphi^* g_{ij}, \varphi^* f, a\tau) = \mathcal{W}(g_{ij}, f, \tau).$$

Now we set

$$\mu(g_{ij}, \tau) = \inf\{\mathcal{W}(g_{ij}, f, \tau) | f \in \mathcal{C}^\infty(M), \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f} dV = 1\}.$$

Note that if we let $u = e^{-f/2}$, then the functional $\mathcal{W}$ can be expressed as

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2](4\pi\tau)^{-\frac{n}{2}} dV$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$ becomes $\int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1$.

Lemma 2.3.

$$\mu(g_{ij}, \tau) = \inf\{\mathcal{W}(g, f, \tau) | f \in \mathcal{C}^\infty(M), \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f} dV = 1\}$$

is finite and nondecreasing, where g_{ij} is a Riemannian metric, f is a smooth function on M and τ is a positive scalar parameter.

In order to prove this ,we will use the following Lemmas. Let Ω be a domain (open connected) in M and $C_0^\infty(\Omega)$ be the space of real valued infinitely differentiable functions, compactly supported in Ω. The Sovolev space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\| f \|^2 = \int_\Omega f^2 + \int_\Omega |\nabla f|^2,$$

where the integrations use the volume element arising from the Riemannian structure and $\nabla f, |\nabla f|^2$, are also determined by the Riemannian structure. Let $\triangle$ be the Laplace-Beltrami operator. For any real valued measurable function f on Ω, we say that $f \in L^{p^+}(\Omega)$ if $|f|^q$ is integrable on Ω for some $q > p$. We use $\| f \|_q$ to denote the $L^q(\Omega)$ norm of f. Let H be a non-negative measurable function on Ω, for which $\log H \in L^{\frac{n}{2}+}$. Let ρ be a positive real number and define $a_\rho(H)$ as the infimum of

$$\int (\rho|\nabla|^2 - f^2 \log f^2 + f^2 \log H) \text{ for f} \in \mathrm{H}_0^1,$$

subject to the proviso $\int f^2 = 1$. (The integral is well defined, since $f \in H_0^1 \Rightarrow f \in L^{2n/(n-2)}$.)

Lemma 2.4. *$\int(\rho|\nabla f|^2 + f^2 \log H)$ is bounded below if $f \in H_0^1$ and $\int f^2 = 1$.*

Proof. We may write $\log H = U + V$, where U has its $L^{\frac{n}{2}}$ norm as small as we want, say

$$\| U \| \frac{n}{2} \leq \epsilon$$

and V is bounded, say $|V| \leq D$. By the Sobolev theory, there exists a constant C independent of f such that

$$\| f \|_{\frac{2n}{(n-2)}} \leq C \| f \| .$$

But now

$$\int f^2 U \leq \| f \|^2_{\frac{2n}{(n-2)}} \| V \|_{\frac{n}{2}} \leq C^2 \in \| f \|^2 .$$

So

$$\begin{aligned} &\int (\rho|\nabla f|^2 + f^2 \log H) \\ &= \rho \| f \|^2 - \rho + \int f^2 V \geq \rho \| f \|^2 - \rho - D - C^2\epsilon \| f \|^2, \end{aligned}$$

which is bounded below as long as $C^2\epsilon < \rho$.

□

Lemma 2.5. *For $f \in H_0^1, \int f^2 = 1$, the functional $\int(\rho|\nabla f|^2 - f^2 \log f^2)$ is bounded below.*

Proof. Pick ϵ satisfying $0 < \epsilon < \frac{2}{(n-2)}$. Then by Jensen's inequality for the logarithm,

$$\int f^2 \log f^2 = \frac{1}{\epsilon} \int f^2 \log |f|^{2\epsilon} \leq (2 + 2\epsilon) \log \| f \|_{2+2\epsilon},$$

and by the Sobolev theory,

$$\| f \|_{2+2\epsilon} \leq C \| f \| .$$

Hence

$$\int(\rho|\nabla f|^2 - f^2\log f^2) = \rho \parallel f \parallel^2 -\rho - \int f^2 \log f^2$$
$$\geq \rho \parallel f \parallel^2 -\rho - \frac{(2+2\epsilon)}{\epsilon}\log C \parallel f \parallel,$$

which is bounded below since $\parallel f \parallel \geq 1$. $\square$

Lemma 2.6. *$a_\rho(H)$ is finite.*

Proof. It follows from

$$\int(\rho|\nabla f|^2 - f^2\log f^2 + f^2 \log H)$$
$$= \int(\frac{\rho}{2}|\nabla f|^2 + f^2\log H) + \int(\frac{\rho}{2}|\nabla f|^2 - f^2\log f^2).$$

$\square$

The following two results are proved in [3]:

Theorem 2.7. *$a_\rho(H)$ is an attained minimum.*

Theorem 2.8. *A minimizer f for $a_\rho(H)$ is continuous on $\bar{\Omega}$.*

Any $f \in H_0^1$ with $\int f^2 = 1$ which attains the minimum of $a_\rho(H)$ will be called a minimizer for $a_\rho(H)$.

Remark 1. We can show that $\mu(g_{ij}, \tau)$ is achieved by a smooth minimizer f from Theorem 2.8.

Proof of Lemma 2.3. We use Lemma 2.6 to show that $\mu(g_{ij}, \tau - t)$ is finite. We can get that $\mu(g_{ij}(t), \tau - t)$ is nondecreasing along the Ricci flow follows from [6, Corollary 1.5.9.] $\square$

3. No local collapsing theorems

Definition 3.1. Let κ, γ be two positive constants and let $g_{ij}(t), 0 \leq t < T$, be a solution to the Ricci flow on an n-dimensional manifold M. We call the solution $g_{ij}(t)$ is κ-noncollapsed at $(x_0, t_0) \in M \times [0, T)$ on the scale γ if it satisfies the following property:
Whenever

$$|R_m|(x, t) \leq r^{-2}$$

for all $x \in B_{t_0}(x_o, r)$ and $t \in [t_0 - r^2, t_0]$, we have

$$Vol(B_{t_0}(x_0, r))) \geq \kappa r^n.$$

Here $B_{t_0}(x_0, r)$ is the geodesic ball centered at $x_0 \in M$ and of radius r with respect to the metric $g_{ij}(t_0)$. We now use the $\mathcal{W} - functional$ to prove the no local collapsing theorem.

3.1. No local collapsing theorem I.

Theorem 3.2 (No local collapsing theorem I)**.** *Suppose that $M$is a compact Riemannian $manifold$ and $g_{ij}(t), 0 \leq t < T < \infty$, is a solution to the Ricci $flow$. Then the solution $g_{ij}(t)$ is κ-noncollapsed at $(x_0, t_0) \in M \times [0, T)$ on the scale $\gamma \in (0, \sqrt{T}]$.*

Proof. We want to prove

$$Vol_{t_0}(B_{t_0}(x_0, a)) \geq \kappa a^n \tag{10}$$

for all $0 < a \leq r$. Recall that

$$\mu(g_{ij}, \tau) = \inf\{W(g_{ij}, f, \tau) | \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dv = 1\}. \tag{11}$$

Note that: Since $\mu(g_{ij}(t), \tau - t)$ is nondecreasing in t by Lemma 2.3, if we assume that $g_{ij}(t) = g_{ij}(0)$ for all $t \in \mathbb{R}$ then $\mu(g(0), 2T) \leq \mu(g(0), \tau)$ for all $0 \leq \tau \leq 2T$.

Let f be the minimizer of $\mu(g(0), 2T)$. By Theorem 2.8, we know that f is smooth. Since M is compact,we get $|\mu(g(0), 2T)| \leq a$ for some $a \in \mathbb{R}$. Let

$$\mu_0 = \inf_{0 \leq \tau \leq 2T} \mu(g_{ij}(0), \tau) \geq \mu(g_{ij}(0), 2T) > -\infty.$$

By Lemma 2.3, we have

$$\mu(g_{ij}(t_0), b) \geq \mu(g_{ij}(0), t_0 + b) \geq \mu_0 \tag{12}$$

for $0 < b \leq r^2$. Let $0 < \zeta \leq 1$ be a positive smooth function on $\mathbb{R}$ where $\zeta(s) = 1$ for $|s| \leq \frac{1}{2}, |\zeta'|^2/\zeta \leq 20$ and $\zeta(s)$ is very close to zero for $|s| \geq 1$. Define a function f on M by

$$(4\pi r^2)^{-\frac{n}{2}} e^{-f(x)} = e^{-c}(4\pi r^2)^{-\frac{n}{2}} \zeta(\frac{d_{t_0}(x, x_0)}{r}),$$

where the constant c is chosen so that $\int_M (4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} = 1$. Then we use (12) to get
(13)
$$\mathcal{W}(g_{ij}(t_0), f, r^2) = \int_M [r^2(|\nabla f|^2 + R) + f - n](4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \geq \mu_0.$$

By (13), we get

$$(c-n) + \int_M [r^2(|\nabla f_0|^2 + R) - \log \zeta](4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \geq \mu_0. \tag{14}$$

Since

$$\begin{aligned}
&\int_M (r^2 R)(4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \\
&= \int_{B_{t_0}(x_0,r)} (r^2 R)(4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} + \int_{M\setminus B_{t_0}(x_0,r)} r^2 R(4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \\
&= \int_{B_{t_0}(x_0,r)} (4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} + \int_{M\setminus B_{t_0}(x_0,r)} r^2 R(4\pi r^2)^{-\frac{n}{2}} e^{-c}\zeta dv_{t_0} \leq 2,
\end{aligned}$$

$$|\nabla f_0|^2 = |\nabla(-\log \zeta)|^2 = \frac{(\zeta')^2}{\zeta^2} \cdot \frac{1}{r^2}$$

and

$$\begin{aligned}
&\int_M (\frac{(\zeta')^2}{\zeta^2} - \log \zeta)(4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \\
=& \int_{B_{t_0}(x_0,r)} (\frac{(\zeta')^2}{\zeta} - \zeta \log \zeta)(4\pi r^2)^{-\frac{n}{2}} e^{-c} dv_{t_0} \\
\leq& \ 2(20+e)(4\pi r^2)^{-\frac{n}{2}} e^{-c} Vol(B_{t_0}(x_0,r),
\end{aligned}$$

thus (13) is reduced to

$$c \geq -2(20+e)\frac{Vol(B_{t_0}(x_0,r))}{Vol(B_{t_0}(x_0,\frac{r}{2}))} + (n-2) + \mu_0.$$

Note that

$$\begin{aligned}
1 &= \int_M (4\pi r^2)^{-\frac{n}{2}} e^{-c} \zeta(\frac{d_{t_0}(x,x_0)}{r}) dv_{t_0} \\
&\geq \int_{B_{t_0}(x_0,\frac{r}{2})} (4\pi r^2)^{-\frac{n}{2}} e^{-c} dv_{t_0} \\
&= (4\pi r^2)^{-\frac{n}{2}} e^{-c} Vol(B_{t_0}(x_0,\frac{r}{2})).
\end{aligned}$$

Note also that

$$\begin{aligned} 1 &= \int_M (4\pi r^2)^{-\frac{n}{2}} e^{-f} dv_{t_0} \\ &= \int_M (4\pi r^2)^{-\frac{n}{2}} e^{-c} \zeta(\frac{d_{t_0}(x,x_0)}{r}) dv_{t_0} \\ &\leq 2\int_{B_{t_0}(x_0,r)} e^{-c}(4\pi r^2)^{-\frac{n}{2}} dv_{t_0}. \end{aligned}$$

Thus we get

$$Vol(B_{t_0}(x_0,r)) \geq \frac{1}{2} e^c (4\pi r^2)^{\frac{n}{2}}.$$

Let us set

$$\mathcal{K} = \min\{\frac{1}{2}\exp(-2(20+e)3^{-n} + (n-2) + \mu_0), \frac{1}{2}\alpha_n\}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. Then we obtain

$$\begin{aligned} Vol(B_{t_0}(x_0,r))) &\geq \frac{1}{2} e^c (4\pi r^2)^{\frac{n}{2}} \\ &\geq \frac{1}{2}(4\pi)^{\frac{n}{2}} exp(-2(20+e)3^{-n} + (n-2) + \mu_0) r^n \\ &\geq \mathcal{K} r^n \end{aligned}$$

provided that $Vol(B_{t_0}(x_0, \frac{r}{2})) \geq 3^{-n} Vol(B_{t_0}(x_0,r))$.

The above argument also works for any smaller radius $a \leq r$. Thus we have proved

$$Vol(B_{t_0}(x_0,a)) \geq \mathcal{K} a^n \tag{15}$$

for $a \in (0,r]$ and $Vol(B_{t_0}(x_0, \frac{a}{2})) \geq 3^{-n} Vol(B_{t_0}(x_0,a))$. Now we argue by contradiction to prove the assertion (9) for any $a \in (0,r]$.

Suppose $(10)_a$ fails for some $a \in (0,r]$. Then by (15) we have

$$\begin{aligned} Vol(B_{t_0}(x_0, \frac{a}{2})) &< 3^{-n} Vol(B_{t_0}(x_0,a)) \\ &< 3^{-n} \mathcal{K} a^n \\ &< \mathcal{K}(\frac{a}{2})^n. \end{aligned}$$

Thus $(10)_{\frac{a}{2}}$ also fails. By induction we get

$$Vol(B_{t_0}(x_0, \frac{a}{2^k})) < \mathcal{K}(\frac{a}{2^k})^n$$

for all $k \geq 1$. But this contradicts to the limit

$$\lim_{\kappa\to\infty} Vol(B_{t_0}(x_0, \frac{a}{2^k}))/(\frac{a}{2^k})^n = \alpha_n.$$

$\square$

3.2. **No Local Collapsing Theorem II..** In this section we will use a cut-off argument to extend the no local collapsing theorem to any complete solution with bounded curvature. In some sense, the second no local collapsing theorem gives a good relative estimate of the volume element for the Ricci flow.

Lemma 3.3 (Perelman)**.** *Suppose we have a solution to the Ricci flow* $(g_{ij})_t = -2R_{ij}$*.*
(a) Suppose that $Ric(x, t_0) \leq (n-1)K$ *for* $\mathrm{dist}_{t_0}(x, x_0) < r_0$*. Then the distance function* $d(x,t) = \mathrm{dist}_t(x, x_0)$ *satisfies at* $t = t_0$ *outside* $B(x_0, r_0)$ *the differential inequality*

$$d_t - \Delta d \geq -(n-1)(\frac{2}{3}Kr_0 + r_0^{-1}).$$

(The inequality is understood in the barrier sense when necessary.)
(b) Suppose $Ric(x, t_0) \leq (n-1)K$ *when*

$$\mathrm{dist}_{t_0}(x, x_0) < r_0 \quad or \quad \mathrm{dist}_{t_0}(x, x_1) < r_0.$$

Then at $t = t_0$*,*

$$\frac{d}{dt}\mathrm{dist}_t(x_0, x_1) \geq -2(n-1)(\frac{2}{3}Kr_0 + r_0^{-1}).$$

Proof of Lemma (a). Let $r : [0,\ d(x, t_0)] \to M$ be a shortest normal geodesic from x_0 to x with respect to the metric $g_{ij}(t_0)$. Let $\{X,\ e_1, \ldots,\ e_{n-1}\}$ be an orthonormal basis of $T_{x_0}M$. Extend this basis parallelly along γ to form a parallel orthonormal basis $\{\ X,\ e_1, \cdots,\ e_{n-1}\}$ along γ. We consider that x and x_0 are not conjugate to each other in the metric $g_{ij}(t_0)$.

Let $X_i(s)$, $i = 1, \ldots,\ n-1$, be the Jacobi fields along γ such that $X_i(0) = 0$, $X_i(d(x,\ t_0)) = e_i(d(x, t_0))$ and $[X_i, X] = 0$ for $i = 1, \ldots,\ n-1$. Then we have (see for example [4])

$$\Delta_{t_0}(x, x_0) = \sum_{i=1}^{n-1} \int_0^{d(x,t_0)} (|\nabla_X X_i|^2 - R(X, X_i, X, X_i))ds$$

Define vector fields Y_i, $i = 1, \ldots, n-1$, along γ as follows:

$$Y_i(s) = f(s)e_i(s),$$

where $f(s) = \frac{s}{r_0}$ if $s \in [0, r_0]$ and $f(s) = 1$ if $s \in [r_0, d(x, t_0)]$ then we can see that

$$Y_i(0) = 0 = X_i(0),$$
$$Y_i(d(x,t_0)) = e_i(d(x,t_0)) = X_i(d(x,t_0)).$$

Thus by using the standard index comparison theorem (see for example [5]) we have

$$\begin{aligned}
&\nabla d_{t_0}(x, x_0) \\
&= \sum_{i=1}^{n-1} \int_0^{d(x,t_0)} (|\nabla_X \nabla X_i|^2 - R(X, X_i, X, X_i))ds \\
&\leq \sum_{i_1}^{n-1} \int_0^{d(x,t_0)} (|\nabla_X Y|^2 - R(X, Y_i, X, Y_i))ds \\
&= \int_0^{r_0} \frac{1}{r_0^2}(n - 1 - s^2 Ric(X, X))ds + \int_{r_0}^{d(X,t_0)} (-Ric(X, X))ds \\
&= -\int_r Ric(X, X) + \int_0^{r_0} (\frac{(n-1)}{r_0^2} + (1 - \frac{s^2}{r_0^2} Ric(X, X))ds \\
&\leq -\int_r Ric(X, X) + (n-1)(\frac{2}{3} K r_0 + r_0^{-1}).
\end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial t} d_t(x, x_0) = \frac{\partial}{\partial t} \int_0^{d(x,t_0)} \sqrt{g_{ij} X^i X^j} ds = -\int_r Ric(X, X)ds.$$

Thus we get the desired result. □

Proof of Lemma (b). The proof is divided into three cases.

Case 1: $d_{t_0}(x_0, x_1) \geq 2r_0$.

Let γ be a normalized minimal geodesic from x_0 to x_1 and $X(s) = \frac{dr}{ds}$. If any piecewise-smooth vector field V along γ that vanishes at the endpoints, the second variation formula gives

$$\int_0^{d(x_0,x_1)} (|\nabla_X V|^2 + \langle R(V, X)V, X \rangle)ds \geq 0.$$

Let $e_i(s)_{i=1}^{n-1}$ be a parallel orthonormal frame along γ that is perpendicular to X. Put $V_i(s) = f(s)e_i(s)$, where

$$f(s) = \begin{cases} \frac{s}{r_0} & \text{if} \quad 0 \leq s \leq r_0, \\ 1 & \text{if} \quad r_0 \leq s \leq d(x_0, x_1) - r_0, \\ \frac{d(x_0,x_1)-s}{r_0} & \text{if} \quad d(x_0, x_1) - r_0 \leq s \leq d(x_0, x_1). \end{cases}$$

Then $|\nabla_X V_i| = |f'(s)|$ and

$$\begin{aligned}
&\int_0^{d(x_0,x_1)} \langle R(V_i,X)V_i,X\rangle ds \\
&= \int_0^{r_0} \frac{s^2}{r_0^2}\langle R(e_i,X)e_i,X\rangle ds + \int_{r_0}^{d(x_0,x_1)-r_0} \langle R(e_i,X)e_i,X\rangle ds \\
&\quad + \int_{d(x_0,x_1)-r_0}^{d(x_0,x_1)} \frac{(d(x_0,x_1)-s)^2}{r_0^2}\langle R(e_i,X)e_i,X\rangle ds.
\end{aligned}$$

Then

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n-1}\int_0^{d(x_0,x_1)} (|\nabla_X V_i|^2 + \langle R(V_i,X)V_i,X\rangle) ds \\
&= \frac{2(n-1)}{r_0} - \int_0^{d(x_0,x_1)} Ric(X,X)ds + \int_0^{r_0}(1-\frac{s^2}{r_0^2})Ric(X,X)ds \\
&+ \int_{d(x_0,x_1)-r_0}^{d(x_0,x_1)} (1-\frac{(d(x_0,x_1)-s)^2}{r_0^2})Ric(X,X)ds.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\frac{d}{dt}\mathrm{dist}_t(x_0,x_1) &= -\int_0^{d(x_0,x_1)} Ric(X,X)ds \\
&\geq -\frac{2(n-1)}{r_0} - \int_0^{r_0}(1-\frac{s^2}{r_0^2})Ric(X,X)ds \\
&- \int_{d(x_0,x_1)-r_0}^{d(x_0,x_1)} (1-\frac{(d(x_0,x_1)-s)^2}{r_0^2})Ric(X,X)ds \\
&\geq -\frac{2(n-1)}{r_0} - 2(n-1)K\frac{2}{3}r_0 \\
&= -2(n-1)(\frac{2}{3}Kr_0 + r_0^{-1})
\end{aligned}$$

Case 2: $2\sqrt{\frac{3}{2K}} \leq d_{t_0}(x_{0,1}) \leq 2r_0$.

Let $r_1 = \sqrt{\frac{3}{2K}}$ and applying Case(1) with r_0 replaced by r_1, we get

$$\begin{aligned}
\frac{d}{dt}(d_t(x_0,x_1)) &\geq -2(n-1)(\frac{2}{3}Kr_1 + r_1^{-1}) \\
&\geq -2(n-1)(\frac{2}{3}Kr_0 + r_0^{-1})
\end{aligned}$$

Case 3: $d_{t_0}(x_0,x_1) \leq \min\{2\sqrt{\frac{3}{2K}}, 2r_0\}$.

In this Case,

$$\int_0^{d(x_1,t_0)} Ric(X,X)ds \leq (n-1)K2\sqrt{\frac{3}{2K}} = (n-1)\sqrt{6K}$$

and

$$2(n-1)(\frac{2}{3}Kr_0 + r_0^{-1}) \geq (n-1)\sqrt{\frac{32}{3}K}.$$

□

Theorem 3.4 (No local collapsing theorem II). *For any $A > 0$ there exists $\kappa = \kappa(A) > 0$ with the following property: if $g_{ij}(t)$ is a complete solution to the Ricci flow on $0 \leq t \leq r_0^2$ with bounded curvature and satifying*

$$|Rm|(x,t) \leq r_0^{-2} \quad \text{on} \quad B_0(x_0,r_0) \times [0,r_0^2]$$

and

$$Vol(B_0(x_0,r_0)) \geq A^{-1}r_0^n,$$

then $g_{ij}(t)$ is κ-noncollapsed on all scales less than r_0 at every point (x,r_0^2) with $d_{r_0}^2(x,x_0) \leq Ar_0$.

Proof. From the evolution equation of the *Ricci flow.* we know that the metrics $g_{ij}(\cdot,t)$ are equivalent to each other on $B_0(x_0,r_0) \times [0,r_0^2]$. Thus without loss of generality, we may assume that the curvature of the solution is uniformly bounded for all $t \in [0,r_0^2]$ and all points in $B_t(x_0,r_0)$. Fix a point $(x,r_0^2) \in M \times r_0^2$. By scaling we may assume $r_0 = 1$. We may also assume $d_1(x,x_0) = A$. Let $p = x, \bar{\tau} = 1-t$ and consider Perelman's reduced volume

$$\tilde{V}(\bar{\tau}) = \int_M (4\pi\bar{\tau})^{-\frac{n}{2}} e^{-\ell(q,\bar{\tau})} dV_{1-\bar{\tau}}(q),$$

where

$$\ell(q,\bar{\tau}) = \inf\{\frac{1}{2\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}} \sqrt{\tau}(R+|\dot{\gamma}|^2)d\tau \;\Big|\; \gamma : [0,\bar{\tau}] \longrightarrow M \text{ with } \gamma(0)=p, \gamma(\bar{\tau})=q\}$$

is the Li-Yau-Perelman distance.

We argue by contradiction. Suppose for some $0 < r < 1$ we have

$$|Rm|(y,t) \leq r^{-2}$$

whenever $y \in B_1(x,r)$ and $1-r^2 \leq t \leq 1$, but $\epsilon = r^{-1}Vol_1(B_1(x,r))^{\frac{1}{n}}$ is very small. Then arguing as in the proof of the no local collapsing theorem I (Theorem 3.3.2) [6], we see that Perelman's reduced volume

$$\tilde{V}(\epsilon r^2) \leq 2\epsilon^{\frac{n}{2}}.$$

On the other hand, from the monotonicity of Perelman's reduced volume we have

$$(4\pi)^{-\frac{n}{2}} \int_M e^{-\ell(q,1)} dV_0(q) = \tilde{V}(1) \leq \tilde{V}(\epsilon r^2).$$

Thus once we bound the function $\ell(q,1)$ over $B_0(x_0,1)$ from above, we will get the desired contradiction and will prove the theorem.

For any $q \in B_0(x_0,1)$, exactly as in the proof of the no local collapsing theorem I, we choose a path $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = x,\ \gamma(1) = q,\ \gamma(\frac{1}{2}) = y \in B_{\frac{1}{2}}(x_0,a)$ and $\gamma(\tau) \in B_{1-\tau}(x_0,1)$ for $\tau \in [\frac{1}{2}]$ such that

$$\mathcal{L}(\gamma|\ [0,\frac{1}{2}]\ ) = 2\sqrt{\frac{1}{2}}\ell(y,\frac{1}{2}) \quad (= L(y,\frac{1}{2})).$$

Now

$$\mathcal{L}(\gamma|\ [\frac{1}{2},1]) = \int_{\frac{1}{2}}^{1} \sqrt{\tau}(R(\gamma(\tau),1-\tau) + |\dot{\gamma}(\tau)|_{g_{ij}(1-\tau)})d\tau$$

is bounded from above by a uniform constant since all geometric quantities in g_{ij} are uniformly bounded on $\{(y,t)|\ t \in [0,1/2],\ y \in B_{t(x_0,1)}\}$ (where $t \in [0,1/2]$ is equivalent to $\tau \in [1/2,1]$). Thus all we need is to estimate the minimum of $\ell(\cdot,\frac{1}{2})$ or equivalently $\bar{L}(\cdot,\frac{1}{2}) = 4\frac{1}{2}\ell(\cdot,\frac{1}{2})$ in the ball $B_{\frac{1}{2}}(x_0,a)$.

Recall that $\bar{L}$ satisfies the differential inequality

$$\frac{\partial \bar{L}}{\partial \tau} + \Delta \bar{L} \leq 2n.$$

We will use this in a maximum principle argument.

Let $\phi = \phi(u)$ be a smooth function that equals 1 on $(-\infty,a)$, equals infinity on (a,∞) and is increasing on (t,a), where $(a,t \in \mathbb{R}, a > t)$ with

$$2(\phi')^2/\phi - \phi'' \geq (2A + 100n)\phi' - C(A)\phi$$

for some constant $C(A) < \infty$. To satisfy this equation, it suffices to take

$$\phi(u) = \frac{1}{e^{(2A+100n)(\frac{1}{10}-u)} - 1}$$

for u near $\frac{1}{10}$. Now put

$$h(y,t) = \phi(d(y,t) - A(2t-1))(\bar{L}(y,1-t) + 2n + 1),$$

where $d(y,t) = \mathrm{dist}_t(y,x_0)$. Since the scalar curvature R evolves by

$$\frac{\partial R}{\partial t} = \triangle R + 2|Rc|^2 \geq \Delta R + \frac{2}{n}R^2,$$

we can apply the maximum principle to deduce

$$R(x,t) \geq -\frac{n}{2t} \quad \text{for} \quad t \in (0,1] \quad \text{and} \quad x \in M.$$

Thus for $\bar{\tau} = 1 - t \in [0, \frac{1}{2}]$,

$$\begin{aligned} \bar{L}(\cdot, \bar{\tau}) &= 2\sqrt{\bar{\tau}} \int_0^{\bar{\tau}} \sqrt{\tau}(R + |\dot{\gamma}|^2) d\tau \\ &\geq 2\sqrt{\bar{\tau}} \int_0^{\bar{\tau}} \sqrt{\tau}(-\frac{n}{2(1-\tau)}) d\tau \\ &\geq 2\sqrt{\bar{\tau}} \int_0^{\bar{\tau}} \sqrt{\tau}(-n) d\tau \\ &> -2n. \end{aligned}$$

That is

$$\bar{L}(\cdot, 1-t) + 2n + 1 \geq 1, \text{ for } t \in [\frac{1}{2}, 1].$$

Also $\min_y h(y,1) \leq h(x,1) = \phi(\mathrm{dist}_1(x,x_0) - A)(2n+1) = 2n+1$.

As ϕ is infinite on (a,∞) and $\bar{L}(., \frac{1}{2}) + 2n + 1 \geq 1$, the minimum of $h(\cdot, \frac{1}{2})$ is achieved at some y satisfying $d(y, \frac{1}{2}) \leq a$. The calculations in Lemma 3.3 (a) give

$$\Box h \geq -(2n + C(A))h$$

at a minimum point of h, where $\Box = \partial_t - \triangle$. Then

$$\frac{d}{dt} h_{min}(t) \geq -(2n + C(A)) h_{min}(t),$$

so

$$h_{min}(\frac{1}{2}) \leq e^{n + \frac{C(A)}{2}} h_{min}(1) \leq (2n+1) e^{n + \frac{C(A)}{2}}.$$

It follows that

$$\min_{y: d(y, \frac{1}{2}) \leq a} \bar{L}(y, \frac{1}{2}) + 2n + 1 \leq (2n+1) e^{n + \frac{C(A)}{2}}.$$

This implies the theorem. $\Box$

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