國立台灣大學數學研究所碩士論文
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A DISCUSSION OF DUALITY PROBLEM IN THE CONTEXT OF INTERSECTION COHOMOLOGY

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A Discussion of Duality Problem in the Context of Intersection Cohomology（題 目），

係由本人指導撰述，同意提付審查。

指導教授 王金龍（簽章）

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中 文 摘 要

眾所皆知，在一般非特異的拓樸空間上，例如緊緻流形， Poincare duality 定理成立，但是在特異空間上則否。Goresky 和 MacPherson 在其論文中定義了一種同調群理論，即所謂的 Intersection 同調群（或是透過 dual，考慮 Intersection 上同調群）。他們證明只要特異空間滿足一些基本的性質，例如具有 Whitney stratification，那麼在這個同調群理論下， Poincare duality 定理仍然成立。他們的理論用到了三角剖分 (triangulation) 的辦法。後來 Deligne 用一種較抽象的建構，在 complexes of sheaves 所形成的 derived category 上運作。此 derived category 具有 triangulated category 的結構。Deligne 在其上定義了所謂的 perverse sheaves 以及 perverse extension。可以證明在適當的情況下此種 perverse sheaf 與 intersection complex (IC)等價。透過 Verdier duality 以及 dual perversity 的定義，很容易得知 Intersection 上同調群滿足 Poincare duality。本人研究此主題的動機乃是想進一步了解 Poincare duality 在 perverse sheaves 以及 intersection complex 之間所扮演的角色。文中包括了對於 Goresky 和 MacPherson 以及 Deligne 的建構之介紹，並給予一些例子與說明。

Ackowledgements

This is an expository work, which follows in a large part the *Faisceaux pervers* by A.A.Beilinson, J.Bernstein, and P.Deligne ([1]). Since the materials are very neatly written, in order to introduce the Deligne's construction of perverse sheaves, I have almost translated the required parts to make the idea clear. Refefences involved or maybe useful for the readers are cited at the end of this note.

In the course of my study of the related topics, I had chances to report them to an audience consisting of professors and scholars. Let me thank them first for their condescending to listen to me. Also I'd like to take advantage of this opportunity to express my heartily thanks to Professor A.N.Wang, Professor S.W.Yang and Professor I.H.Tsai for their kindly help and correction, and for the questions they posed. In particular, I have to thank my advisor, Assistant Professor C.L.Wang, under whose guidance I have learned a great deal, not only of the subject matter, but of the method of study, and viewpoints toward mathematics.

Finally let me thank all those who have helped me during my graduate study. I would like dedicate this work to my friends, my teachers and my family.

Contents

A DISCUSSION Of DUALITY PROBLEM
IN THE CONTEXT OF INTERSECTION COHOMOLOGY

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§0 Introduction

As is well-known, there exists for X a manifold a cohomology theory which gives the so-called Poincaré duality. What about the case when X is allowed to be a singular space (e.g. singular algebraic variety)? Does Poincaré duality persist in such context? In their papers, Goresky and MacPherson gave an affirmative answer to this question, when the space satisfies some conditions and a certain cohomology theory is chosen (i.e. the intersection cohomology). Goresky-MacPherson's first proof of Poincaré duality is by means of triangulation approach. Later Deligne gave a construction which can describe intersection cohomology systematically and put the theory in a more general context which of course has many applications. He used sheaf-theoretic approach, working on the *derived category*, where Verdier duality, a generalized form of Poincaré duality, is satisfied. The Deligne's construction and the operations in the derived category are also mentioned in Goresky-MacPherson's second paper on intersection homology.

In what follows, we will first give a brief description on Goresky-MacPherson's triangulation approach and their definition of intersection cohomology. Then, in order to introduce Deligne's construction, we have to set up some foundational background on the derived category. After that we introduce the triangulated category. In the meantime, we will give the definitions of the six operations and account the properties they satisfy. Using the six operations, we can describe the perverse sheaves and Deligne's perverse extension. All these being done, we will mention the self-duality for the middle perversity and we'd like to look closer the relations between the perverse sheaves and intersection cohomology. For example, we'd like to know under what conditions is the extension sheaf of some kinds just the intersection cohomology sheaf (e.g. could the condition requiring Poincaré duality be enough?).

§1 Intersection cohomology due to Goresky and MacPherson

For an oriented n-manifold X, Poincaré duality is satisfied, i.e. if $i + j = n$ then the pairing

$$H_i(X) \times H_j(X) \xrightarrow{\cap} H_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is nondegenerate when tensored with the rational numbers. (Here, ε is the augmentation which counts the points of a zero cycle according to their multiplicities.) In case X is singular, this pairing might not exist and Poincaré duality could fail, as can be seen from the following example.

Example 1.1: Let X be the suspension of the torus T (see Fig.1). Clearly $V \cap W$ is a nonvanishing zero-cycle, but since $W = \partial C$, the homology class $V \cap W$ does not only depend on the homology class of W. Thus the above pairing is not well-defined. Also by comparing the Betti numbers in complememtary dimensions, we can easily see that Poincaré duality is false (since

$$H_2(X) = H_1(T) = \mathbf{Z} \oplus \mathbf{Z}, H_1(X) = H_0(T) = \mathbf{Z}).$$

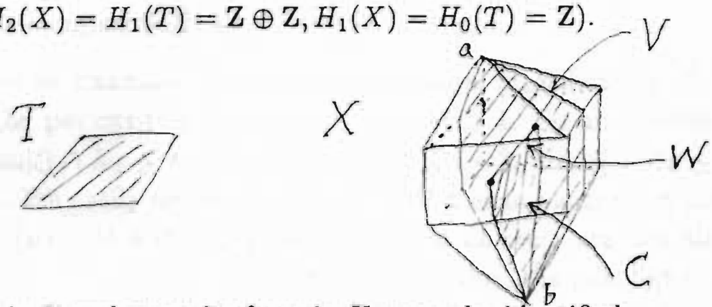


Fig.1. Opposite sides in T and opposite faces in X are to be identified.

To rescue the Poincaré duality on a singular space, Goresky and MacPherson modify the definition of homology in the usual sense by calculating the homology of a subclass of the chain complexes consisting of chains which are "allowable".

To state Goresky-MacPherson's construction, let's first give some definitions.

A *pseudomanifold* of dimension n is a compact space for which there exists a subspace Σ of dimension $\leq n-2$ such that $X - \Sigma$ is a non-singular oriented manifold of dimension n which is dense in X. For example, every complex irreducible algebraic variety has a piecewise linear structrure making it a pseudomanifold.

A *stratification* of a pseudomanifold of dimension n is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \ldots \supset X_1 \supset X_0$$

such that for every point $p \in X_i - X_{i-1}$ there exists a filtered space

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_i = \textit{ a point}$$

and for each j a homeomorphic map between $V_j \times \Delta^i$ and a neighborhood of p in X_j. Here Δ^i is the simplex of dimension i. We see that $X_i - X_{i-1}$ is non-singular of dimension i, which we will call the stratum of dimension i. Every pseudomanifold admits a stratification. In our discussion of this section, we will assume that X is a pseudomanifold of dimension n with a stratification.

A *perversity* is an $(n-1)$-tuple of integers $\bar{p} = (p_2, p_3, \ldots, p_n)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. We will use four special perversities $\bar{0} = (0, 0, ..., 0)$, $\bar{t} = (0, 1, ..., n-2)$ and if $n = 2n'$, $\bar{m} = (0, 0, 1, 1, ..., n'-2, n'-1)$ and $\bar{m'} = (0, 1, 1, 2, ..., n'-1, n'-1)$. If $p_k + q_k = r_k$ for all $2 \leq k \leq n$, we write $\bar{p} + \bar{q} = \bar{r}$. For example $\bar{m} + \bar{m'} = \bar{t}$. If i is an integer and p a perversity, a subspace Y of X is called $(\bar{p}, i)$-allowable if $Y \cap X_{n-k}$ is of dimension $\leq i - k + p_k$ for all k.

If T is a triangulation of X, we denote $C_i^T(X)$ the simplicial i-chains of X with respect to T. Let $C_i(X)$ be the union of $C_i^T(X)$ for all T modulo the following equivalent relations: $c \in C_i^T(X)$

and $c' \in C_i^{T'}(X)$ are equivalent if their canonical image in $C_i^{T''}(X)$ are the same for a common refinement T'' of T and T'. The $C_i(X)$s form a chain complex $C_*(X)$ whose homology is the homology of X. We can define the support $|c|$ of a chain $c \in C_i(X)$ as follows: if c is T- simplicial, $|c|$ is the union of all the i simplexes for which the coefficient of $c \in C_i^T(X)$ is non-zero.

If $\bar{p}$ is a perversity, and $IC_i^{\bar{p}}(X) \subset C_i(X)$ is the subgroup of all the chains such that $|c|$ is $(\bar{p}, i)$-allowable and $|\partial c|$ is $(\bar{p}, i-1)$-allowable. Clearly $IC_*^{\bar{p}}(X)$ is a subcomplex of $C^*(X)$.

Definition 1.2: The i-th intersection homology group of perversity $\bar{p}$, written as $IH_i^{\bar{p}}(X)$, is the i-th homology group of the complex $IC_*^{\bar{p}}(X)$.

Remark 1.3: In the case of Example 1.1, we have a natural stratification $X = X_3 \supset X_1 = X_0 = \{a, b\} = \Sigma$. Here the perversity p is: $p_2 = 0$, $p_3 = 0$ or 1. So an allowable i-chain must satisfy the conditioins: $\dim|\xi| \cap X_1 \le i-2$, $\dim|\partial\xi| \cap X_1 \le i-3$, $\dim|\xi| \cap X_0 \le i-3+p_3$, and $\dim|\partial\xi| \cap X_0 \le i-4+p_3$. We easily see that C and V in this example are not allowable 2-chains for the zero perversity $\bar{0}$. $\{a\}$, $\{b\}$ and $\{a,b\}$ (resp. the line $\overline{ab}$ etc.) are not allowable 0-chains (resp. 1-chains). By excluding the chains which are not allowable, the pairings of the intersection homologies are just the Poincaré duality pairings (with compact support) in $X - \Sigma$.

Proposition 1.4: $IH_i^{\bar{p}}(X)$ is a finite group which is independent of the choice of the stratification of X.

Proposition 1.5 (transversality): If $\bar{p} + \bar{q} = \bar{r}$, and $i + j - n = l$, there exists a unique product

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \longrightarrow IH_l^{\bar{r}}(X)$$

which satisfies $[c] \cap [c'] = [c \cap c']$ for all c and c' transversal in dimension, where $[c]$ denotes the class of c.

Theorem 1.6 (generalized Poincaré duality): If $\bar{p} + \bar{q} = \bar{t}$ and $i + j = n$, the pairing

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \longrightarrow IH_0^{\bar{t}}(X) \xrightarrow{\varepsilon} \mathbf{Z}$$

tensored with the rational numbers is non-degenerate.

§2 Derived categories and triangulated categories

When one considers the derived functors, the generalization of a category to the derived category naturally arises. Before getting into the construction of the derived category, let us mention some ideas of the derived category:

- An object X of an abelian category should be identified with all its resolutions.
- The main reason for such an identification is that some most important functors, such as Hom, $\otimes$, Γ can be redefined.
- The above redefinition of such functors as Γ, $\otimes$ and others makes some semi-exact functors in some sense "exact". However we have to point out that the notion of exactness in a derived category is not so obvious.

Lemma (Definition) 2.1: Let $K^{\cdot}$ and $L^{\cdot}$ be two complexes over R and $\Phi = (\varphi^i)$, where $\varphi^i : K^i \longrightarrow L^i$ are a collection of morphisms φ in R. Then the maps

$$h = \varphi d + d\varphi : K^{\cdot} \longrightarrow L^{\cdot}$$

form a morphism of complexes. The morphism h is said to be homotopic to 0 ($h \sim 0$). We have the following result:
If $f \sim g$ (i.e. $(f - g) \sim 0$), then $H^{\cdot}(f) = H^{\cdot}(g)$, where $H^{\cdot}(\cdot)$ is the mapping induced on the homology of a complex.

Definition 2.2: A morphism $f : K^{\cdot} \longrightarrow L^{\cdot}$ of complexes in an abelian category $\mathcal{A}$ is said to be a quasi-isomorphism if the corresponding homology morphism

$$H^n(f) : H^n(K^{\cdot}) \longrightarrow H^n(L^{\cdot})$$

is an isomorphism for any n.

Remark 2.3: a) For any two projective (injective) resolutions of an object there exists some quasi-isomorphism between them.

b) Any object X of an abelian category $\mathcal{A}$ can be considered as a complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

(with X at the 0-th place).

It is a kind of 0-complex (acyclic outside zero). The augmentation ε_X of a left resolution $P^{\cdot} \xrightarrow{\varepsilon_X}$ determines a quasi-isomorphism of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Hence the notion of a resolution is only a special case of a quasi-isomorphism.

c) Let $0^{\cdot}$ be the complex with all terms equal to the zero object of $\mathcal{A}$. Then a unique morphism $K^{\cdot} \longrightarrow 0^{\cdot}$ (and $0^{\cdot} \longrightarrow K^{\cdot}$) is a quasi-isomorphism iff $K^{\cdot}$ is acyclic.

Definition (Theorem) 2.4: Let $\mathcal{A}$ be an abelian category, $\mathcal{K}om(\mathcal{A})$ the category of complexes over $\mathcal{A}$. There exists a category $\mathcal{D}(A)$ and a functor $Q : \mathcal{K}om(\mathcal{A}) \longrightarrow \mathcal{D}$ with the following properties:

(a) $Q(f)$ is an isomorphism for any quasi-isomorphism (q.i. for short) f.

(b) Any functor $F : \mathcal{K}om(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ transforming q.i. into q.i. can be uniquely factorized through $\mathcal{D}(\mathcal{A})$, i.e. there exists a unique functor $G : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}$ with $F = G \circ Q$.

The category $\mathcal{D}(A)$ is called the *derived category* of the abelian category $\mathcal{A}$

Definition 2.5: A class of morphisms $S \subset \text{Mor}\mathcal{B}$ is said to be localizing if the following conditions are satisfied :

(a) S is closed under composition: $id_X \in S$ for every $X \in \text{Ob}\mathcal{B}$ and $s \circ t \in S$ for any $s, t \in S$ whenever the composition is defined.

(b) Extension conditions: for any $f \in \mathrm{Mor}\mathcal{B}$, $s \in S$ there exists $g \in \mathrm{Mor}\mathcal{B}$, $t \in S$ such that the following square

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{f} & Y \end{array} \quad \left(resp. \begin{array}{ccc} W & \xleftarrow{g} & Z \\ \uparrow t & & \uparrow s \\ X & \xleftarrow{f} & Y \end{array} \right)$$

is commutative.

(c) Let f, g be two morphisms from X to Y; the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

In general, q.i. in $\mathcal{K}om(\mathcal{A})$ do not form a localizing class. However, through constructing the category $\mathcal{K}(\mathcal{A})$ of complexes modulo homotopic equivalence, we can show that q.i. in this new category already form a localizing class of morphisms.

Definition 2.6: Let $\mathcal{A}$ be an abelian category. The homotopic category $\mathcal{K}^+(\mathcal{A})$ is defined as follows:

$\mathrm{Ob}\mathcal{K}(\mathcal{A}) = \mathrm{Ob}\mathcal{K}om(\mathcal{A})$, $\mathrm{Mor}\mathcal{K}(\mathcal{A}) = \mathrm{Mor}\mathcal{K}om(\mathcal{A})$ modulo homotopic equivalence. By $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, $\mathcal{K}^b(\mathcal{A})$ we denote the full subcategories of $\mathcal{K}(\mathcal{A})$ formed by complexes with the corresponding booundedness conditions.

Definition 2.7: Let $f : K^\cdot \to L^\cdot$ be a morphism of complexes. A *cone* of f is the following complex $C(f)$:

$$C(f)^i = K[1]^i \oplus L^i,$$

$$d_{C(f)}(k^{i+1}, \ell^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i).$$

For difference of taste, one may as well define a cone as

$$C_f^\cdot = L^\cdot \oplus K[1]^\cdot,$$

with analogous differential.

Remark 2.8: $C(f)$ can also be written as columns of height 2 and morphisms as matrices, so that

$$d_{C(f)} = \begin{pmatrix} d_{K[1]} & 0 \\ f[1] & d_L \end{pmatrix}.$$

Clearly $d^2_{C(f)} = 0$.

Definition 2.9: The *cylinder* $Cyl(f)$ of a morphism f is the following complex:

$$Cyl(f) = K^\cdot \oplus K[1]^\cdot \oplus L^\cdot,$$

$$d^i_{Cyl(f)}(k^i, k^{i+1}, \ell^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L \ell^i).$$

Lemma 2.10: For any morphism $f : K^\cdot \to L^\cdot$ there exists the following commutative diagram in KomA with exact rows:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & L^\cdot & \xrightarrow{\bar{\pi}} & C(f) & \xrightarrow{\delta=\delta(f)} & K[1]^\cdot & \longrightarrow & 0 \\ & & & & \downarrow \alpha & & & & & & \\ 0 & \longrightarrow & K^\cdot & \xrightarrow{\bar{f}} & Cyl(f) & \xrightarrow{\pi} & C(f) & \longrightarrow & 0 & & \\ & & \| & & \downarrow \beta & & & & & & \\ & & K^\cdot & \xrightarrow{f} & L^\cdot & & & & & & \end{array} \qquad (*)$$

It is functorial in f and has the following property: α and β are quasi-isomorphisms; moreover $\beta\alpha = id_L$ and $\alpha\beta$ is homotopic to $id_{Cyl(f)}$, so that $L^\cdot$ and $Cyl(f)$ are canonically isomorphic in the derived category.

Proof : a) the definition of morphisms in the first row and the verification that they commute with d:

$$\begin{array}{ccc} \ell^i & \xrightarrow{\pi} & (0,\ell^i) \\ \downarrow{\scriptstyle d_L} & & \downarrow{\scriptstyle d_{C(f)}} \\ d_L\ell^i & \xrightarrow{\pi} & (0, d_L\ell^i) \end{array}$$

$$\begin{array}{ccc} (k^{i+1},\ell^i) & \xrightarrow{\delta} & k^{i+1} \\ \downarrow{\scriptstyle d_{C(f)}} & & \downarrow{\scriptstyle d_{K[1]}} \\ (-d_K k^{i+1}, f(k^{i+1}) + d_L\ell^i) & \xrightarrow{\delta} & -d_K k^{i+1} \end{array}$$

The exactness of the first row is clear.

b) the definition of morphisms in the second row and the verification that they commute with d:

$$\begin{array}{ccc} (k^i, k^{i+1}, \ell^i) & \xrightarrow{\pi} & (k^{i+1},\ell^i) \\ \downarrow{\scriptstyle d_{Cyl(f)}} & & \downarrow{\scriptstyle d_{C(f)}} \\ (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L\ell^i) & \xrightarrow{\pi} & (-d_K k^{i+1}, f(k^{i+1}) + d_L\ell^i) \end{array}$$

$$\begin{array}{ccc} k^i & \xrightarrow{\bar f} & (k^i,0,0) \\ \downarrow{\scriptstyle d_K} & & \downarrow{\scriptstyle d_{Cyl(f)}} \\ d_K k^i & \xrightarrow{\bar f} & (d_K k^i, 0, 0) \end{array}$$

The exactness of the second row is clear.

c) the definition of morphisms α and β and the verification that they commute with d:

$$\begin{array}{ccc} \ell^i & \xrightarrow{d_L} & d_L\ell^i \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ (0,0,\ell^i) & \xrightarrow{d_{Cyl(f)}} & (0,0,d_L\ell^i) \end{array}$$

$$\begin{array}{ccc} (k^i, k^{i+1}, \ell^i) & \xrightarrow{d_{Cyl(f)}} & (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L\ell^i) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ f(k^i) + \ell^i & \xrightarrow{d_L} & f(d_K k^i) + d_L\ell^i \end{array}$$

The commutativity of the squares $\pi\alpha = \pi$, $\beta\bar f = f$ is clear.

d)The formula $\beta\alpha = id_L$ is clear.

Define $h^i : Cyl(f)^i \to Cyl(f)^{i-1}$ by the formula

$$h^i(k^i, k^{i+1}, \ell^i) = (0, k^i, 0).$$

We have

$$\alpha\beta(k^i, k^{i+1}, \ell^i) = (0, 0, f(k^i) + \ell^i),$$
$$d_{Cyl(f)} h^i(k^i, k^{i+1}, \ell^i) = (-k^i, -d_K k^i, f(k^i)),$$
$$h^i d_{Cyl(f)}(k^i, k^{i+1}, \ell^i) = (0, d_K k^i - k^{i+1}, 0).$$

Hence

$$\alpha\beta = id_{Cyl(f)} + (dh + hd).$$

Since $\alpha\beta$ and $\beta\alpha$ induce the identity mappings on cohomology, α and β are q.i.. QED

Proposition 2.11: An exact triple of complexes in $\mathcal{K}om(\mathcal{A})$ is q.i. to the middle row of an appropriate diagram of the form (*).

Proof : Let

$$0 \longrightarrow K^{\cdot} \xrightarrow{f} L^{\cdot} \xrightarrow{g} M^{\cdot} \longrightarrow 0$$

be an exact triple. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\cdot} & \xrightarrow{f} & L^{\cdot} & \xrightarrow{g} & M^{\cdot} & \longrightarrow & 0 \\ & & \| & & \uparrow \beta & & \uparrow \gamma & & \\ 0 & \longrightarrow & K^{\cdot} & \xrightarrow{\bar{f}} & Cyl(f) & \xrightarrow{\bar{\pi}} & C(f) & \longrightarrow & 0 \end{array} \quad , \quad (**)$$

where β is the map in (*) and γ is defined by $\gamma(k^{i+1}, \ell^i) = g(\ell^i)$.

a) Let us verify that γ is a morphism of complexes:

$$\begin{array}{ccc} (k^{i+1}, \ell^i) & \xrightarrow{d_{C(f)}} & (-d_K k^{i+1}, f(k^{i+1} + d_L \ell^i) \\ \downarrow \gamma & & \downarrow \gamma \\ g(\ell^i) & \xrightarrow{d_M} & d_M g(\ell^i) = g(f(k^{i+1})) + g(d_L \ell^i) \\ & & = g(d_L \ell^i) \end{array}$$

b) Let us verify that the right square in (**) is commutative:

$$\begin{array}{ccc} (k^i, k^{i+1}, \ell^i) & \xrightarrow{\pi} & (k^{i+1}, \ell^i) \\ \downarrow \beta & & \downarrow \gamma \\ f(k^i) + \ell^i & \xrightarrow{g} & g(f(k^i)) + g(\ell^i) = g(\ell^i) \end{array}$$

c) To complete the proof we have to show that γ is a q.i..

Since g is an epimorphism, γ is also an epimorphism and $\mathrm{Ker}\gamma$ is the following complex:

$$K[1]^{\cdot} \oplus \mathrm{Ker} g = K[1]^{\cdot} \oplus \mathrm{Im} f$$
$$d(k^{i+1}, f(k^i)) = (-d_K k^{i+1}, f(k^{i+1} + d_K k^i)).$$

This complex has zero cohomology because its identity mapping is homotopic to the zero one: $\chi d + d\chi = id$, where

$$\chi = \{\chi^i : k^{i+1} \oplus (\mathrm{Im} f)^i \to k^i \oplus (\mathrm{Im} f)^{i-1}\},$$

$$\chi^i(k^{i+1}, f(k^i)) = (k^i, 0).$$

Hence the long exact sequence of complexes

$$0 \to \mathrm{Ker}\gamma \to C(f) \xrightarrow{\gamma} M^{\cdot} \to 0$$

implies that γ is a q.i.. QED

Theorem 2.12: *The class of q.i. in categories $\mathcal{K}^*(\mathcal{A})$ for $* = +, -, b, \emptyset$ is localizing.*

Proof: Let's verify that the conditions (a) to (c) of 2.5 are satisfied.
(a) Obvious.

b) First let us verify the extension condition. For this we must imbed the diagram

$$K^{\cdot} \underset{q.i}{\xrightarrow{f}} L^{\cdot} \xleftarrow{g} M^{\cdot}$$

into a commutative square

$$\begin{array}{ccc} N^{\cdot} & \underset{q.i}{\xrightarrow{k}} & M^{\cdot} \\ \downarrow h & & \downarrow g \\ K^{\cdot} & \underset{q.i}{\xrightarrow{f}} & L^{\cdot} \end{array}.$$

The required square is a part of the following diagram, which is commutative in $\mathcal{K}^*(\mathcal{A})$:

$$\begin{array}{ccccccc} C(\pi g)[-1] & \xrightarrow{k} & M^{\cdot} & \xrightarrow{\pi g} & C(f) & \longrightarrow & C(\pi g) \\ \downarrow h & & \downarrow g & & \| & & \downarrow h[1] \\ K^{\cdot} & \xrightarrow{f} & L^{\cdot} & \xrightarrow{\pi} & C(f) & \longrightarrow & K[1]^{\cdot} \end{array} .(***)$$

An elememt of $C(\pi g)[-1]^i = C(\pi g)^{i-1}$ is a triple

$$(m^i, k^i, \ell^{i-1}), m^i \in M^i, k^i \in K^i, \ell^{i-1} \in L^{i-1},$$
$$k : (m^i, k^i, \ell^{i-1}) \to (-1)^i m^i.$$

Define $h(m^i, k^i, \ell^{i-1}) = (-1)^{i+1} k^i$. Then

$$g \circ k - f \circ h : (m^i, k^i, \ell^{i-1}) \to (-1)^i (g(m^i) + f(k^i)).$$

The last difference equals

$$\chi d_{C(\pi g)[-1]} + d_L \chi,$$

where $\chi = \{\chi^i\}$ and

$$\chi^i : C(\pi g)[-1]^i = C(\pi g)^{i-1} \to L^{i-1}$$

is given by

$$\chi^i(m^i, k^i, \ell^{i-1}) = (-1)^{i-1} \ell^{i-1}$$

(Use the formula :

$$d_{C(\pi g)[-1]}(m^i, k^i, \ell^{i-1}) = (-d_M m^i, -d_K k^i, d_L \ell^{i-1} + f(k^i) + g(m^i))).$$

Hence the left square in diagram (***) is commutative modulo homotopy.

It remains to prove that k is a q.i.. Since f is a q.i., $C(f)$ is acyclic. But the upper row in (***) is a distinguished triangle and k is a q.i..

The sencond extension condition can be proved in the same way: imbed

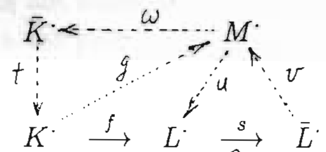
$$M^\cdot \xleftarrow{g} K^\cdot \xrightarrow[q.i.]{f} L^\cdot$$

in

$$\begin{array}{ccccccc}
C(f)[-1] & \xrightarrow{\tau} & K^\cdot & \xrightarrow{f} & L^\cdot & \longrightarrow & C(f) \\
\| & & \downarrow g & & \downarrow h & & \| \\
C(f)[-1] & \xrightarrow{g\tau} & M^\cdot & \xrightarrow{k} & C(g\tau) & \longrightarrow & C(f)
\end{array}$$

c) Let $f : K^\cdot \to L^\cdot$ be a morphism in $\mathcal{K}^*(\mathcal{A})$. We will show that the existence of q.i. $s : L^\cdot \to \bar{L}^\cdot$ with $sf = 0$ in $\mathcal{K}^*(\mathcal{A})$ implies the existence of q.i. $t : \bar{K}^\cdot \to K^\cdot$ such that $ft = 0$.

First construct the triangle $(M^\cdot, L^\cdot, \bar{L}^\cdot; u, s, v)$.



We have an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(K^\cdot, M^\cdot) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(K^\cdot, L^\cdot) \xrightarrow{s_*} \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(K^\cdot, \bar{L}^\cdot) \longrightarrow \cdots.$$

The condition $sf = 0$ means that $f \in \mathrm{Ker} s_*$, hence there exists $g \in \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(K^\cdot, M^\cdot)$ such that $u_*(g) = f$.

Next we construct the triangle $(\bar{K}^\cdot, K^\cdot, M^\cdot; t, g, w)$.

Now we have the following exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(M^\cdot, L^\cdot) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(K^\cdot, L^\cdot) \xrightarrow{t^*} \mathrm{Hom}_{\mathcal{K}^\cdot(\mathcal{A})}(\bar{K}^\cdot, L^\cdot) \longrightarrow \cdots.$$

Since $f \in \mathrm{Im}(g^*)$, $t^*(f) = 0$; thus we have $ft = 0$.

To prove that t is a q.i., consider the long exact sequence associated with the above triangles. From the first sequence we deduce that $H^*(M^\cdot) = 0$, since s_* is a q.i.. So it follows from this and the second sequence that t^* is an isomorphism. QED

Definition 2.13: A *triangulated category* is an additive category $\mathcal{D}$ with an additive automorphism $T : \mathcal{D} \longrightarrow \mathcal{D}$ called translation functor (sometimes written as $X \longrightarrow X[1]$ instead of $T(X)$ and $X \longrightarrow X[n]$ for $T^n(X)\ldots$) and the class of distinguished triangles, which satisfy axioms TR1–TR4 below:

TR1.a) Any triangle isomorphic to a distinguished one is itself distinguished.

b) Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

c) $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$ is a distinguished triangle.

TR2. A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is distinguished if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished .

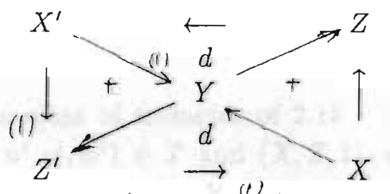
TR3. Assume that we are given two distinguished triangles and two morphisms f, g as in the diagram below

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

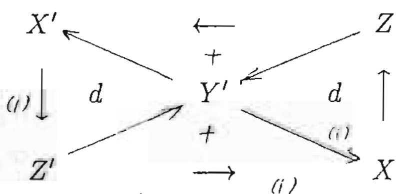
This diagram can be completed(not necessarily uniquely) to a morphism of triangles (i.e. the above diagram is commutative) by a morphism

$$h : Z \longrightarrow Z'.$$

Before stating the fourth axiom, let us describe the *octahedron diagram* as shown below.



(*upper cap*)



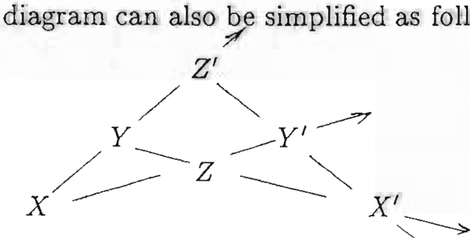
(*lower cap*)

Let an octahedron be represented by two caps as above with common brim. In these diagram X, Y, etc. are objects of $\mathcal{D}$; arrows of the type $X' \longrightarrow Z'$ represent morphisms $X' \longrightarrow Z'[1]$ in $\mathcal{D}$; triangles marked d are distingushed triangles, those marked $+$ are commutative. Further one requires that the two composite morphisms $Y \longrightarrow Y'$(through Z and Z') coincide, so do the composite morphisms $Y' \rightarrow Y[1]$ (through $X[1]$ and X').

TR4. Any diagram of the type "upper cap" can be completed to an octahedron diagram. (or equivalently

TR4': Any diagram of the type "lower cap" can be completed to an octahedron diagram.

Remark 2.14: the octahedron diagram can also be simplified as follows:



Before stating the following theorem, we give a

Definition 2.15: A triangle is said to be distinguished if it is isomorphic to the middle row

$$K^{\cdot}\xrightarrow{\bar{f}}Cyl(f)\xrightarrow{\pi}C(f)\xrightarrow{\delta}K[1]$$

of some diagram shown in lemma 2.10.

Theorem 2.16: Let $\mathcal{A}$ be an abelian category. The category $\mathcal{K}(\mathcal{A})$ with the translation functor and with distinguished triangles as in the above definition is a triangulated category. The same is true for $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, and $\mathcal{K}^b(\mathcal{A})$.

Proof : Let us verify the axioms TR1-TR4 of the triangulated category. For convenience of proof, we expand the octahedron diagram into the following form

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z' & \xrightarrow{w} & T(X) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id_{T(X)}} \\
X & \xrightarrow{u''} & Z & \xrightarrow{v''} & Y' & \xrightarrow{w''} & T(X) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle id_Z} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle T(u)} \\
Y & \xrightarrow{u'} & Z & \xrightarrow{v'} & X' & \xrightarrow{w'} & T(Y) \\
 & & & & \downarrow{\scriptstyle T(v)w'} & \swarrow{\scriptstyle T(v)} & \\
 & & & & T(Z') & &
\end{array}$$

Let T be the class of triangles of 2.15. Then TR4 is equivalent to: given $(X,Y,Z';u,v,w) \in T$, $(Y,Z,X';u',v',w') \in T$ and $(X,Z,Y';u'',v'',w'') \in T$, such that $u'' = u'u$, there exist $f \in \mathrm{Hom}(Z',Y')$ and $g \in \mathrm{Hom}(Y',Z')$ such that $(Z',Y',X';f,g,T(v)w') \in T$ and (id_X,u',f) and (u,id_Z,g) are morphisms of triangles.

TR1 : a) is obvious by transitivity of isomorphisms.

b) It suffices to take the standard diagram $(X^{\cdot},Y^{\cdot},C_u^{\cdot};u,q,p)$.

c) We need two simple lemmas:

Lemma 2.16.1: The composition of two consecutive morphisms in the triangle $(X^{\cdot},Y^{\cdot},C_u^{\cdot};u,q,p)$ is homotopic to 0.

Proof: By definition we have $pq = 0$. To prove that $qu \sim 0$ (resp. $up \sim 0$) we take the homotopy operator to be $q_{X^{\cdot}} : X^{\cdot} \to Y^{\cdot} \oplus T(X^{\cdot})$ (resp. $p_{Y^{\cdot}} : Y^{\cdot} \oplus T(X^{\cdot}) \to Y^{\cdot}$) as morphism of degree -1.

Lemma 2.16.2: For each object $X^{\cdot}$ in $\mathcal{K}om^*(\mathcal{A})$, the cone $C^{\cdot}_{id_{X^{\cdot}}}$ is homotopic to the zero complex $0^{\cdot}$.

Proof: The morphisms $id_{C^{\cdot}_{id_{X^{\cdot}}}}$, $0 \in \mathrm{Hom}^{-1}(X^{\cdot} \oplus T(X^{\cdot}), X^{\cdot} \oplus T(X^{\cdot}))$ given by $k(x',x'') = (0,x')$.

By the above two lemmas, the diagram

$$\begin{array}{ccccccc}
X^{\cdot} & \xrightarrow{id_X} & X^{\cdot} & \xrightarrow{0} & 0^{\cdot} & \xrightarrow{0} & T(X^{\cdot}) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle id_{T(X^{\cdot})}} \\
X^{\cdot} & \xrightarrow{id_X} & X^{\cdot} & \xrightarrow{q} & C^{\cdot}_{id_{X^{\cdot}}} & \xrightarrow{p} & T(X^{\cdot})
\end{array}$$

is commutative in $\mathcal{K}(\mathcal{A})$ and the third vertical arrow is an isomorphism in $\mathcal{K}(\mathcal{A})$.

TR2 : $(\Longrightarrow)$ Given $(X^{\cdot},Y^{\cdot},C_u^{\cdot};u,q,p)$, we have to prove that

$(Y^\cdot, C_u^\cdot, T(X^\cdot); q, p, -T(u))$ is distinguished. Let $C_q^\cdot$ be the cone of the morphism $q : Y^\cdot \to C_u^\cdot$. Consider the following diagram in $\mathcal{K}(\mathcal{A})$:

$$\begin{array}{ccccccc}
Y^\cdot & \xrightarrow{q} & C_u^\cdot & \xrightarrow{p} & T(X^\cdot) & \xrightarrow{-T(u)} & T(Y^\cdot) \\
\downarrow id_{Y^\cdot} & & \downarrow id_{C_u^\cdot} & & r \Downarrow\Uparrow r' & & \downarrow id_{T(Y^\cdot)} \\
Y^\cdot & \xrightarrow{q} & C_u^\cdot & \xrightarrow{\bar{q}} & C_q^\cdot & \xrightarrow{\bar{p}} & T(Y^\cdot)
\end{array}$$

where $\bar{q} = q_{C_u^\cdot} : C_u^\cdot \to C_u^\cdot \oplus T(Y^\cdot)$ and $\bar{p} = p_{T(Y^\cdot)} : C_u^\cdot \oplus T(Y^\cdot) \to T(Y^\cdot)$.

The morphism r, r' are defined as follows:

$$\begin{array}{cccccc}
r & : & T(X^\cdot) & \longrightarrow & C_q^\cdot \\
& & x & \mapsto & (0, x, -u(x)) \\
r' & : & C_q^\cdot & \longrightarrow & T(X^\cdot) \\
& & (y, x, y') & \mapsto & x
\end{array}.$$

Clearly $r'r = id_{T(X^\cdot)}$.

Moreover $rr' \sim id_{C_q^\cdot}$, i.e. $id_{C_q^\cdot} - rr' = kd + dk$, where k is given by $k(y, x, y') = (0, 0, y)$. Hence r is a homotopic equivalence.

Furthermore the 1st and 3rd squares are commutative, and the 2nd one is commutative up to homotopy, with $rp \sim \bar{q}$ given by $k(y, x) = (0, 0, y)$.

$$((\bar{q} - rp)(y, x) = (kd + dk)(y, x))$$

($\Longleftarrow$) By applying the first implication twice.

TR3: Given a diagram in $\mathcal{K}om(\mathcal{A})$

$$\begin{array}{ccccccc}
X^\cdot & \xrightarrow{u} & Y^\cdot & \xrightarrow{q} & C_u^\cdot & \xrightarrow{p} & T(X^\cdot) \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\
X'^\cdot & \xrightarrow{u'} & Y'^\cdot & \xrightarrow{q'} & C_{u'}^\cdot & \xrightarrow{p'} & T(X'^\cdot)
\end{array}$$

with the 1st square commutative up to homotopy. (let $gu - u'f = kd + dk$, with $k : X^\cdot \to Y'^\cdot$)
We define $h : C_u^\cdot \to C_{u'}^\cdot$ by

$$h(y, x) = (g(y) + k(x), f(x)).$$

It is easily seen that h is a morphism, making 2nd, 3rd squares commutative.

TR4: Given $u \in \text{Hom}(X^\cdot, Y^\cdot)$, $v \in \text{Hom}(Y^\cdot, Z^\cdot)$ and $w \in \text{Hom}(X^\cdot, Z^\cdot)$ such that $w \sim vu$ ($k : X^\cdot \to Z^\cdot$ the homotopy from vu to w).

Define $f : C_u^\cdot \to C_w^\cdot$ and $g : C_w^\cdot \to C_v^\cdot$ by

$$f(y, x) = (v(y) + k(x), x)$$

and

$$g(z, x) = (z - k(x), u(x)),$$

then the squares 2,3,5 and 6 are commutative.

$$
\begin{array}{ccccccc}
X^{\cdot} & \xrightarrow{u} & Y^{\cdot} & \xrightarrow{q} & C_u^{\cdot} & \xrightarrow{p} & T(X^{\cdot}) \\
\downarrow{\scriptstyle id_X} & 1 & \downarrow{\scriptstyle v} & 2 & \downarrow{\scriptstyle f} & 3 & \downarrow{\scriptstyle id_{T(X^{\cdot})}} \\
X^{\cdot} & \xrightarrow{w} & Z^{\cdot} & \xrightarrow{q''} & C_w^{\cdot} & \xrightarrow{p''} & T(X^{\cdot}) \\
\downarrow{\scriptstyle u} & 4 & \downarrow{\scriptstyle id_Z} & 5 & \downarrow{\scriptstyle g} & 6 & \downarrow{\scriptstyle T(u)} \\
Y^{\cdot} & \xrightarrow{v} & Z^{\cdot} & \xrightarrow{q'} & C_v^{\cdot} & \xrightarrow{p'} & T(Y^{\cdot}) \\
 & & & & \downarrow{\scriptstyle T(q)p'} & \swarrow{\scriptstyle T(q)} & \\
 & & & & T(C_u^{\cdot}) & &
\end{array}
$$

Now consider the diagram

$$
\begin{array}{ccccccc}
C_u^{\cdot} & \xrightarrow{f} & C_w^{\cdot} & \xrightarrow{g} & C_v^{\cdot} & \xrightarrow{T(g)p'} & T(C_u^{\cdot}) \\
\downarrow{\scriptstyle id_{C_u^{\cdot}}} & & \downarrow{\scriptstyle id_{C_w^{\cdot}}} & & \omega \Downarrow\Uparrow \theta & & \downarrow{\scriptstyle id_{T(C_u^{\cdot})}} \\
C_u^{\cdot} & \xrightarrow{f} & C_w^{\cdot} & \xrightarrow{\bar{q}} & C_f^{\cdot} & \xrightarrow{\bar{p}} & T(C_u^{\cdot})
\end{array}
$$

The morphism $\omega : C_v^{\cdot} \to C_f^{\cdot}$ is defined by

$$\omega(z, y) = (z, o, y, 0),$$

making the 3rd square commutative. Moreover the 2nd square commutes up to homotopy, with homotopy given by $\bar{k} \in \mathrm{Hom}^{-1}(C_w^{\cdot}, C_f^{\cdot})$

$$(\bar{k}(z, x) = (0, 0, 0, x), \bar{q} - \omega q = \bar{k}d + d\bar{k})$$

It remains to prove that ω is a homotopic equivalence.
Let $\theta : C_f^{\cdot} \to C_v^{\cdot}$ be given by

$$\theta(z, x, y, x') = (z - k(x), u(x) + y).$$

Clearly $\theta\omega = id_{C_v^{\cdot}}$. In addition, we have $\omega\theta \sim id_{C_f^{\cdot}}$, with homotopy given by $\tilde{k}$, where $\tilde{k}$ is defined by $\tilde{k}(z, x, y, x') = (0, 0, 0, x)$ $(id_{C_f^{\cdot}} - \omega\theta = \tilde{k}d + d\tilde{k})$.QED

Proposition 2.17: Let (X, Y, Z) and (X', Y', Z') be two distinguished triangles and $g : Y \longrightarrow Y'$:

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow &
\end{array}
$$

Then the following conditions are equivalent:

(a) $v'gu = 0$,

(b) $\exists f$ such that (1) is commutative,(b') $\exists h$ such that (2) is commutative,

(c) $\exists$ a morphism of triangles (f, g, h).

Moreover if these conditions are satisfied, and $Hom^{-1}(X, Z') = 0$, then the morphism f of (b) (resp.h of (b')) is unique.

Proof: The exactness of the sequence

$$\mathrm{Hom}^{-1}(X, Z') \longrightarrow \mathrm{Hom}(X, X') \longrightarrow \mathrm{Hom}(X, Y') \longrightarrow Hom(X, Z'),$$

applied to gu in $\mathrm{Hom}(X, Y')$, shows that $(a) \Longleftrightarrow (b)$, with uniqueness of f if $\mathrm{Hom}^{-1}(X, Z') = 0$.

The implication (b)$\Longrightarrow$(c) follows from TR2: if f satisfies (b), then $\exists h$ such that (f, g, h) is a morphism of triangle. The converse is trivial.

Finally, a dual argument shows that $(a) \Longleftrightarrow (b')$, and the uniqueness of h if

$\mathrm{Hom}^{-1}(X, Z') = 0$.

Corollary 2.18: Let $X \longrightarrow Y \longrightarrow Z \longrightarrow$ be a distinguished triangle. If $\mathrm{Hom}^{-1}(X, Z) = 0$, then

(i) the cone of u is unique up to an isomorphism;

(ii) d is the unique morphism $x : Z \longrightarrow X[1]$ such that the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ is distinguished.

Proof : If in the above proposition, let $X = X'$, $Y = Y'$ and f, g are identities, then Z is isomorphic to Z'. Hence $\mathrm{Hom}^{-1}(X, Z') = 0$ and (i) follows from the uniqueness of h.

As for (ii), applying (2.17) to

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{d} & \\ \| & & \| & & \downarrow h & & \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{x} & \end{array} .$$

We necessarily have $h = id_Z$, hence $d = x$. QED

Definition 2.19: A functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$ is called a cohomological functor if it is additive and the sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(w)} H(Z)$$

in $\mathcal{A}$ is exact for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{D}$.

Proposition 2.20: (i) If

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

is a distinguished triangle, then $gf = 0$.

(ii) For any $W \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ and $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ are cohomological functors.

Proof: (i) By TR1,

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished. Therefore by TR4 there exists a morphism $\phi : 0 \longrightarrow Z$ which makes the following diagram commutative:

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow id_X & & \downarrow f & & \downarrow \phi & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & X[1] \end{array} .$$

Hence $g \circ f = \phi \circ 0 = 0$.

(ii) Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

be a distinguished triangle. In order to show that $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ is a cohomological functor, it is enough to show that, for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ with $g \circ \phi = 0$, we can find $\psi \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, with $\phi = f \circ \psi$. This follows from TR1, TR3, and TR4 which imply that the dotted arrow below can be completed:

$$\begin{array}{ccccccc} W & \longrightarrow & W & \longrightarrow & 0 & \longrightarrow & W[1] \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & X[1] \end{array}$$

The proof of $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ is similar. QED

Corollary 2.21: In the diagram below if ϕ, ψ are isomorphisms, then θ is also an isomorphism.

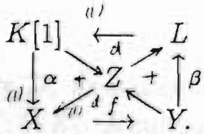
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

2.22. Abelian Subcategories

2.22.0. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a full subcategory of $\mathcal{D}$. We suppose that $\mathrm{Hom}^i(X, Y) := \mathrm{Hom}(X, Y[i])$ is zero for $i < 0$ and for X, $Y \in \mathcal{C}$.

Proposition 2.22.1: Let $f : X \to Y$ be a map in $\mathcal{C}$. Complete f to a distinguished triangle $X \to Y \to Z \to$, and suppose that Z is a part of some distinguished triangle $K[1] \to Z \to L \to$, with K and L in $\mathcal{C}$:

(2.22.1.1)



Then $\alpha[-1] : K \to X$ is a kernel of f in $\mathcal{C}$, and $\beta : Y \to L$ is a cokernel.

Proof: For $W \in \mathcal{C}$ and under the condition of 2.22.0, the long exact sequence of Hom gives the exact sequences

$$0 \to \mathrm{Hom}^{-1}(W, Z) \to \mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$$

and

$$0 \to \mathrm{Hom}(W, K) \to \mathrm{Hom}^{-1}(W, Z) \to 0.$$

They show that $(K, \alpha[-1])$ is a kernel of f. It follows from a dual argument that $(Z, \beta)$is a cokernel.

Definition 2.22.2: A morphism $f : X \to Y$ in $\mathcal{C}$ is called $\mathcal{C}$-admissible (or simply admissible if no confusion arises) if it is the base of a diagram (2.22.1.1).

Remark 2.22.3: If f is a monomorphism, then $K = 0$ by 2.22.1. hence $Z \xrightarrow{\sim} \mathcal{C}$ and (2.22.1.1) reduces to a distinguished triangle (X, Y, Z). If f is an epimorphism, we have $L = 0$, hence $K[1] \xrightarrow{\sim} Z$, and (2.22.1.1) reduces to a distinguished triangle (K, X, Y). Conversely, for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{d}$ with X, Y, $Z \in \mathcal{C}$, f and g are admissible, f is a kernel of g, and g a cokernel of f. By 2.22.0 and 2.18, d is determined by f and g.

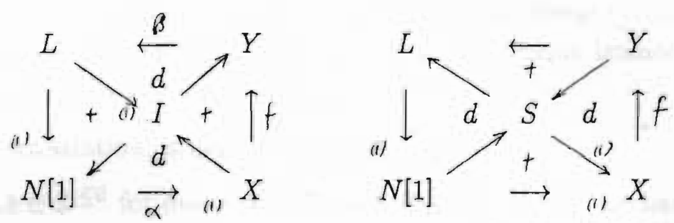
Definition 2.22.4: A sequence $X \to Y \to Z$ in $\mathcal{C}$ is called an admissible short exact sequence if it is obtained from a distinguished triangle by suppressing the map of degree 1.

Proposition 2.22.5: Suppose $\mathcal{C}$ is stable under finite direct sums. Then the following conditions are equivalent:

(i) $\mathcal{C}$ is abelian, and its short exact sequences are admissible.

(ii) every morphism of $\mathcal{C}$ is $\mathcal{C}$-admissible.

Proof: (ii)$\Longrightarrow$(ii). By 2.22.1, every morphism of $\mathcal{C}$ has a kernel and cokernel. To prove the abelianness of $\mathcal{C}$, it remains to show that $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$. Take (2.22.1.1) for the lower cap of an octahedron and apply TR4' to complete it to an octahedron:



By 2.22.1, β is an epimorphism, since it is a cokernel of f. By (2.22.2), the triangle (I, Y, L) is distinguished; hence I is in $\mathcal{C}$ and is the image of f. Dually, the distinguished triangle (K, X, I) (obtained from the distinguished triangle in the upper cap by translation) shows that I is the coimage of f. Finally, by 2.22.2, the short exact sequences of $\mathcal{C}$ are admissible.

(ii)$\Longrightarrow$(i). The kernel K, cokernel L and image I of $f: X \to Y$ give two short exact sequences $0 \to K \to X \to I \to 0$ and $0 \to I \to Y \to L \to 0$. Let them be the two triangles forming the upper cap in the above diagram, Applying TR4, we get the lower cap. Hence f is admissible.

Definition 2.22.6: A full subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be admissible if it satisfies 2.22.0 and the equivalent conditions of 2.22.5.

Definition 2.22.7: In a triangulated category $\mathcal{D}$, we sometimes call an object Y *extension of Z by X* if there exists a distinguished triangle (X, Y, Z). A subcategory $\mathcal{D}'$ of $\mathcal{D}$ is stable if $\forall$ triangle (X, Y, Z) with X, $Z \in \mathcal{D}' \Longrightarrow Y \in \mathcal{D}'$.

§3 The t-structure

First let us state some facts:

- An important discovery in the homological algebra was that the derived category of two absolutely different abelian category can be equivalent as triangulated category.

- t-structure is a technique that allows us to see various abelian subcategory inside a given triangulated category.

Definition 3.1: A t-category is a triangulated category $\mathcal{D}$ with two full subcategory $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}$ such that (writting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$)

(i) For all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$, we have $\mathrm{Hom}(X, Y) = 0$.

(ii) We have $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

(iii) For every $X \in \mathcal{D}$, there exists a distinguished triangle (A, X, B) with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

We say that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$. And its core is the full subcategory $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Example 3.2: Let $\mathcal{A}$ be an abelian category and $\mathcal{D}= \mathcal{D}^*(\mathcal{A})$ be its derived category. Denote by $\mathcal{D}^{\geq n}$(resp.$\mathcal{D}^{\leq n}$ the full subcategory of $\mathcal{D}$ formed by complexes $K^\cdot$ with $H^i(K^\cdot) = 0$ for $i < n$(resp. for $i > n$). Then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure with core $\mathcal{A}$.

Proposition 3.3: (i) The inclusion of $\mathcal{D}^{\leq n}$ in $\mathcal{D}$ has a right adjoint $\tau_{\leq n}$, and that of $\mathcal{D}^{\geq n}$ has a left adjoit $\tau_{\geq n}$.

(ii) For all $X \in \mathcal{D}$, there exists a unique morphism $d \in \mathrm{Hom}(\tau_{\geq 1}X, \tau_{\leq 0}X)$ such that the triangle $\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \longrightarrow$ is distinguished. Up to a unique isomorphism, this triangle is the unique triangle (A, X, B) with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

Proof: By duality and translation, it suffices to prove (i) for $\mathcal{D}^{\leq 0}$.

It suffices to find $A \in \mathcal{D}^{\leq 0}$ for every $X \in \mathcal{D}$ such that $\forall T \in \mathcal{D}^{\leq 0}$ we have

$$\mathrm{Hom}(T, A) \xrightarrow{\sim} \mathrm{Hom}(T, X).$$

Let (A, X, B) be a triangle in the condition (iii) of the definition of t-category. By the long exact sequence of Hom and conditions (ii) and (i), we have

$$\mathrm{Hom}(T, A) \xrightarrow{\sim} \mathrm{Hom}(T, X).$$

Hence we can set $\tau_{\leq 0}X = A$. This proves (i), the existence of a distiguished triangle $(\tau_{\leq 0}X, X, \tau_{\geq 1}X)$ and the fact that every distinguished triangle 3.1(iii) (A, X, B) is uniquely isomorphic to this triangle, without considering the map of degree 1. The uniqueness of this last triangle follows from 3.1(i), (ii) and 2.18(ii).

Corollary 3.4: The distinguished triangle $(\tau_{\leq 0}X, X, \tau_{\geq 1}X)$ shows the equivalence of the following conditions:

(a) $\tau_{\leq 0}X = 0$, i.e. (a') $\mathrm{Hom}(T, X) = 0 \ \forall T \in \mathcal{D}^{\leq 0}$

(b) $X \xrightarrow{\sim} \tau_{\geq 1}X$, i.e. (b') $X \in \mathcal{D}^{\geq 1}$.

Remark 3.5:

- (a') $\iff$ (b') shows that $\mathcal{D}^{\geq 1}$ is right orthogonal to $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ is stable under extensions.
- Dually,

$$\tau_{\geq 1}X = 0 \iff X \in \mathcal{D}^{\leq 0};$$

  $\mathcal{D}^{\leq 0}$ is left orthogonal to $\mathcal{D}^{\geq 1}$ and stable under extensions. In particular, $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are stable under finite direct sums.

Lemma 3.6: For $a \leq b$, we have $\mathcal{D}^{\leq a} \subset \mathcal{D}^{\leq b}$ and there exists a unique morphim from $\tau_{\leq a}X \to \tau_{\leq b}X$ making the diagram

$$\begin{array}{ccc} \tau_{\leq a}X & \longrightarrow & \tau_{\leq b}X \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

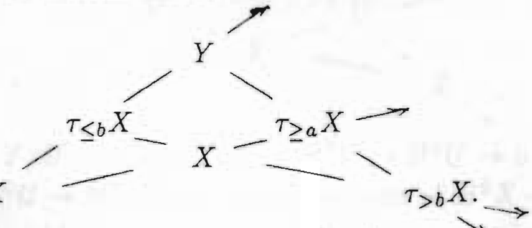
commutative. It indentifies $\tau_{\leq a}X$ and $\tau_{\leq a}\tau_{\leq b}X$. Dually, we have $\tau_{\geq a}X \to \tau_{\geq b}X$, identifying $\tau_{\geq b}X$ and $\tau_{\geq b}\tau_{\geq a}X$.

Proposition 3.7: Let $a \leq b$. $\forall X \in \mathcal{D}$, $\exists!$ morphism $\tau_{\geq a}\tau_{\leq b}X \longrightarrow \tau_{\leq b}\tau_{\geq a}X$ making the diagram commutative.

$$\begin{array}{ccccc} \tau_{\leq b}X & \to & X & \to & \tau_{\geq a}X \\ \downarrow & & & & \uparrow \\ \tau_{\geq a}\tau_{\leq b}X & & \xrightarrow{\sim} & & \tau_{\leq b}\tau_{\geq a}X \end{array}$$

It is an isomorphism.
Proof: (1) The map $\tau_{\leq b}X \to \tau_{\geq a}X \in \mathcal{D}^{\geq a}$ can be uniquely factored through $\tau_{\geq a}\tau_{\leq b}X$.

(2) Since $\tau_{\geq a}\tau_{\leq b}X$ is in $\mathcal{D}^{\leq b}$, it factors uniquely through $\tau_{\leq b}\tau_{\geq a}X$. Applying TR4 to $\tau_{<a}X \to \tau_{\leq b}X \to X$, we have



In this octahedron, Y is at the same time $\tau_{\geq a}\tau_{\leq b}X$ ($\because (\tau_{<a}\tau_{\leq b}X = \tau_{<a}X,\ \tau_{\leq b}X,\ Y)$) and $\tau_{\leq b}\tau_{\geq a}X$ ($\because$ $(Y,\ \tau_{\geq a}X,\ \tau_{>b}X)$).

• Set $\tau_{[a,b]}X := \tau_{\geq a}\tau_{\leq b}X \xrightarrow{\sim} \tau_{\leq b}\tau_{\geq a}X$.

Theorem 3.8: The core $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ of the t-category $\mathcal{D}$ is an admissible subcategory of $\mathcal{D}$, stable under extensions. The functor $H^0 := \tau_{\geq 0}\tau_{\leq 0}$ from $\mathcal{D}$ to $\mathcal{C}$ is a cohomological functor.
Proof: Let $X,\ Y \in \mathcal{C}$, $f : X \to Y$, and S the cone of f. The distinguished triangle $(Y,\ S,\ X[1])$ shows that $S \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$, hence $\tau_{\geq 0}S \in \mathcal{C}[1]$ and the distinguished triangle $(\tau_{\leq -1}S,\ S,\ \tau_{\geq 0}S)$ gives a diagram (2.22.1.1). Therefore $\mathcal{C}$ is admissible abelian. The stability by extensions follows from the remark after the above lemma.

It remains to show that for every distinguished triangle $(X,\ Y,\ Z)$, the sequence $H^0X \to H^0Y \to H^0Z$ is exact.

Case1. $X, Y, Z \in \mathcal{D}^{\leq 0} \longrightarrow H^0X \to H^0Y \to H^0Z \to 0$ is exact :

For $U \in \mathcal{D}^{\leq 0}$, $V \in \mathcal{D}^{\geq 0}$, we have $H^0U = \tau_{\geq 0}U$, $H^0V = \tau_{\leq 0}V$ and

$\mathrm{Hom}(H^0U, H^0V) \xrightarrow{\sim} \mathrm{Hom}(U, H^0V) \xrightarrow{\sim} Hom(U, V)$.

Let $T \in \mathcal{C}$(hence $T \in \mathcal{D}^{\geq 0}$), the long exact sequence gives

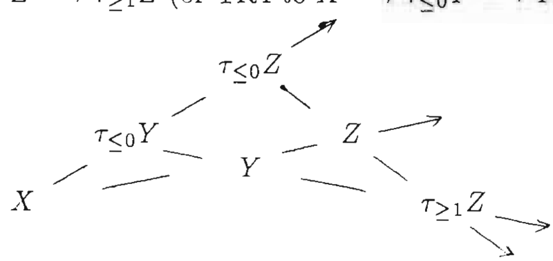
$0 \longrightarrow \mathrm{Hom}(Z,T) \longrightarrow \mathrm{Hom}(Y,T) \longrightarrow \mathrm{Hom}(X,T)$

$0 \longrightarrow \mathrm{Hom}(H^0Z,T) \longrightarrow \mathrm{Hom}(H^0Y,T) \longrightarrow \mathrm{Hom}(H^0X,T)\ \forall T$. Hence the exactness.

Case2. $X \in \mathcal{D}^{\leq 0}$: $\forall T \in \mathcal{D}^{\geq 1}$, the long exact sequence of Hom gives

$\mathrm{Hom}(Z,T) \xrightarrow{\sim} \mathrm{Hom}(Y,T)$, hence $\tau_{\geq 1}Y \xrightarrow{\sim} Z$.

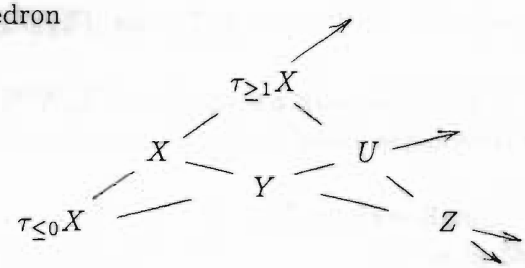
Applying TR4' to $Y \longrightarrow Z \longrightarrow \tau_{\geq 1}Z$ (or TR4 to $X \longrightarrow \tau_{\leq 0}Y \longrightarrow Y$), we have



We obtain a distinguished triangle $(X,\ \tau_{\leq 0}Y,\ \tau_{\leq 0}Z)$, and thus back to *Case1.*

Case2.* Dually, if $Z \in \mathcal{D}^{\geq 0}$, $0 \longrightarrow H^0X \longrightarrow H^0Y \longrightarrow H^0Z$ is exact.

General case. By TR4, we have an octahedron



Applying *Case2.* to $(\tau_{\leq 0}X, Y, U)$, we get $H^0X \to H^0Y \to H^0U \to 0$. And applying *Case2** to $(U, Z.(\tau_{\geq 1}X)[1])$ gives $0 \to H^0U \to H^0Z$, hence the exactness of $H^0X \to H^0Y \to H^0Z$. QED

Definition 3.9: A t-structure is said to be non-degenerate if the intersection of the $\mathcal{D}^{\leq n}$, and that of $\mathcal{D}^{\geq n}$ reduce to the zero objects. We set $H^iX; = H^0(X[i])$.

Proposition 3.10: Suppose the system of the functors H^i is conservative and the t-structure of $\mathcal{D}$ is non-degenerate. Then an object X of $\mathcal{D}$ is in $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) if and only if $H^iX = 0$ for $i > 0$ (resp. for $i < 0$).

Proof: Let $X \in \mathcal{D}$. We want to show that $X = 0$ if all the H^iX are zero. If X is in $\mathcal{D}^{\leq 0}$, then the hypothesis $H^0X = 0$ ensures that X is in $\mathcal{D}^{\leq 1}$; continually, we find that X is in theintersection of $\mathcal{D}^{\leq n}$, hence is zero. Dually, if $X \in \mathcal{D}^{\geq 0}$, then $X = 0$. In general, since the H^i of $\tau_{\leq 0}X$ and $\tau_{\geq 1}X$ are zero. They are both zero themselves. Hence we conclude that X is zero from the distinguished triangle $(\tau_{\leq 0}X, X, \tau_{\geq 1}X)$.

If a morphism $f : X \to Y$, of which Z is a cone, induces the isomorphisms of $H^i(X) \xrightarrow{\sim} H^i(Y)$, then the cohomological long exact sequence shows that $H^i(X)$ are zero (therefore $Z = 0$) and f is an isomorphism. Finally, if $H^i(X) = 0$ for $i > 0$, all the $H^i(\tau_{>0}X)$ are zero, $\tau_{>0}X = 0$ and by (3.5) X is in $\mathcal{D}^{\leq 0}$. The case for $\mathcal{D}^{\geq 0}$ is dual.

Definition 3.11: Let $T : \mathcal{C} \to \mathcal{C}'$ be a functor between two triangulated categories. T is said to be an exact functor if T is additive, graded by translation, and transforms distinguished triangles into distinguished triangles. The morphisms of exact functors are the morphisms of the graded functors.

Definition 3.12: Let $\mathcal{D}_i (i = 1, 2)$ be two t-categories, $\mathcal{C}_i$ the core of $\mathcal{D}_i$, and ε the functor of inclusion of $\mathcal{C}_i$ in $\mathcal{D}_i$. Let $T : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor from the triangulated category $\mathcal{D}_1$ to the triangulated category $\mathcal{D}_2$. We say that T is right t-exact if $T(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$, left t-exact if $T(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$, and t-exact if it is both right t-exact and left t-exact.

Proposition 3.13: (i) If T is left t-exact (resp. right t-exact), then the additive functor ${}^pT := H^0 \circ T \circ \varepsilon : \mathcal{C}_1 \to \mathcal{C}_2$ is left (resp. right) exact.

(ii) For a left (resp. right) t-exact functor$T$ and for K in $\mathcal{D}_1^{\geq 0}$ (resp. $\mathcal{D}_1^{\leq 0}$), we have ${}^pTH^0K \xrightarrow{\sim} H^0TK$ (resp. $H^0TK \xrightarrow{\sim} {}^pTH^0K$).

(iii) Let (T^*, T_*) be a pair of adjoint exact functors: $T : \mathcal{D}_2 \to \mathcal{D}_1$ and $T_* : \mathcal{D}_1 \to \mathcal{D}_2$, right adjoint of T^*. Then T^* is right t-exact if and only if T_* if left t-exact, and in this case $({}^pT^*, {}^pT_*)$ form a pair of adjoint functors $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$

(iv) If $T_1 : \mathcal{D}_1 \to \mathcal{D}_2$ and $T_2 : \mathcal{D}_2 \to \mathcal{D}_3$ are left (resp. right) t-exact, then so is $T_2 \circ T_1$ and we have ${}^p(T_2 \circ T_1) = {}^pT_2 \circ {}^pT_1$.

Proof: If T is left t-exact, then for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, the cohomological long exact sequence of the distinguished triangle $(T(X), T(Y), T(Z))$ gives

$0 \to H^0T(X) \to H^0T(Y) \to H^0T(Z)$, since $T(Z)$ is in $\mathcal{D}^{\geq 0}$. This (resp. the dual argument) proves (i).

For K in $\mathcal{D}_1^{\geq 0}$, the triangle $(H^0K, K, \tau_{>0}K)$ gives a triangle $(TH^0K, TK, T\tau_{>0}K)$ with $T\tau_{>0}K$ in $\mathcal{D}_2^{>0}$. Hence it follows from the cohomological long exact sequence that $H^0TH^0K \xrightarrow{\sim} H^0TK$. This (resp. the dual argument) proves (ii).

If T_* is left t-exact, then for $U \in \mathcal{D}_1^{>0}$ and $V \in \mathcal{D}_2^{\leq 0}$, we have $\mathrm{Hom}(T^*V, U) = \mathrm{Hom}(V, T_*U) = 0$. Since this is true for all U, we have $\tau_{>0}T^*V = 0$, i.e. T^*V is in $\mathcal{D}_1^{\leq 0}$: T^* is right t-exact. For $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, we therefore have $H^0T^*B = \tau_{\leq 0}T^*B$ and $H^0T_*A = \tau_{\leq 0}T_*A$, hence a functorial isomorphism

$$\mathrm{Hom}(H^0T^*B, A) \xrightarrow{\sim} \mathrm{Hom}(T^*B, A) = \mathrm{Hom}(A, T_*B) \xleftarrow{\sim} \mathrm{Hom}(A, H^0T_*B).$$

This and the the dual argument prove (iii).

If T_1 and T_2 are left t-exact and $A \in \mathcal{C}_1$, we have $T_1A_1 \in \mathcal{D}^{\geq 0}$ and ${}^p(T_2 \circ T_1)A = H^0T_2T_1A = H^0T_2H^0T_1A$ by (ii). This and the dual argument prove (iv). QED

Remark 3.14: (i) Let $\mathcal{D}_1^+ = \cup\mathcal{D}_1^{\geq n}$ and $\mathcal{D}_2^- = \cup\mathcal{D}_2^{\leq n}$. The result (iii) also holds for $T^* : \mathcal{D}_2^- \to \mathcal{D}_1$ and $T_* : \mathcal{D}_1^+ \to \mathcal{D}_2$, where the adjoints are realized as $\mathrm{Hom}(T^*V, U) = \mathrm{Hom}(V, T_*U)$ functorially, for V in $\mathcal{D}_2^-$ and U in $\mathcal{D}_2^+$. The proof is the same.
(ii) For A in $\mathcal{C}_1$ and B in $\mathcal{C}_2$, the adjunction maps for (T^*, T_*) and $({}^pT^*, {}^pT_*)$ are related by the following commutative diagrams:

$$\begin{array}{ccc} T^*{}^pT_*A & \longrightarrow & {}^pT^*{}^pT_*A \\ \downarrow & & \downarrow \\ T^*T_*A & \longrightarrow & A \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & T_*T^*B \\ \downarrow & & \downarrow \\ {}^pT_*{}^pT^*B & \longrightarrow & T_*{}^pT^*B. \end{array}$$

3.15. Let $T : \mathcal{D}' \to \mathcal{D}$ be a fully faithful exact functor between two triangulated category: for a triangle tr of $\mathcal{D}'$ to be distinguished, it suffieces to require that its image Ttr by T is distinguished: if tr_1 is a distinguished triangle of the same base as tr, and Ttr and Ttr_1 are distinguished, of the same base (hence isomorphic), then tr and tr_1 are isomorphic.

Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are with t-structures and T is t- exact. For X in $\mathcal{D}'$ to be in $\mathcal{D}'^{\leq 0}$ (resp. $\mathcal{D}'^{\geq 0}$), it suffices that TX is in $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$): we have $X \in \mathcal{D}'^{\leq 0} \iff \tau_{>0}X = 0$, and T commutes with $\tau_{>0}$ (resp. the dual argument). Conversely, if $\mathcal{D}'$ is a full triangulated subcategory of a triangulated category $\mathcal{D}$, and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, then $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0} := (\mathcal{D}' \cap \mathcal{D}^{\leq 0}, \mathcal{D}' \cap \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}'$ if and only if $\mathcal{D}'$ is stable under the functor $\tau_{\leq 0}$. If this condition is satisfied, this t-structure on $\mathcal{D}'$ is called the induced t-structure. For $\mathcal{D}'$ with the induced t-structure, the functor of inclusion of $\mathcal{D}'$ in $\mathcal{D}$ is t-exact: we have $\mathcal{C}' = \mathcal{D}' \cap \mathcal{C}$, and the restriction to $\mathcal{D}'$ of the functor $\tau_{\leq p}$, $\tau_{\geq p}$ or H^p of $\mathcal{D}$ coincides with the functor on $\mathcal{D}'$ with the same notation.

§4 Perverse sheaves

Let X be a topological space (or even a topos), with sheaf of ring $\mathcal{O}$. And let $\mathcal{D}(X, \mathcal{O})$ be the derived category of the abelian category $\mathcal{M}(X, \mathcal{O})$ of sheaves of left $\mathcal{O}$-modules over X. We denote $\mathcal{D}^+(X, \mathcal{O})$ the full subcategory of complexes which are bounded below.

Let U be an open set of X, F its closed complement, j the inclusion of U in X, and i the inclusion of F in X. We retain the notation $\mathcal{O}$ for sheaf of ring on U or F which is the inverse image of O on X. We'd like to describe a construction which can, given a t−structure on $\mathcal{D}^+(U,\mathcal{O})$ and a t-structure on $\mathcal{D}^+(F,\mathcal{O})$, produce a t-structure on $\mathcal{D}^+(X,\mathcal{O})$.

The categories $\mathcal{M}(X,\mathcal{O})$, $\mathcal{M}(U,\mathcal{O})$ and $\mathcal{M}(F,\mathcal{O})$ are related by the functors:

$j_! : \mathcal{M}(U,\mathcal{O}) \to \mathcal{M}(X,\mathcal{O})$: extension by 0 (exact);

$j^* : \mathcal{M}(X,\mathcal{O}) \to \mathcal{M}(U,\mathcal{O})$: restriction (exact), also denoted $j^!$;

$j_* : \mathcal{M}(U,\mathcal{O}) \to \mathcal{M}(X,\mathcal{O})$: direct image (left exact);

$i^* : \mathcal{M}(X,\mathcal{O}) \to \mathcal{M}(F,\mathcal{O})$: restriction (exact);

$i_* : \mathcal{M}(F,\mathcal{O}) \to \mathcal{M}(X,\mathcal{O})$: direct image (exact), also denoted $i_!$;

$i^! : \mathcal{M}(X,\mathcal{O}) \to \mathcal{M}(F,\mathcal{O})$: sections with support in F (left exact).

For the convenience of the readers, we give defintions of the six operations introduced above. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a sheaf on W (= F or U). We have the

Definition 4.1: (1) For a sheaf $\mathcal{G}$ on U. Define $j_!\mathcal{G}$ to be the sheaf on X whose sections over the open set V of X are given by

$$\Gamma(V, j_!\mathcal{G}) = \{s \in \Gamma(U \cap V, \mathcal{G}) \mid \text{supp}(s) \text{ is closed relative to } V\}.$$

(2) For a sheaf $\mathcal{G}$ on W, $f_*\mathcal{G}$ ($f = i$ or j) is defined to be the sheaf:

$$V \longmapsto f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}(V)),\ V \text{ open in } X.$$

(3) For a sheaf $\mathcal{F}$ on X, $f^*\mathcal{F}$ ($f = i$ or j) is defined to be the sheaf on W associated to the presheaf:

$$V \longmapsto \varinjlim_{U'} \mathcal{F}(U'),$$

V open in W and U' ranges through the family of open neighborhoods of $f(V)$ in X.
(4) For a sheaf on X, denote $\mathcal{F}^F$ the subsheaf of $\mathcal{F}$ such that its sections over the open set V are

$$\Gamma(V, \mathcal{F}^F) = \{s \in \Gamma(V, \mathcal{F} \mid \text{supp}(s) \subseteq W\}.$$

We define $i^!\mathcal{F}$ as $i^*\mathcal{F}^F$.

Remark 4.2 : 1) j^*, i^* may be described as

$$f^*(\mathcal{F})_x = \mathcal{F}_{f(x)}, \forall x \in U\ (resp. F), and\ f = j\ (resp. i).$$

They are clearly exact.

2) $j_!$ may be described by $j_!(\mathcal{F})_x = \mathcal{F}_x$ $\forall x \in U$ and zero elsewhere. It is clearly exact.

3) Direct images are defined as usual. We have $i_* = i_!$ (extension by zero).

4) j_* is right adjoint to j^* as usual. Also $i^!$ is right adjoint to $i_! = i_*$, since the adjunction map $\mathcal{G} \to i^! i_* \mathcal{G}$ is an isomorphism. Therefore j_* and $i^!$ are both left exact.

5) We keep the notations $\mathcal{M}(X, \mathcal{O})$ (etc.) as in [1]. To avoid confusion, we have to point out that we have only used their properties as sheaves of abelian goups, though the formulae concerned are true in a more general context. When necessary, we will mention the extra assumptions to insure the validity of, say, Verdier duality.

The six operations form two sets of 3 adjoint functors $(j_!, j^! = j^*, j_*)$ and $(i^*, i_* = i_!, i^!)$. One has $j^*i_* = 0$, whence by adjunction $i^*j_! = 0$, and $i^!j_* = 0$. For every sheaf $\mathcal{F}$ on X, the adjunction maps give the exact sequences

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

(4.1.1) and

$$0 \to i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$$

(which can be completed by a 0 at right for injective $\mathcal{F}$).

For $\mathcal{F}$ on F (resp. U), they also give the isomorphisms

(4.1.2) $$i^*i_*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \xrightarrow{\sim} i^!i_*\mathcal{F},$$

(4.1.3) $$j^*j_*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \xrightarrow{\sim} j^*j_!\mathcal{F}.$$

For each pair of adjoint functors (T^*, T_*), the adjunction map $T^*T_* \to Id.$ (resp. $Id \to T_*T^*$) is an isomorphism if and only if T_* (resp. T^*) is fully faithful. Hence the assertions (4.1.2), (4.1.3) are equivalent to : i_*, j_* and $j_!$ are fully faithful. When a functor T between the abelian categories are exact, it passes trivially to the derived categories. We usually use the same notation for its extension to the derived categories. This extension is at the same time the left derived functor LT and the right derived functor RT of T.

4.3 Let $\mathcal{D}$, $\mathcal{D}_U$, and $\mathcal{D}_F$ be three derived categories, and i_*, j^* be the exact functors:

$i_* : \mathcal{D}_F \to \mathcal{D}$, $j^* : \mathcal{D} \to \mathcal{D}_U$.

We write $i_! := i_*$ and $j^! := j^*$. And we assume the following conditioins are satisfied.

(4.3.1) i_* has a left adjoint and a right adjoint, denoted i^* and $i^!$.

(4.3.2) j^* has a left adjoint and a right adjoint, denoted $j_!$ and j_*.

(4.3.3) One has $j^*i_* = 0$ and by adjunction $i^*j_! = 0$ and $i^!j_* = 0$. For $A \in \mathcal{D}_F$ and $B \in \mathcal{D}_U$,

$$\mathrm{Hom}(j_!B, i_*A) = 0 \text{ and } \mathrm{Hom}(i_*A, j_*B) = 0.$$

(4.3.4) For every $K \in \mathcal{D}$, there exists $d : i_*i^*K \to j_!j^*K[1]$(resp. $d : j_*j^* \to i_*i^!K[1]$), unique by (4.3.3) and (2.18), such that the triangle $j_!j^*K \to K \to i_*i^*K \to$ (resp. $i_*i^!K \to K \to j_*j^*K \to$) is distinguished.

(4.3.5) i_*, $j_!$ and j_* are fully faithful: the adjunction morphisms $i^*i_* \to Id \to i^!i_*$ and $j^*j_* \to Id \to j^*j_!$ are isomorphisms.

The above formulation is self-dual; the duality exchanges $j_!$ and j_* as well as i^* and $i^!$.

4.4. a) Since the functor i_* is fully faithful, the composotion of the adjunction morphisms $i_*i^! \to Id \to i_*i^*$ is the i_* of the unique morphism of functors

(4.4.1) $i^! \to i^*$.

When this morphism applies to i_*X, and when we identify $i^!i_*X$ and i^*i_*X with X, we obtain the identity automorphism of X.

b) The composition of adjunction morphisms $j_!j^* \to Id \to j^*j^*$ comes from a unique morphism of functors

(4.4.2) $j_! \to j_*$.

If we identify $j^*j_!$ and j^*j_* with the identity functor, then j^* of (4.4.2) is the identity automorphism.

c) For $X \in \mathcal{D}_U$, the cone over $j_!X \to j_*X$ is therefore annihilated by j^*: it is in $i_*\mathcal{D}_F$.

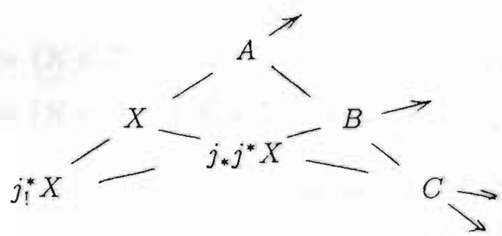
By (4.3.3) and 2.18, the distinguished triangle of base $j_!X \to j_*X$ is unique up to unique isomorphism, hence we have a functor $j_*/j_! : \mathcal{D}_U \to \mathcal{D}_F$ such that

(4.4.3) $(j_!, j_*, i_*(j_*/j_!))$

is a distinguished triangle. The dual construction gives a functor $T : \mathcal{D}_U \to \mathcal{D}_F$ which is characterized by a functorial distinguished triangle $(i_*T, j_!, j_*)$. A triangle of this type is deduced from (4.4.3) by translation, hence we have an isomorphism $T = (j_*/j_!)[-1]$. By applying i^* and $i^!$ to (4.4.3), and using $i^*j_! = i^!j_* = 0$, we obtain the isomorphism

(4.4.4) $i^*j_* \xrightarrow{\sim} j_*/j_! \xrightarrow{\sim} i^!j_![1]$.

4.5 Let $X \in \mathcal{D}$ and apply TR4 to the adjunction morphisms $j_!j^*X \to X \to j_*j^*X$. We want to show that the octahedron thus obtained



is unique up to unique isomorphism, and is functorial in X.

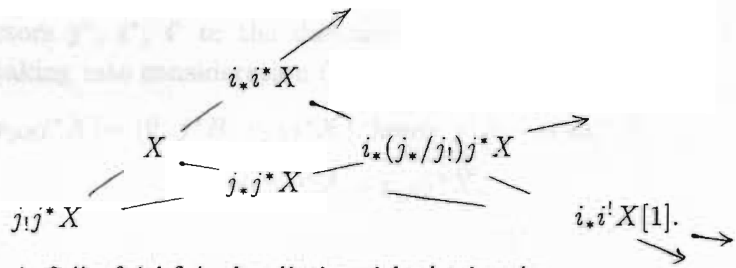
(a) By (4.3.3), (4.3.4) and 2.18, there exists a unique isomorphism $A = i_*i^*X$, which identifies $X \to A$ with the adjunction map. It also identifies $(j_!j^*X, X, A)$ to the distinguished triangle $(j_!j^*X, X, i_*i^*X)$ of (4.3.4).

(b) By applying the same argument to j_*j^*X, we can identify B with $i_*i^*j_*j^*X = i_*(j_*/j_!)j^*X$ (4.4.4). By 2.17, a unique morphism $A \to B$ makes the upper cap of the octahedron diagram commutative. The morphism $A \to B$, i.e. $i_*i^*X \to i_*i^*j_*j^*X$ is therefore the one deduced from $X \to j_*j^*X$ by functoriality.

(c) Dually, there exists a unique isomorphism of C with $i_*i^!X[1]$, identifying the map of degree 1 of (X, j_*j^*X, C) with the adjunction morphism $i_*i^!X \to X$. (Here the triangle (X, j_*j^*X, C) is deduced from the triangle (4.3.4) $(i_*i^!X, X, j_*j^*X)$ by translation (TR2), by changing the sign of $j_*j^*X \xrightarrow{(1)} i_*i^!X$.) The morphism $B \to C$ is the unique morphism making the square $(B, C, j_!j^*X, X)$ commutative. By the isomorphism (4.4.4) of $j_*/j_!$ with $i^!j_![1]$, it is the morphism $i_*(j_*/j_!)j^*X = i_*i^!j_!j^*X[1] \to i_*i^!X[1]$ deduced from $j_!j^*X \to X$ by functoriality.

(d) We want to determine all the members, and all the maps of the octahedron (by $C \xrightarrow{(1)} A$ we mean the composition $C \xrightarrow{(1)} X \to A$) and prove its functoriality. If we replace A, B, C by the above representations, the octahedron can be written as

(4.5.1)



Since the functor i_* is fully faithful, the distinguished triangle $(i_*i^*X, i_*(j_*/j_!)j^*X, i_*i^!X[1])$ is the i_* of a triangle $(i_*X, (j_*/j_!)j^*X, i^!X[1])$, which is automatically distinguished by 3.15. The i_* of the map d of degree 1 of this triangle is the composition $i_*i^!X[1] \xrightarrow{(1)} X \to i_*i^*X$ such that d is (4.4.1) for $X[1]$ (= the transform by [1] of (4.4.1) for X). By turning the triangle (TR2), changing the sign of the new map of degree 1 and irascing i_* (3.15), we obtain a functorial distinguished triangle

(4.5.2) $(i^!, i^*, (j_*/j_!)j^*)$

of base (4.4.1). In the framework of the discussion before 4.3 of this section, this triangle arises because for each flabby sheaf $\mathcal{F}$, the sequence $0 \to i_!\mathcal{F} \to i^*\mathcal{F} \to i^*j_*j^*\mathcal{F} \to 0$ is exact.

Now we can state the construction of glueing two given t−structures. Suppose that the hypothese of 4.4 are satisfied, and let $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ be a t−structure on $\mathcal{D}_U$, and $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ be a t−structure on $\mathcal{D}_F$

Definition 4.6:

$$\mathcal{D}^{\leq 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\leq 0} \ \textit{and} \ i^*K \in \mathcal{D}_F^{\leq 0}\}$$
$$\mathcal{D}^{\leq 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\geq 0} \ \textit{and} \ i^!K \in \mathcal{D}_F^{\geq 0}\}.$$

Theorem 4.7: With the foregoing hypotheses and notations, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\leq 0})$ is a t-structure on $\mathcal{D}$.

We say that it is the t−structure deduced from $\mathcal{D}_U$ and $\mathcal{D}_F$ by *glueing*.

Proof: We have to verify axioms of 3.1.

Axiom (i). Let $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$. The first triangle of (4.4.4) for X gives an exact sequence

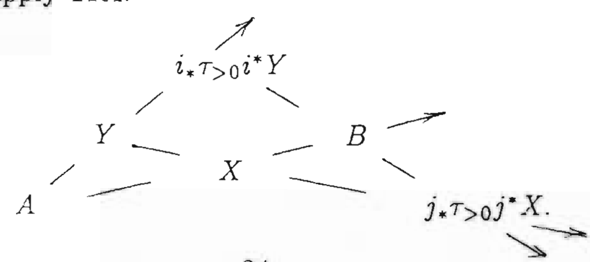
$$\mathrm{Hom}(i_*i^*X, Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(j_!j^*X, Y).$$

We have $Hom(i_*i^*X, Y) = Hom(i^*X, i^!Y) = 0$, by 3.1 for $\mathcal{D}_F$, and

$\mathrm{Hom}(j_!j^*X, Y) = \mathrm{Hom}(j^*X, j^*Y) = 0$, by 3.1 for $\mathcal{D}_U$. The assertion follows.

Axiom (ii). It follows trivially from 3.1 (i) for $\mathcal{D}_U$ and $\mathcal{D}_F$.

Axiom (iii). Let $X \in \mathcal{D}$. Choose Y, then A, to make the distinguished triangles (Y, X, $j_*\tau_{>0}j^*X$) and (A, Y, $i_*\tau_{>0}i^*Y$), and apply TR4:



We apply the functors j^*, i^*, $i^!$ to the distinguished triangles of this octahedron, in the following manner, taking into consideration (4.4.1) to (4.4.5) :

$j^*(i_*\tau_{>0}i^*Y,\ B,\ j_*\tau_{>0}j^*X)= (0,\ j^*B,\ \tau_{>0}j^*X)$, hence $j^*B \xrightarrow{\sim} \tau_{>0}j^*X$,

$j^*(A,\ X,\ B)=(j^*A,\ j^*X,\ \tau_{>0}j^*X)$, hence $j^*A \simeq \tau_{\leq 0}j^*X$,

$i^*(A,\ Y,\ i_*\tau_{>0}i^*Y)=(i^*A,\ i^*Y,\ \tau_{>0}i^*Y)$, hence $i^*A \simeq \tau_{\leq 0}i^*Y$,

$i^!(i_*\tau_{>0}i^*Y,\ B,\ j_*\tau_{>0}j^*X)= (\tau_{>0}i^*Y,\ i^!B,\ 0)$, hence $\tau_{>0}i^*Y \simeq i^!B$.

Therefore we have $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$, and $(A,\ X,\ B)$ satisfies 3.1 (iii).

4.8. Suppose that we have only a t-structure on $\mathcal{D}_F$, and we apply 4.7 to the degenerate t-structure $(\mathcal{D}_U, 0)$ on $\mathcal{D}_U$ and the given t-structure on $\mathcal{D}_F$. We denote the functors relative to the t-structure on $\mathcal{D}$ by $\tau^F_{\leq p}$. The functor $\tau^F_{\leq p}$ is right adjoint to the inclusion of the full subcategory of $\mathcal{D}$ consisiting of X such that i^*X is in $\mathcal{D}_F^{\geq p}$. As in the proof of axiom 3.1(iii) of 4.7, we know that $(\tau^F_{\leq p}X, X, i_*\tau_{>p}i^*X)$ is a distinguished triangle (with the notations of 4.7, we have $X = Y$). THe recursive H^p for this t-structure are therefore equal to $i_*H^pi^*X$.

Dually, we define $\tau^F_{\geq p}$ in terms of the degenerate t-structure $(0, \mathcal{D}_U)$ on $\mathcal{D}_U$; it is left adjoint to the inclusion of $(i^!)^{-1}(\mathcal{D}_F^{\geq p})$ in $\mathcal{D}$; we have a distinguished triangle $(i_*\tau_{<p}i^!X, X, \tau^F_{\geq p}X)$. And the H^p are $i_*H^pi^!X$.

In the same way, if we are only given a t-structure on $\mathcal{D}_U$ and we let $\mathcal{D}_F$ be the degenerate t-structure $(\mathcal{D}_F, 0)$ (resp.$(0, \mathcal{D}_F)$), then we can define on $\mathcal{D}$ a t-structure for which the functors $\tau_{\leq p}$ (resp.$\tau_{\geq p}$) are denoted by $\tau^U_{\leq p}$ (resp.$\tau^U_{\geq p}$); they give rise to the distinguished triangle $(\tau^U_{\leq p}X, X, j_*\tau_{>p}j^*X)$ (resp. $(j_!\tau_{<p}j^*X, X, \tau^U_{\geq p}X)$) for which the H^pX are $j_*H^pj^*X$ (resp. $j_!H^pj^*X$).

From the proof of the axiom(iii) of 4.7, it follows that $\tau_{\leq 0} = \tau^F_{\leq 0}\tau_{\leq} 0^U$. By translation and duality, we obtain

(4.8.1) $\tau_{\leq p} = \tau^F_{\leq p}\tau^U_{\leq p}$ and $\tau_{\geq p} = \tau^F_{\geq p}\tau^U_{\geq p}$.

Definition 4.9: Let Y be an object of $\mathcal{D}_U$. An object of $\mathcal{D}$ is said to be an extension of Y if there is an isomorphism $j^*X \simeq Y$. such isomorphism gives by adjunction the morphisms $j_!Y \to X \to j_*Y$. If an extension X is isomorphic to $\tau^F_{\geq p}j_!Y$ (resp. $\tau^F_{\leq p}j_*Y$), then the isomorphism is unique, and we simply let $X = \tau^F_{\geq p}j_!Y$ (resp. $\tau^F_{\leq p}j_*Y$).

Proposition 4.10: Let $Y \in \mathcal{D}_U$, and p an integer. Then there exists a unique(up to a unique isomorphism) extension X of Y such that $i^*X \in \mathcal{D}_F^{\leq p-1}$ and $i^!X \in \mathcal{D}_F^{\geq p+1}$; in fact $X = \tau^F_{\leq p-1}j_*Y = \tau^F_{\geq p+1}j_!Y$.

Proof : The distinguished triangle $(i^*X,\ (j_*/j_!)Y,\ i^!X[1])$ shows that the following conditions are equivalent:

(a) $i^*X \in \mathcal{D}_F^{\leq p-1}$, $i^!X \in \mathcal{D}_F^{\geq p+1}$;

(b) $i^!X[1] = \tau_{\geq p}(j_*/j_!)Y = \tau_{\geq p}i^*j_*Y$

(b') $i^*X = \tau_{\leq p-1}(j_*/j_!)Y$.

The distinguished triangle $(X,\ j_*Y,\ i_*i^!X[1])$ shows that

(b) $\iff$ $X = \tau^F_{\leq p-1}j_*Y$ and dually

$(b') \iff X = \tau^F_{\geq p+1} j_! Y$. QED

4.10.1. Let $\mathcal{D}_m$ be the full subcategory of $\mathcal{D}$ consisting of objects X satisfying the conditions $i^*X \in \mathcal{D}_F^{\leq p-1}$ and $i^!X \in \mathcal{D}_F^{\geq p+1}$ of 4.10. The functor j^* induces an equivalence of categories $\mathcal{D}_m \to \mathcal{D}_U$. It admits in fact $\tau^F_{\leq p-1} j_*$ as quasi-inverse. We sometimes denote this quasi- inverse by ${}^p j_{!*}$.

Definition 4.11: Let $\mathcal{C}$, $\mathcal{C}_U$, $\mathcal{C}_F$ be cores of the t-categories $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_F$; $\mathcal{D}_U$, $\mathcal{D}_F$, are with given t-structures, and $\mathcal{D}$ the t-structure of 4.7. Let ε be the inclusion of $\mathcal{C}$, $\mathcal{C}_U$, $\mathcal{C}_F$ in $\mathcal{D}$, $\mathcal{D}_U$, $\mathcal{D}_F$. And for $T = j_!$, j^*, j_*, i^*, i_*,and $i^!$, let ${}^pT = H^0 \circ T \circ \varepsilon$.

Since by definition of t-structure of $\mathcal{D}$, j^* is t-exact, i^* is right t-exact and $i^!$ is left t-exact. Applying 3.13(iii), we have

Proposition 4.12: (i) The functors $j_!$, and i^* (resp. j^* and i_*, resp. j_* and $i^!$) are right t-exact(resp. t-exact, resp. left t-exact).

(ii) $({}^pj_!, {}^pj^*, {}^pj_*)$ and $({}^pi^*, {}^pi_*, {}^pi^!)$ form two sets of three adjoint functors.

Proposition 4.13: (i) The composition ${}^pj^* \circ {}^pi_*$, ${}^pi^* \circ {}^pj_!$, and ${}^pi^! \circ {}^pj_*$ are zero; for $A \in \mathcal{C}_F$ and $B \in \mathcal{C}_U$, $\mathrm{Hom}({}^pj_!B, {}^pi_*A) = \mathrm{Hom}({}^pi_*A, {}^pj_*B) = 0$.
(ii) For every A in $\mathcal{C}$, the sequences

$$ {}^pj_!{}^pj^*A \to A \to {}^pi_*{}^pi^*A \to 0 $$

and

$$ 0 \to {}^pi_*{}^pi^!A \to A \to {}^pj_*{}^pj_*A $$

are exact.
(iii) ${}^pi_*$, ${}^pj_!$ and ${}^pj_*$ are fully faithful: the adjunction morphisms ${}^pi^*{}^pi_* \to Id \to {}^pi^!{}^pi_*$ and ${}^pj^*{}^pj_* \to Id \to {}^pj^*{}^pj_!$ are isomorphism.
Proof: They are consequences of 4.3.3, 4, 5 and 3.13(iv).

4.13.1. From (i) and either of the exact sequences of (ii), it follows that an object X in $\mathcal{C}$ is in the essential image $\bar{\mathcal{C}}_F$ of ${}^pi_*$ if and only if ${}^pj^*X = 0$. Since the functor ${}^pj^*$ is exact, this essential image is a thick subcategory (i.e. stable by extensions and by taking quotients) of $\mathcal{C}$. If we identify $\mathcal{C}_F$ via the fully faithful functor ${}^pi_*$ with the subcategory $\bar{\mathcal{C}}_F$ of $\mathcal{C}$, then the adjunctions $({}^pi^*, {}^pi_*)$ and $({}^pi_*, {}^pi^!)$ (4.12(ii)) show that for X in $\mathcal{C}$, ${}^pi^*X$ is the largest quotient of X which is in $\mathcal{C}_F$, and ${}^pi^!X$ is the largest sub-object of X which is in $\mathcal{C}_F$.

Proposition 4.14: The functor ${}^pj^*$ identifies $\mathcal{C}_U$ with the quotient of $\mathcal{C}$ by the thick subcategory $\mathcal{C}_F$ (or more precisely $\bar{\mathcal{C}}_F$; cf.4.13.1).
Proof: Let $Q : \mathcal{C} \to \mathcal{C}/\mathcal{C}_F$ be the quotient functor. THe exact functor ${}^pj^*$ admits a factorization $T \circ Q$, and T is faithful: if f in $\mathcal{C}/\mathcal{C}_F$ comes from f_1 in $\mathcal{C}$, and f is killed by T, then f_1 is killed by ${}^pj^*$, i.e. $\mathrm{Im}(f_1)$is in the image of $\mathcal{C}_F$, and f_1 is killed by Q. Since $Id \xrightarrow{\sim} {}^pj^*{}^pj_! = T \circ Q \circ {}^pj_!$, T is surjective over the isomorphism classes of objects. It remains to show that T is fully faithful, therefore an equivalence of categories. We need a lemma which is a result of (4.3.4) and 4.12(i).

Lemma 4.15: For every A in $\mathcal{C}$, the sequences

$$0 \to {}^p i_* H^{-1} i^* A \to {}^p j_! {}^p j^* A \to A \to {}^p i_* {}^p i^* A \to 0$$

and

$$0 \to {}^p i_* {}^p i^! A \to A \to {}^p j_* {}^p j^* A \to {}^p i_* H^1 i^! A \to 0$$

are exact.

Now we complete the proof of 4.14. By the above lemma, the kernel and cokernel of ${}^p j_! {}^p j^* A \to A$ are in the image of ${}^p i_*$. Therefore every object of $\mathcal{C}/{}^i_*\mathcal{C}_F$ has a representative in the image of ${}^p j_!$. For ${}^p j_! X$ and ${}^p j_! Y$ in this image,

$$T : \mathrm{Hom}(Q^p j_! X, Q^p j_! Y) \to \mathrm{Hom}(TQ^p j_! X, TQ^p j_! Y) = \mathrm{Hom}(X, Y)$$

is true for every section of $Q^p j_!$, hence is surjective. QED.

4.16. (a) Since the functor ${}^p i_*$ is fully faithful, the composition of the adjunction morphisms ${}^p i_* {}^p i^! \to Id \to {}^p i_* {}^p i^*$ is the ${}^p i_*$ of a unique morphism of functors

(4.16.1) ${}^p i^! \to {}^p i^*$.

The diagrams 3.14(ii) for (i^*, i_*) and $(i_*, i^!)$, and the t-exactness of i_* give for A in $\mathcal{C}$ a commutative diagram

$$\begin{array}{ccccc} {}^p i_* {}^p i^! A & \longrightarrow & A & \longrightarrow & {}^p i_* {}^p i^* A \\ \| & & \| & & \| \\ i_* {}^p i^! A & & \| & & i_* {}^p i^* A \\ \downarrow & & \| & & \uparrow \\ i_* i^! A & \longrightarrow & A & \longrightarrow & i_* i^* A \end{array}$$

from which it follows that for A in $\mathcal{C}$, the morphism (4.16.1): ${}^p i^! A \to {}^p i * A$ is the composition

(4.16.2) ${}^p i^! A \to i^! A \xrightarrow{(4.4.1)} i^* A \to {}^p i^* A$.

When we apply (4.16.1) to $i_* A$ (A in $\mathcal{C}_F$), we obtain the identity automorphism of A.

(b) Since ${}^p j^*$ is a quotient functor (4.14), the composition of the adjunction morphisms ${}^p j_! {}^p j^* \to Id \to T^p j_* {}^p j^*$ comes from a unique morphism of functors

(4.16.3) ${}^p j_! \to {}^p j_*$.

The diagrams 3.14(ii) for (j^*, j_*) and $(j_!, j^*)$, and the t-exactness of j^* give for A in $\mathcal{C}$ a commutative diagram

$$\begin{array}{ccccc} {}^p j_! {}^p j^* A & \longrightarrow & A & \longrightarrow & {}^p j_* {}^p j^* A \\ \| & & \| & & \| \\ {}^p j_! j^* A & & \| & & {}^p j_* j^* A \\ \uparrow & & \| & & \downarrow \\ j_! j^* A & \longrightarrow & A & \longrightarrow & j_* j^* A \end{array}$$

from which it follows that for B in $\mathcal{C}_U$, the morphism (4.4.2) of $j_! B$ in $j_* B$ is the composition

(4.16.4) $j_! B \to \tau_{\geq 0} j_! B = {}^p j_! B \xrightarrow{(4.16.3)} {}^p j_* B = \tau_{\leq 0} j_* B \to j_* B$.

When we apply ${}^p j^*$ to (4.16.3), we obtain the identity automorphism of the identity functor. In particular, for B in $\mathcal{C}_U$, the kernel and cokernel of (4.16.3): ${}^p j_! B \to {}^p j_* B$ are in ${}^p i_* \mathcal{C}_F$.

Definition 4.17: The functor $j_{!*}$ assigns $B \in \mathcal{C}_U$ the image of ${}^p j_! B$ in ${}^p j_* B$. We thus have a series of maps

$$j_!B \to {}^pj_!B \to j_{!*}B \to {}^pj_*B \to j_*B$$

Proposition 4.18: For $B \in \mathcal{C}_U$, we have

$${}^pj_!B = \tau^F_{\geq 0}j_!B = \tau^F_{\leq -2}j_*B$$
$$j_{!*}B = \tau^F_{\geq 1}j_!B = \tau^F_{\leq -1}j_*B$$
$${}^pj_*B = \tau^F_{\geq 2}j_!B = \tau^F_{\leq 0}j_*B.$$

More precicely, ${}^pj_!B$ with the map $j_!B \to {}^pj_!B$ is $\tau^F_{\geq 0}j_!B$, etc.

Proof : Since

Since $j^*j_!B \in \mathcal{C}_U$, we have $j_!B = \tau^U_{\geq 0}j_!B$. By 4.8.1, we have ${}^pj_!B = \tau_{\geq 0}j_!B = \tau^F_{\geq 0}j_!B$; by 4.10, $\tau^F_{\geq 0}j_!B = \tau^F_{\leq -2}j_*B$. Likewise, ${}^pj_*B = \tau^F_{\geq 2}j_!B = \tau^F_{\leq 0}j_*B$.

4.8 gives a distinguished triangle $(i_*H^0i^!j_!B,\ \tau^F_{\geq 0}j_!B,\ \tau^F_{\geq 1}j_!B)$. It shows that $\tau^F_{\geq 1}j_!B \in \mathcal{D}^{[-1,0]}$.

A dual argument gives a distinguished triangle $(\tau^F_{\leq -1}j_*B,\ \tau^F_{\leq 0}j_*B,\ i_*H^0i^*B)$ which shows that $\tau^F_{\leq -1}j_*B \in \mathcal{D}^{[0,1]}$.

By 4.10, we find that $\tau^F_{\geq 1}j_!B = \tau^F_{\leq -1}j_*B \in \mathcal{C}$, and the above triangles become short exact sequences

$$0 \to i_*H^0i^!j_!B \to {}^pj_!B \to \tau^F_{\geq 1}j_!B \to 0$$
$$0 \to \tau^F_{\leq -1}j_*B \to {}^pj_*B \to i_*H^0i^*j^*B \to 0$$

They show that $\tau^F_{\geq 1}j_!B = \tau^F_{\leq -1}j_*B$ is just the image $j_{!*}$ of ${}^pj_!B$ in ${}^pj_*B$.

Definition 4.19: The subcategory ${}^p\mathcal{D}^{\leq 0}(X,\mathcal{O})$ (resp. ${}^p\mathcal{D}^{\geq 0}(X,\mathcal{O})$) of $\mathcal{D}(X,\mathcal{O})$ is the subcategory consisting of complexes K (resp. K in $\mathcal{D}^+(X,\mathcal{O})$) such that for each stratum S (we denote by i_S the inclusion of S in X), we have $H^n i_S^* K = 0$ for $n > p(S)$ (resp. $H^n i_S^! K = 0$ for $n < p(S)$).

Remark 4.20: If a, b are integers such that $a \leq p \leq b$, then it can be shown that (cf.[1] p.56)

$$\mathcal{D}^{\leq a}(X,\mathcal{O}) \subset {}^p\mathcal{D}^{\leq 0}(X,\mathcal{O}) \subset \mathcal{D}^{\leq b}(X,\mathcal{O})$$

(4.20.1) and

$$\mathcal{D}^{\geq a}(X,\mathcal{O}) \supset {}^p\mathcal{D}^{\geq 0}(X,\mathcal{O}) \supset \mathcal{D}^{\geq b}(X,\mathcal{O}).$$

We denote by ${}^p\mathcal{D}^{+,\leq 0}(X,\mathcal{O})$ the intersection of $\mathcal{D}^+(X,\mathcal{O})$ and ${}^p\mathcal{D}^{\leq 0}(X,\mathcal{O})$. In the same way we define the categories with $+$ replaced by $-$, b, and with 0 replaced by n.

Proposition 4.21: For each perversity p, $({}^p\mathcal{D}^{+,\leq 0}(X,\mathcal{O}), {}^p\mathcal{D}^{\geq 0}(X,\mathcal{O}))$ is a t-structure on $\mathcal{D}^+(X,\mathcal{O})$.

Proof: By induction on the number N of strata: if $N = 0$, we have $X = \emptyset$ and the assertion is clear. If $N = 1$, we are back to the natural t-structure, with $p(X)$ as translation. For $N \geq 2$, let F be a proper closed subset which is a union of strata, and U the open complement. For example we can take a closed stratum as F. By induction hypothesis applied to F, and U, with induced stratifications, we have the t-structures on $\mathcal{D}^+(U,\mathcal{O})$ and $\mathcal{D}^+(F,\mathcal{O})$. Then the t-structure on $\mathcal{D}^+(X,\mathcal{O})$ results from 4.7.

Corollary 4.22: $({}^p\mathcal{D}^{\leq 0}(X,\mathcal{O}), {}^p\mathcal{D}^{\geq 0}(X,\mathcal{O}))$ is a t-structure on $\mathcal{D}(X,\mathcal{O})$. It induces a t-structure on $\mathcal{D}^*(X,\mathcal{O})$ for $* = +,-,b$.

Remark 4.23: Denote by ${}^p\tau$ the corresponding functors τ. Since ${}^p\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq b}$ and ${}^p\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq a}$, the (usual) cohomological long exact sequence of the distinguished triangle $({}^p\tau_{\leq 0}K, K, {}^p\tau_{\geq 1}K)$ shows that $H^i{}^p\tau_{\leq 0}K = H^iK$ for $i < a$. Similarly, $H^i{}^p\tau_{\leq 0}K \xrightarrow{\sim} H^iK$ for $i > b$. It follows that ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$ respect $\mathcal{D}^*$ $(* = +,-,b)$.

Definition 4.24: The category $\mathcal{M}(p,X,\mathcal{O})$ of p-perverse sheaves of $\mathcal{O}$-modules on X is defined to be ${}^p\mathcal{D}^{\leq 0}(X,\mathcal{O}) \cap {}^p\mathcal{D}^{\geq 0}(X,\mathcal{O})$. It is an admissible abelian subcategory of $\mathcal{D}^b(X,\mathcal{O}$.

Proposition 4.25: Let U be a locally closed subset of X, which is a union of strata, and let j be the inclusion of U in X. Then for each perversity p, the functors $j_! : \mathcal{D}(U,\mathcal{O}) \to \mathcal{D}(X,\mathcal{O})$ and $j^* : \mathcal{D}(X,\mathcal{O} \to \mathcal{D}(U,\mathcal{O})$ are right t-exact.

Proof: It follows directly from the definitions. We omit the details.

4.26 Notation: We omit $(X,\mathcal{O})$ in the notation. We sometimes write $\mathcal{D}^{\leq p}$ (resp. $\mathcal{D}^{\geq p}$) instead of ${}^p\mathcal{D}^{\leq 0}$ (resp. ${}^p\mathcal{D}^{\geq 0}$). For p of a constant value a, we have $\mathcal{D}^{\leq p} = \mathcal{D}^{\leq a}$ (in the sense of the natural t-structure), and $\mathcal{D}^{\geq p} = \mathcal{D}^{\geq a}$. For each integer$n$, we have $\mathcal{D}^{\leq p+n} = {}^p\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq p+n} = {}^p\mathcal{D}^{\geq n}$. Finally, for $p \leq q$, we have $\mathcal{D}^{\leq p} \subset \mathcal{D}^{\leq q}$ and $\mathcal{D}^{\geq p} \supset \mathcal{D}^{\geq q}$. This generalizes (4.20.1). Similarly, we write $\tau_{\leq p}$ and $\tau_{\geq p}$ for ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$, and H^p for the H^0 in the sense of the t-structure of perversity p. In the situation of 4.25, we have used the same notations $j_!$, $j^!$, j_*, and j^* in the category of sheaves for the derived functors. We will use p in the left exponent of a functor to mean that it is deduced by passing to p-perverse sheaves. For example, for A in $\mathcal{M}(p,U,\mathcal{O})$ we set ${}^pj_!A = \tau_{\geq p}j_!A = H^p(j_!A)$. By 3.14(i), $({}^pj_!, {}^pj^!)$ and $({}^pj^*, {}^pj_*)$ are two pairs of adjoint functors.

The functors $j_!$, $j^!$, j_*, j^* for the usual sheaves will be denoted with 0 in the left exponent: they correspond to the 0 perversity.

For a p-perverse sheaf A on U, $j_!A$ is in $\mathcal{D}^{\leq p}(X,\mathcal{O})$ and j_*A is in $\mathcal{D}^{\geq p}(X,\mathcal{O})$. The natural morphism $\alpha : j_!A \to j_*A$ admits a factorization

$$j_!A \to {}^pj_!A \xrightarrow{\beta} {}^pj_*A \to j_*A \quad (\beta = {}^pH^0(\alpha)).$$

The functor ${}^pj_{!*}$ or simply $j_{!*}$, is denfined by

$$j_{!*}A = \operatorname{Im}({}^pj_!A \to {}^pj_*A).$$

For a p-perverse sheaf A on X, we define a canonical morphism: ${}^pj^!A \to {}^pj^*A$ as the composotion ${}^pj^!A \to j^!A \to j^*A \to {}^pj^*A$.

Remark 4.27: For inclusions $U \xrightarrow{k} V \xrightarrow{j} X$ of locally closed sets which are unions of strata, the transitivity formulae $(jk)_! = j_!k_!$, $(jk)_* = j_*k_*$, $(jk)^! = k^!j^!$, $(jk)^* = k^*j^*$ give (3.13 (iv)) the analogous formulae for the p-perverse sheaves. By applying ${}^pj_!$, ${}^pj_{!*}$ and ${}^pj_*$ to the morphisms ${}^pk_! \longrightarrow {}^pk_{!*} \hookrightarrow {}^pk_*$, we obtain a chain of maps

$${}^p(jk)_! = {}^pj_!{}^pk_! \twoheadrightarrow {}^pj_!{}^pk_{!*} \twoheadrightarrow {}^pj_{!*}{}^pk_{!*} \hookrightarrow {}^pj_*{}^pk_{!*} \hookrightarrow {}^pj_*{}^pk_* = {}^p(jk)_*$$

which gives an isomorphism of functors

(4.27.1) ${}^p(jk)_{!*} = {}^pj_{!*}{}^pk_{!*}$.

Proposition 4.28: For B in $\mathcal{M}(p,U,\mathcal{O})$, $j_{!*}B$ is the unique extension P of B in $\mathcal{D}(X,\mathcal{O})$ such that for each stratum $S \subset F$ (denote by s its inclusion in X), we have $H^i s^* P = 0$ for $i \geq p(S)$ and $H^i s^! P = 0$ for $i \leq p(S)$.

Proof: It follows from 4.10.1 that if $\mathcal{D}'$ is the subcategory of $\mathcal{D}(X,\mathcal{O})$ formed by K such that $H^i s^* K = 0$ for $i \geq p(S)$ and $H^i s^! K = 0$ for $i \leq p(S)$ for each stratum $s : S \to F$, then j^* induces an equivalence $\mathcal{D}' \to \mathcal{D}(U,\mathcal{O})$. Its restriction to $\mathcal{D}' \cap \mathcal{M}(p,X,\mathcal{O})$ is an equivalence $\mathcal{D}' \cap \mathcal{M}(p,X,\mathcal{O}) \to \mathcal{M}(p,U,\mathcal{O})$, of inverse $j_{!*}$.

Remark 4.29: ${}^p j_! B$ and ${}^p j_* B$ can be charaterized analogously (cf. 4.18, [1]1.3.14).

To state the following proposition, we first make the assumptions below:

4.30.

- If $S \subset \bar{T}$,then $p(S) \geq p(T)$.
- For each n, the union F_n(resp. U_n) of strata S such that $p(S) \geq n$ (resp. $p(S) \leq n$) is therefore closed (resp. open).
- Let $j_n : U_{n-1} \hookrightarrow U_n$ be the inclusions.

Proposition 4.31: With the hypothese and notations of 4.30, let A be a p-perverse sheaf on U_k, a an integer $\geq k$ such that $p \leq a$, and $j : U_k \hookrightarrow X = U_a$. We have

$j_{!*}A = \tau_{\leq a-1} j_{a*} \cdots \tau_{\leq k} j_{k+1*} A$. ($\tau_{\leq i}$ are relative to the natural t-structure)

Proof: By (4.27.1), we are led to verify that $(j_{k+1})_{!*}A = \tau_{\leq k} j_{k+1*} A$. Let $F = U_{k+1} - U_k$. By 4.18, we have

$$(j_{k+1})_{!*}A = {}^p\tau^F_{\leq -1}(j_{k+1})_* A.$$

On F, the function p is constant, of value $k+1$, and ${}^p\tau^F_{\leq -1}$ is just $\tau^F_{\leq k}$(for the natural t-structure on F). Since on U_k we have $p \leq k$, A is in $\mathcal{D}^{\leq k}(U,O)$ (4.20.1) and

and $\tau_{\leq k}(j_{k+1})_* A \xleftarrow{\sim} \tau^F_{\leq k}\tau^F_{\leq k} j_{k+1*} A$.

§5 Duality

5.1. We list some conditions and notations here:

a) $\mathcal{O}$ is the constant sheaf of value R, with R a field.

b) The stata are topological varieties, everywhere of the same dimension. If a stratum S is contained in the closure of a stratum T, dim$S$ <dim$T$.

c) For $j : S \to X$ a stratum, the functor ${}^0 j_*$ is of finite cohomological dimension over the category of sheaves of R-modules. For a locally constant sheaf of R-module $\mathcal{F}$ on S which is of finite type, the $R^i j_* F$ are again locally constant of finite type on each stratum.

For a locally closed set U which is union of strata, we write $\mathcal{D}_C(U,R)$ (or $\mathcal{D}_S(U,R)$ if we want to specify the stratification) for the full triangulated subcategory of $\mathcal{D}(U,R)$ formed by constructible K such that $H^i K$ are locally constant of finite type on each stratum. $\mathcal{D}^*_C(U,R)$ $(* = +,-,b)$ is similarly defined.

The condition c) insures that for $j : U \hookrightarrow V$ with U and V locally closed unions of strata, the functors $j_!$, $j^!$, j_* and j^* respect these subcategories. (cf.[1], p.61)

For closed set $F \subset U$ which is a union of strata, $\tau^F_{\leq a}$ respects trivially $\mathcal{D}_C(U,R)$. The proof of 4.7 shows therefore that for each perversity p, $\tau_{\leq p}$ and $\tau_{\geq p}$ respect $\mathcal{D}_C(X,R)$.

5.2. Besides the conditions of 5.1, suppose further that X admits a triangulation (locally finite) such that each stratum S of $\mathcal{S}$ is a union of (open) simplexes. For example: real algebraic variety with a Whitney stratification. We then have the theory of Verdier duality. The Verdier duality is an involutive automorphism of $\mathcal{D}_C(X,R)$, and for $j : U \to X$ locally closed, which is a union of strata, the duality interchanges the functors $j_!$ and j_*, as well as $j^!$ and j^*.

For each stratum S, of dimension d, with orientation sheaf or, the Verdier dualizing functor D on $\mathcal{S}$ ($K \longmapsto R\mathrm{Hom}(K, R \otimes or[d])$ satisfies: for $K \in \mathcal{D}_{\mathcal{S}}(S,R)$,

$$H^i\mathrm{D}K = (H^{-d-i}K)^{\vee} \otimes or.$$

Here it is essential that the cohomological sheaves of K are locally constant of finite rank, and R is a field (or a commutative local artinian ring, eg. $Z/\ell^n Z \ldots$).

Definition 5.3: The dual perversity p^* of p is defined by

$$p^*(S) = -p(S) - dim(S).$$

The foregoing discussion shows that D exchanges $\mathcal{D}^{\geq p}$ and $\mathcal{D}^{\leq p^*}$(as well as $\mathcal{D}^{\leq p}$ and $\mathcal{D}^{\geq p^*}$: we have $p = p^{**}$). In particular it exchanges p-perverse sheaves and p^*-perverse sheaves; ${}^{p^*}H^i$ and ${}^pH^{-i}$. For j inclusion of a locally closed set which is a union of strata, it exchanges ${}^pj_!$ and ${}^{p^*}j_*$, ${}^pj^!$ and ${}^{p^*}j^*$, and ${}^pj_{!*}$ with ${}^{p^*}j_{!*}$. In particular, in $\mathcal{D}_c(X,R)$, the defining conditions of p-perverse sheaves can be rewritten : for each stratum $j : S \hookrightarrow X$, we have

$$H^ij^*K = 0 \;\; \forall i > p(S)$$

and

$$H^ij^*\mathrm{D}K = 0 \;\; \forall i > p^*(S).$$

If all the strata are of even dimension, there exists a self-dual perversity: the one given by

$$p_{1/2}(S) = -\frac{1}{2}dimS.$$

Proposition 5.4: Under the hypotheses 5.2, if all the strata are of even dimension and for the self-dual perversity $p_{1/2}$, if $j : U \longrightarrow X$ is an open set of X which is union of strata and A a self-dual perverse sheaves on U, then $j_{!*}A$ is the unique self-dual extension P of A (in $\mathcal{D}_c(X,R)$) such that for each stratum $S \subset X - U$, H^iP are zero for $i \geq -\frac{1}{2}dimS$.
Proof: That $j_{!*}A$ is self-dual follows from the self-duality of A, and that of $j_{!*}$. With these observations, the proposition is an immediate application of 4.28 and 5.2.

Remark 5.5: If U is orientable and smooth, of pure dimension d, we can take for A the constant sheaf R, put in degree $-d/2$. For this choice $j_{!*}A$ is the intersection complex $IC^{\cdot}$ and 5.4 is the Verdier characterization of $IC^{\cdot}$.

§6 Conclusion

The construction of perverse sheaves is very abstract, and the process involved is sometimes very complicated. So one may ask for an explanation for all these troubles, since there already exist theories on intersection cohomology which satisfies the Poincaé duality, Lefschetz's hyperplane section theorem, etc. as described in Goresky-MacPherson's papers. As we have only touched on the beginning of this subject, the above question is a hard one to answer. However, let us say something pertinent about it. Goresky-MacPherson's construction is also very complicated, if not difficult. It seems not easy to get the essential points of the constructions involved. As for Deligne's construction, a great amount of homological algebra is employed. By using the well-developed sheaf languages and the etale topology, Deligne's construction can provide results which would be impossible by Goresky-MacPherson's construction (eg. there is no such concept as triangulation in the étale topology). Besides, Deligne's construction is algebraic and in many cases functorial, it applies to a very wide range of different fields. This kind of tool can help us proceed further.

In this note we have only restricted to the introduction and some discussion of the duality problem. A further work is still required. Although it may be redundant, we hope this note can be of help for those who want to get a feeling of what's going on for this topics and then can quickly recourse to the original exposition of the related topics.

§7 References

[1] Beilinson A., Bernstein J., Deligne P.: Faiseaux pervers Astérisque, 1982, t.100.

[2] Sergei I. Gelfand, Yuri I. Manin: Methods of Homological Algebra, Springer 1996

[3] Borel, A. et al.: Intersection Cohomology. Boston· Basel· Stuttgart: Birkhäuser 1984

[4] Borel, A. et al.: Algebraic D-modules. Academic Press,Inc.1987

[5] Verdier, J.L.: Catégories derivées, État 0. SGA $4\frac{1}{2}$. Lecture Notes in Mathematics vol.569. Berlin-Heidelberg-New York: Springer 1977

[6] Goresky, M. and MacPherson, R.: Intersection Homology Theory. Topology **19**, 135-162 (1980)

[7] Goresky, M. and MacPherson, R.: Intersection Homology II. Invent.Math. **71**, 77-129

[8] Iversen, B.: Cohomology of Sheaves, Berlin-Heidelberg-New York-Tokyo: Springer 1986

[9] Kashiwara, M., Schapira, P.: Sheaves on Manifolds, Springer-Verlag Berlin Heidelberg 1990