A Supplementary Note

on

DISCRETE DIFFERENTIAL GEOMETRY

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Preface

This is a supplementary note based on the note of Keenan Crane, Discrete Differential Geometry, and the note of Albert Chern and Peter Schröder. The figures of this supplementary note are copied from their notes.

Discrete differential geometry (DDG) is a relatively new subject in mathematical community and is becoming a basic language in computer graphics and computer-aided design. It is also used in elasticity and fluid mechanics, structure chemistry and condensate matter physics. The objects to be studied are those discrete surfaces, which are triangulated surfaces. People study topology, geometry and physics on these objects. The applications include curve and surface smoothing, surface parametrization, vector field design, computation of geodesic distance. DDG short course was introduced in ACM SIGGRAPH in 2013 and Annual meeting of American Math Society in 2018.

The main resources of this course come from

- Keenan Crane, Discrete Differential Geometry: An Applied Introduction, CMU (2020), `https://www.cs.cmu.edu/~kmcrane/Projects/DDG/`
- Albert Chern, Discrete Differential Geometry, UCSD (2020).
- Albert Chern and Peter Schröder, Discrete Differential Geometry, Caltech, 2017.
- Shizuo Kaji, NCTS Course on Image and Shape Manipulation 2018. `https://goo.gl/Joxorp`

For topics of differential geometry, I refer to

- Chern, Chen, Lam, Lectures on Differential Geometry (1998).

Topics: Combinatorial surfaces, curves and surfaces, curvature, connections and parallel transport, exterior algebra, exterior calculus, Stokes theorem, simplicial homology, de Rham cohomology, Helmholtz-Hodge decomposition, conformal mapping, finite element methods, and numerical linear algebra.

Applications: approximation of curvature, curve and surface smoothing, surface parameterization, vector field design, and computation of geodesic distance.

Coding exercises refer to a supplementary C++ framework, available from `https://github.com/dgpdec/course`. Java code is also available, provided by Crane. Matlab code will also be acceptable, Mei-Heng Yueh will provide code training lecture.

— I-Liang Chern 2019

Contents

Chapter 1

Curves

This chapter is based on A. Chern and P. Schröder's note.

1.1 Plane curves

1.1.1 Regular curves

- **Planar curve** A planar curve is a Lipschitz continuous function $\gamma : I \to \mathbb{R}^2$ with $I = [a, b]$ called the parameter space.
- **Regular curve**: A regular curve is a C^1-curve with $\gamma'(t) \neq 0$ for all t in parameter interval. Note that a regular curve cannot have cusps, for example, the cycloid is not a regular curve. But a regular curve allows to have self intersections.
- **Re-parameterization** of γ: if $\phi : I \to \tilde{I}$ is strictly increasing, then $\tilde{\gamma} = \gamma \circ \phi^{-1} : \tilde{I} \to \mathbb{R}^2$ is a re-parameterization of γ.

The arclength of a curve $\tilde{\gamma}$ is

$$s(t) = \int_a^t |\tilde{\gamma}'(t)|\, dt.$$

This defines a re-parameterization on $\tilde{\gamma}$. Namely, $s : [a, b] \to [0, L]$ is monotonic increasing. We can take its inverse, called $\phi : [0, L] \to [a, b]$. Then $\gamma := \tilde{\gamma} \circ \phi$ is the arclength parameterized.

Lemma 1.1. *A regular curve $\tilde{\gamma} : [a, b] \to \mathbb{R}^2$ can be re-parameterized by arclength.*

Below, we consider γ to be arclength parameterized.

Remark There is a gap between Lipschitz continuous curve and C^1 curve. See the reference: Simmon Blatt, "Curves Between Lipschitz and C^1 and Their Relation to Geometric Knot Theory" (2018).

Examples

1. Cycloid:
$$C_\alpha := \begin{cases} x = t - \alpha \sin t \\ y = 1 - \alpha \cos t \end{cases} \qquad \alpha > 0 \text{ is a parameter.} \tag{1.1}$$

2. Heart curve: $r = 1 - \sin\theta$.

3. Spiral curve: $x = r(\varphi)\cos(\varphi)$, $y = r(\varphi)\sin(\varphi)$, with

   (a) $r(\varphi) = a\varphi$ (Archimedean spiral),

   (b) $r(\varphi) = ae^{k\varphi}$ (logarithmic spiral),

   (c) ... see Spiral on wiki.

4. Folium of Descartes: $x^3 + y^3 - 3axy = 0$.

We may see more examples from list of curves and gallery of curves on wiki.

Homework A circular cycloid is the trajectory of a point on a disk which rolls around another disk. Write down the equation of such circular cycloid.

1.1.2 Geometric quantities

- **A geometric quantity** is a quantity that is independent of re-parameterization.
- **Tangent vector**: $T(t) := \gamma'(t)/|\gamma'(t)|$ is a geometric quantity. (Prove it.) For arclength parametrized curve γ, we have $|\gamma'| = 1$.* Thus, $T(s) = \gamma'(s)$. In Cartesian coordinate, we express $\gamma(s) = (x(s), y(s))$, then $T(s) := \gamma'(s) = (x'(s), y'(s))$. We can also express the curve in complex plane as $\gamma(s) = x(s) + iy(s)$, then $T(s) = x'(s) + iy'(s)$.
- **Normal:** $N(s) := (-y'(s), x'(s))$. In the complex plane, $N(s) = iT'(s)$.
- **Curvature:** $\kappa := \langle T', N\rangle = -\langle N', T\rangle$. The last equality is due to $\langle T, N\rangle = 0$. The term $\langle N', T\rangle$ measures how N varies along γ. You can show†
$$\boxed{N'(s) = -\kappa(s)T(s), \quad T'(s) = \kappa(s)N(s).} \tag{1.2}$$

Exercise

1. Show that curvature formula for $\tilde\gamma(t) = (x(t), y(t))$, $t \in [a, b]$ is given by
$$\kappa = \frac{\ddot y \dot x - \ddot x \dot y}{(\dot x^2 + \dot y^2)^{3/2}}.$$

2. Show that a circle with radius R has constant curvature $1/R$.

*Because $s = \int_0^s |\gamma'(\tau)|\,d\tau$. Differentiate in s both sides, we get $|\gamma'(s)| = 1$.

†Use $\langle T, T\rangle = \langle N, N\rangle = 1$ and $\langle T, N\rangle = 0$

Osculating circle approach of curvature Given a regular curve $\gamma(s)$. Let q, p, r be three consecutive points on γ, defined by $\gamma(s-h)$, $\gamma(s)$ and $\gamma(s+h)$. The points q, p, r determines a circle with radius $R_h(s)$. Show that $\lim_{h\to 0+} R_h(s) = 1/\kappa(s)$. This limiting circle is called the osculating circle of γ at p.

Turning angle approach of curvature We may treat $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and express $\gamma(s) = x(s) + iy(s)$. Since $|T| = 1$, we can write $T = e^{i\theta}$. The angle θ is the angle between T and the real axis, called the incline angle of γ. The corresponding normal is $N = iT$. Now, $T' = \kappa N$ reads

$$(e^{i\theta})' = i\theta' e^{i\theta} = \theta' N = \kappa N,$$

which shows

$$\boxed{\kappa = \theta'.}$$

As a consequence,

$$T(t_2) = e^{i(\theta_2-\theta_1)} T(t_1) = e^{i\int_{t_1}^{t_2} \kappa(t)\,dt} T(t_1).$$

For a closed curve with length L,

$$T(L) = e^{i\int_0^L \kappa(t)\,dt} T(0).$$

But $T(0) = T(L)$ for a closed curve. Thus, the total turning angle

$$\int_0^L \kappa(t)\,dt = 2\pi m,$$

for some integer m. This integer m is called *turning number* for a closed regular curve. The turning number is a topological quantity. It is invariant under regular deformations of the closed curve, which means that the deformation is continuous and the corresponding curvature form $\kappa(s)\,ds$ stays continuous. This is a theorem of Whitney-Graustein (1937). An example of non-regular deformation is the deformation of the cycloid (1.1). In equation (1.1), the curve has no loop when $\alpha < 1$ and has loop when $\alpha > 1$. It has a cusp when $\alpha = 1$. As α changes from $\alpha < 1$ to $\alpha > 1$, the deformation of C_α is not regular. The curvature κ goes to ∞ and κds has a jump 2π across $\alpha = 1$.

Variational approach of curvature Straight lines have zero curvature. Straight lines are also the shortest distance between two points. Thus, we expect curvature can be characterized by the variation of arclength. Using length as the potential energy, we will see below that the tension force (minus gradient of the potential energy) is in the normal direction with curvature as its magnitude.

Now, let us consider the space of all paths connecting two fixed points $p, q \in \mathbb{R}^2$:

$$X = \{\gamma : [a, b] \to \mathbb{R}^2 \in C^1, \gamma(a) = p, \gamma(b) = q\}.$$

On X, we define the arclength potential energy $\mathcal{L}$ as

$$\mathcal{L}[\gamma] := \int_a^b |\dot{\gamma}(t)|\, dt = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle}\, dt.$$

Given a specific $\gamma \in X$, we want to define the directional derivative of $\mathcal{L}$ at γ in the direction of a vector field $V : [a, b] \to \mathbb{R}^2 \in C^1$. This is similar to the directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point x_0 in a direction $v \in \mathbb{R}^n$. It is defined as $df_{x_0}(v) := \frac{d}{d\varepsilon}|_{\varepsilon=0} f(x_0 + \varepsilon v)$. In order to have $\gamma + \varepsilon V \in X$, we should require $V(p) = V(q) = 0$. We define the direction derivative of $\mathcal{L}$ at γ in the direction V as

$$\delta\mathcal{L}_\gamma(V) := \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{L}(\gamma + \varepsilon V) = \int_a^b \frac{\langle \dot{\gamma}(t), \dot{V}(t)\rangle}{\sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle}}\, dt = \int_a^b \left\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \dot{V}(t) \right\rangle dt..$$

In this integration, we change the dummy variable t to s, the arclength parameter of γ. Then the integration domain becomes $[0, L]$, the vector $\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = T(s)$, the tangent vector of γ, and $\dot{V}(t)\, dt = \frac{dV}{ds}\, ds$. Thus,

$$\delta\mathcal{L}_\gamma(V) = \int_0^L \langle T(s), V'(s)\rangle\, ds = -\int_0^L \langle T'(s), V(s)\rangle\, ds = \int_0^L \langle \kappa(s)N(s), V(s)\rangle\, ds.$$

In the last step, we have used $V(p) = V(q) = 0$. This means that the gradient of $\mathcal{L}$ at γ is

$$\boxed{\frac{\delta\mathcal{L}_\gamma}{\delta\boldsymbol{\gamma}} = \kappa N.}$$

In mechanics, the total length is the potential energy of a string. Its negative gradient (i.e. $-\kappa N$) is the tension.

Theorem 1.1. *Given $\kappa(s)$, there is a curve $\gamma(s)$ whose curvature is $\kappa(s)$. The curve is unique up to translation and rotation.*

Proof. Hint: It is easier to prove by using the incline angle representation for curvature. □

Remark. The curve flow $\gamma(t, \xi)$ defined by

$$\frac{d}{dt}\gamma(t, \xi) := -\kappa(t, \xi)N(t, \xi), \quad \xi \in [a, b],\ t \geq 0,$$

with fixed ends $\gamma(t, a) = p$, $\gamma(t, b) = q$ is a curve fatten flow. It is an arclength minimization process.

Summary

- A geometric quantity associated with a curve is independent of parameterization. Tangent, normal, curvature are geometric quantities, which can be measured.
- Curvature can be defined as (1) how N is turned, (2) how incline angle θ changes, or (3) tension associated with the arclength potential.
- A curve γ is uniquely determined by its curvature up to translation and rotation. In particular, a planar curve is a straight line if and only if has zero curvature.

1.2 Discrete Plane Curves

A *regular* discrete planar curve is a regular polygonal curve determined by an ordered sequence of points $[\gamma_0, ..., \gamma_{n-1}]$ in $\mathbb{R}^2$ with $\gamma_{i+1} \neq \gamma_i$ and $\gamma_{i+2} \neq \gamma_i$ for all i considered. The edges of the curve are $[\gamma_i, \gamma_{i+1}]$ (or $[i, i+1]$ for simplicity of notation), $i = 0, ..., n-2$. If we add the edge $[\gamma_{n-1}, \gamma_0]$, such discrete curve is called closed.

Just like continuous case, it is convenient to consider arclength parameterized curve, where the edge length $|\gamma_{i+1} - \gamma_i| = \ell$ is a constant. In the continuous case, a quantity, say $\gamma(s)$, changes per unit length means that $d\gamma/ds$. In the discrete setting, it would be $\Delta\gamma/\Delta s$, which is $(\gamma_{i+1} - \gamma_i)/\ell$.

There are many ways to define a discrete curvature, which are equivalent in their continuous limits.

1.2.1 Discrete curvature based on angle change

Below, we shall treat $\mathbb{R}^2$ as $\mathbb{C}$. We define the tangent and normal by

- Tangent $T_{i.i+1} := (\gamma_{i+1} - \gamma_i)/|\gamma_{i+1} - \gamma_i|$,
- Normal $N_{i,i+1} := iT_{i,i+1}$.

The angle change α_i at vertex i is defined by:

$$T_{i,i+1} = \exp(i\alpha_i)T_{i-1,i}, \quad \text{or} \quad N_{i,i+1} = \exp(i\alpha_i)N_{i-1,i}.$$

We define the discrete curvature form

$$\kappa_i^a = \alpha_i.$$

Note that κ_i^a is an angle change which a dimensionless quantity, whereas the continuous curvature is the angle change per arclength (i.e. $d\theta/ds$), which has dimension $1/L$.

It is easy to see that, the total turning angle of a closed discrete curve γ is $2m\pi$ for some integer m. This integral m is the turning number of γ, which is a topological quantity.

1.2.2 Discrete curvature based osculating circle

Given an arclength parameterized discrete curve $(\gamma_0, ..., \gamma_{n-1})$, we can construct a circular arc Γ_i which is tangent to the polygonal curve γ at $(\gamma_i + \gamma_{i+1})/2$ and $(\gamma_i + \gamma_{i-1})/2$. Consider the piecewise curve $\Gamma := (\Gamma_1, ..., \Gamma_{i-2})$, which is a C^1 curve. It can be shown that the radius of the circular arc Γ_i is $\frac{\ell}{2}\cot(\frac{\alpha_i}{2})$. Thus, one define the curvature form on Γ_i to be

$$\kappa_i^t := 2\tan\left(\frac{\alpha_i}{2}\right).$$

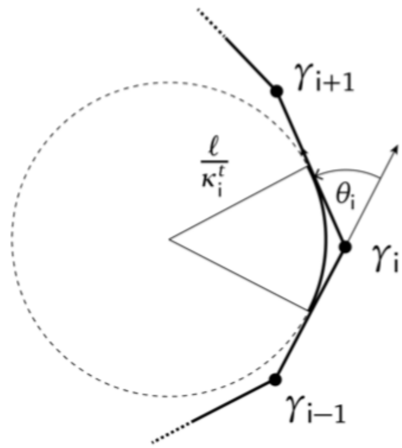


Figure 1.1: The circular arc Γ_i contacts to segments at mid points. Its curvature is defined to be the curvature of the discrete curve at γ_i. Copied from Chern and Schröder's Note.

1.2.3 Discrete curvature based on variational principle

Let us consider a variation of the discrete curve γ by

$$\gamma_i^\varepsilon = \gamma_i + \varepsilon V_i$$

where $(V_0, V_1, ..., V_{n-1})$ is the variation directions. At the ends, we require $V_0 = V_{n-1} = 0$ in order to have fixed ends during variation. The length potential is defined to be

$$\mathcal{L}[\gamma] := \sum_{i=0}^{n-2} |\gamma_{i+1} - \gamma_i|.$$

The change of $\mathcal{L}[\gamma^\varepsilon]$ in the direction of $(V_1, ..., V_{n-2})$ is

$$\frac{\partial}{\partial \varepsilon}|_{\varepsilon=0}\mathcal{L}[\gamma^\varepsilon] = \sum_{i=1}^{n-2} \left\langle \frac{\partial L}{\partial \gamma_i}, V_i \right\rangle$$

$$\frac{\partial L}{\partial \gamma_i} = \frac{\partial |\gamma_i - \gamma_{i-1}|}{\partial \gamma_i} + \frac{\partial |\gamma_{i+1} - \gamma_i|}{\partial \gamma_i} = T_{i-1} - T_i.$$

Thus,

$$dL_\gamma[V] = -\sum_{i=1}^{n-2} \langle T_i - T_{i-1}, V_i \rangle.$$

One can check

$$T_i - T_{i-1} = 2\sin\left(\frac{\alpha_i}{2}\right)\frac{N_i + N_{i-1}}{|N_i + N_{i-1}|}.$$

Thus, we define the discrete curvature form κ_i^s to be

$$\kappa_i^s = 2\sin\left(\frac{\alpha_i}{2}\right).$$

Summary We summarize different definitions of discrete curvature forms:

- exterior angle based: $\kappa_i^a = \alpha_i$;
- osculating circle based: $\kappa_i^t = 2\tan\left(\frac{\alpha_i}{2}\right)$
- variational based: $\kappa_i^s = 2\sin\left(\frac{\alpha_i}{2}\right)$.

They are various approximations of the curvature form $\kappa\, ds$.

1.3 Space Framed Curves

Figure 1.2: Copied from Chern and Schröder's note

- A space curve can be used to describe electric wire, magnetic filament, vortex filament, etc.
- A framed curve is a curve associated with an orthonormal frame with one of the frame coordinate aligned with the tangent of the curve. It can be used to model a band, ribbon, DNA, airplane path, vertebral, etc.

Examples

1. A spinning curve:

$$x = \cos t, \quad y = \sin t, \quad z = t$$

2. A spining ring:
$$x = \cos t, \quad y = \sin t, \quad z = \frac{1+\cos(2t)}{2}.$$

1.3.1 Frenet-Serret frame

Let $\boldsymbol{\gamma} : I = [a, b] \to \mathbb{R}^3$ be a regular curve (i.e. $\dot{\boldsymbol{\gamma}}(t) \neq 0$ for all $t \in [a, b]$. Let us parametrize it by arclength. Along the curve $\boldsymbol{\gamma}$, it is natural to define an orthogonal frame, the Frenet frame, or the Frenet-Serret frame. First, we define the following geometric quantities:

- Tangent: $T(s) := \boldsymbol{\gamma}'(s)$.
- Normal: $N(s) := T'(s)/|T'(s)|$.
- Binormal: $B(s) := T(s) \times N(s)$.

From this definition, we see that (T, N, B) forms an orthonormal frame along $\boldsymbol{\gamma}$, called Frenet-Serret frame.

Remark In the above definition of N, we have used $|T'(s)| \neq 0$. Thus, in addition to the assumption $\boldsymbol{\gamma}'(s) \neq 0$, we also require $\boldsymbol{\gamma}''(s) \neq 0$ in order to define the Frenet-Serret frame. We will see later that such a requirement is not needed for parallel frame.

We can further investigate the variation of the frame along $\boldsymbol{\gamma}$, which will provide us how $\boldsymbol{\gamma}$ is bended and twisted. We have the following definitions.

- Curvature: the curvature κ is defined through $T'(s) = \kappa(s)N(s)$. (This implies $\kappa \geq 0$.)
- Torsion: the torsion τ is defined by $B'(s) = -\tau N(s)$.

In the above definitions, we need to show that $T' \parallel N$ and $B' \parallel N$. The formal one comes from the definition of N. To show $B' \parallel N$, it is equivalent to show $B' \perp B$ and $B' \perp T$. From $\langle B, B\rangle = 1$, we differentiate it to get $\langle B', B\rangle = 0$. To show $B' \perp T$, we differentiate $\langle B, T\rangle = 0$ to get
$$\langle B', T\rangle = -\langle B, T'\rangle = \langle B, \kappa N\rangle = 0.$$
Thus, the above definitions for κ and τ are properly defined.

Theorem 1.2 (Frenet-Serret Formula, 1847, 1951)**.** *For an arclength parameterized curve $\boldsymbol{\gamma}$, the Frenet frame satisfies*
$$\begin{bmatrix} T'(s) & N'(s) & B'(s) \end{bmatrix} = \begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} \begin{bmatrix} & -\kappa(s) & \\ \kappa(s) & & -\tau(s) \\ & \tau(s) & \end{bmatrix}$$

Proof. We only need to show $N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$. Note that $N = B \times T$ because (T, N, B) is an orthonormal frame. We differentiate this formula to get
$$N' = B' \times T + B \times T' = -\tau N \times T + B \times \kappa N = -\kappa T + \tau B.$$
□

Corollary 1.1. *There exists a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ with prescribed curvature $\kappa(s)$ and torsion $\tau(s)$. The curve is unique up to translation and rotation.*

Proof. This follows from the existence and uniqueness of ordinary differential equations. Note that we write $R(s) = [T(s), N(s), B(s)]$. Then R satisfies

$$R'(s) = R(s)A(s), \quad A(s) = \begin{bmatrix} & -\kappa(s) & \\ \kappa(s) & & -\tau(s) \\ & \tau(s) & \end{bmatrix}.$$

The matrix R satisfies

$$\frac{d}{ds}(R(s)R^T(s)) = R'(s)R^T(s) + R(s)R'^T(s) = R(A + A^T)R^T = 0.$$

Since $R(0)$ satisfies $R(0)R^T(0) = I$, we get that $R(s)R^T(s) = I$ for all s. Thus, $R(s)$ stays as an orthogonal matrix for all time. ‡ The solution $R(s)$ is solved from the ODE and is unique up to a rotation. Once $R(s)$ is solved, $T(s)$ is found, and $\boldsymbol{\gamma}(s)$ is solved by integrating $T(s)$, and is unique up to an initial choice $\boldsymbol{\gamma}(0)$. □

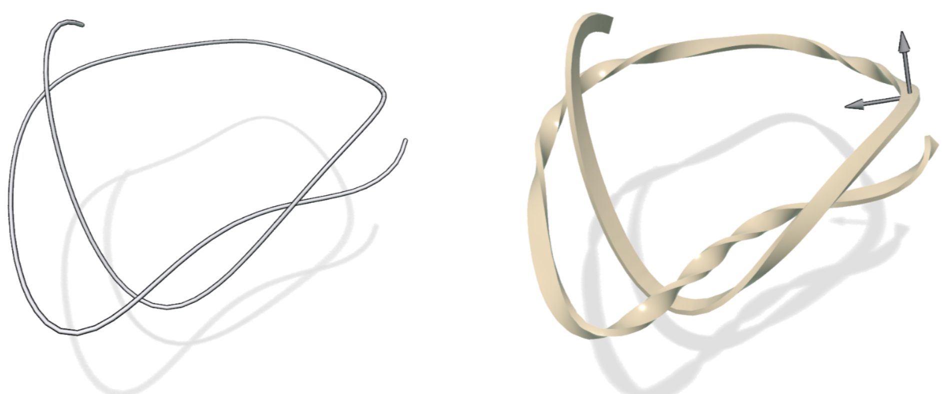


Figure 1.2: *A space curve (left) and a framed space curve (right).*

Figure 1.3: Copied from Chern and Schröder's note

‡ R is orthogonal $\Leftrightarrow R^TR = I$ (i.e. its column vectors are orthonormal) $\Leftrightarrow R^T = R^{-1} \Leftrightarrow R^{-T} = R \Leftrightarrow (R^{-1})^T = (R^{-1})^{-1} \Leftrightarrow R^{-1}$ is orthogonal $\Leftrightarrow R^T$ is orthogonal $\Leftrightarrow RR^T = I$ (i.e. the row vectors of R are orthonormal).

1.3.2 General orthonormal frame

The Frenet frame is only one kind of many frames. Indeed, on the normal plane $\mathcal{N}(s) \perp T(s)$, we are free to choose a unit vector $N_1(s)$, and define $N_2(s) := T(s) \times N_1(s)$, then $(T(s), N_1(s), N_2(s))$ forms an orthonormal frame. The Frenet frame is a special orthonormal frame with N_1 chosen to be $T'/|T'|$. A spatial curve $\boldsymbol{\gamma}$ is determined by two parameters, either by $T(s) \in S^2$ (which has 2 degrees of freedoms), or by $\kappa(s)$, $\tau(s)$. [§] For a framed curve in $\mathbb{R}^3$, the choice of $N_1(s)$ introduces one more degree of freedom, namely, $N_1(s) \in S^1$ on the normal plane. Thus, a framed curve in $\mathbb{R}^3$ has 3 degrees of freedoms. The Frenet frame determines two geometric quantities: curvature κ and torsion τ. Likely, a general orthonormal frame determines three quantities: the complex curvature $\psi \in \mathbb{C}$ and the twist ω, which are introduced below.

1.3.3 Complex curvature and Twist

On $\mathcal{N}(s)$, we are free to choose $N_1(s)$, which is called a gauge. By defining $N_2(s) = T(s) \times N_1(s)$, then $(T(s), N_1(s), N_2(s))$ forms a general orthonormal frame.

Measuring how $(T(s), N_1(s), N_2(s))$ varies along $\boldsymbol{\gamma}$

- $T'(s)$: By taking derivative of $\langle T(s), T(s)\rangle = 1$, we get $\langle T'(s), T(s)\rangle = 0$. That is, $T'(s) \in \mathcal{N}(s)$. We can express it as

$$\boxed{T'(s) = \psi_1 N_1 + \psi_2 N_2.}$$

  Thus, there are two curvatures ψ_1 and ψ_2 for general frame. Usually, we put them together as a complex number $\psi(s) := \psi_1 + i\psi_2$ and call it a *complex curvature* under the frame $(T(s), N_1(s), N_2(s))$. The reason will be illustrated later.

- Project $N_1'(s)$ on the normal plane Next, we check how $N_1(s)$ varies along $\boldsymbol{\gamma}(s)$. By differentiate $\langle N_1(s), N_1(s)\rangle = 1$, we get $N_1'(s) \perp N_1(s)$. If we project $N_1'(s)$ to the normal plane $\mathcal{N}(s)$, we get the *twist*

$$\omega := \langle N_1'(s), N_2(s)\rangle = -\langle N_2'(s), N_1(s)\rangle,$$

  which measures how N_1 twists on the normal plane as it moves along $\boldsymbol{\gamma}$. If $\omega = 0$, we call that $N_1(s)$ (same as $N_2(s)$) parallel transports along $\boldsymbol{\gamma}$. Such a frame with $\omega \equiv 0$ is called a *parallel frame*.

[§] Knowing $T(s)$ we can determine the curve $\boldsymbol{\gamma}$ by integrating $T(s)$. The curve is unique up to a translation, i.e. the initial position. The tangent $T(s) \in S^2$ which has 2 degrees of freedoms. The curve $\boldsymbol{\gamma}$ can also be determined by $\kappa(s)$ and $\tau(s)$, and is unique up to translation and rotation. In short, a curve in $\mathbb{R}^3$ has two degrees of freedoms.

- Project $N_1'(s)$ on $T(s)$: We still need to check the component of $N_1'(s)$ on $T(s)$. Taking derivative of $\langle N_1(s), T(s)\rangle = 0$, we get

$$\langle N_1'(s), T(s)\rangle = -\langle N_1(s), T'(s)\rangle = -\psi_1.$$

  Similarly, we get

$$\langle N_2'(s), T(s)\rangle = -\psi_2.$$

We conclude with the following theorem.

Theorem 1.3. *A general orthonormal frame (T, N_1, N_2) on an arclength parameterized curve $\boldsymbol{\gamma}$ satisfies the frame equation*

$$\begin{bmatrix} T'(s) & N_1'(s) & N_2'(s) \end{bmatrix} = \begin{bmatrix} T(s) & N_1(s) & N_2(s) \end{bmatrix} \begin{bmatrix} & -\psi_1(s) & -\psi_2(s) \\ \psi_1(s) & & -\omega(s) \\ \psi_2(s) & \omega(s) & \end{bmatrix} \tag{1.3}$$

Corollary 1.2. *There exists a framed curve $\boldsymbol{\gamma}(s)$, $(T(s), N_1(s), N_2(s))$ with prescribed complex curvature $\psi(s)$ and twist $\omega(s)$. The framed curve is unique up to translation and rotation.*

1.3.4 Orthogonal Matrix Representation

An orthonormal frame (T, N_1, N_2) can be viewed as a rotation matrix:

$$R(s) = (T(s), N_1(s), N_2(s)).$$

Here, we treat the vectors as column vectors. The matrix $R(s)$ is a 3×3 orthogonal matrix, which means that

$$R^T R = I, \quad \text{its column vectors are orthonormal.}$$

This is equivalent to

$$R^T = R^{-1} \quad \Leftrightarrow \quad R^{-T} = R \quad \Leftrightarrow \quad (R^{-1})^T = (R^{-1})^{-1} \quad \Leftrightarrow \quad R^{-1} \text{ is orthogonal}$$

$$\Leftrightarrow \quad R^T \text{ is orthogonal} \quad \Leftrightarrow RR^T = I, \text{ the row vectors of } R \text{ are orthonormal.}$$

This is just reflecting orthonormality of (T, N_1, N_2). The space of all orthogonal matrices in $\mathbb{R}^3$ with matrix multiplication forms a group, called rotation group, or orthogonal group. It is denoted by $O(3)$. Since $det(R^T R) = (det(R))^2 = 1$. Those orthogonal matrices with $det(R) = 1$ preserve orientation. This subset of $O(3)$ is denoted by $SO(3)$, called special orthogonal group. If $R(s) \in O(3)$ is smooth, then there exists a anti-symmetric matrix $A(s)$ such that

$$R'(s) = R(s)A(s).$$

To show this, let $A(s) = R^{-1}(s)R'(s) = R^T(s)R'(s)$. By differentiate the identity $R^T(s)R(s) = I$, we get

$$0 = R'^T R + R^T R' = (R^T R')^T + R^T R'.$$

Thus, the matrix A satisfies $A + A^T = 0$. Thus, we have the following $SO(3)$ representation of framed curved.

Theorem 1.4. *A one-parameter family of orthogonal matrices $R(s) \in SO(n)$ is determined by an anti-symmetric matrix $A(s)$ through the linear ODE*

$$R'(s) = R(s)A(s). \tag{1.4}$$

It is unique up to an initial choice of $R(0)$. Conversely, given an anti-symmetric matrix $A(s)$, it determines a one-parameter family of orthogonal matrices satisfying (1.4). It is unique up to initial rotation $R(0)$.

Corollary 1.3 (Fundamental Theorem of framed curves)**.** *Given a continuous complex-valued functions $\psi : [0, L] \to \mathbb{C}$ and $\omega : [0, L] \to \mathbb{R}$, there exists a acrlength parametrized framed curve $\boldsymbol{\gamma} : [0, L] \to \mathbb{R}^3$, $[T, N_1, N_2] : [0, L] \to SO(3)$ with complex curvature ψ and twist ω. The framed curve is unique up to translation and rotation.*

1.3.5 Rotation vector representation

To study how frame moves along a curve $\boldsymbol{\gamma}(s)$, we can treat s as time, and investigate the motion of the frame $(T(s), N_1(s), N_2(s))$ on S^2, the unit sphere. It is a motion of rotation on S^2.

Theorem 1.5. *The frame equation*

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} & \psi_1(s) & \psi_2(s) \\ -\psi_1(s) & & \omega(s) \\ -\psi_2(s) & -\omega(s) & \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}.$$

is equivalent to

$$\begin{cases} T'(s) &= \boldsymbol{\omega}(s) \times T(s), \\ N_1'(s) &= \boldsymbol{\omega}(s) \times N_1(s) \\ N_2'(s) &= \boldsymbol{\omega}(s) \times N_2(s) \end{cases} \tag{1.5}$$

where

$$\boldsymbol{\omega}(s) = \omega T - \psi_2 N_1 + \psi_1 N_2. \tag{1.6}$$

If $T'(s) \neq 0$, then

$$\begin{cases} \kappa N = \psi_1 N_1 + \psi_2 N_2, \\ \kappa B = -\psi_2 N_1 + \psi_1 N_2 \end{cases} \tag{1.7}$$

Proof.

1. Suppose $\boldsymbol{\omega} = aT + bN_1 + cN_2$. Use the above equations, you can get a, b, c.
2. When $T'(s) \neq 0$, then $N(s) := T'(s)/|T'(s)|$ is well-defined. In this case, we recall $T' = \psi_1 N_1 + \psi_2 N_2$. Thus,

$$\kappa N = T' = \psi_1 N_1 + \psi_2 N_2,$$

and

$$\kappa B = \kappa T \times N = -\psi_2 N_1 + \psi_1 N_2.$$

□

Remarks The meaning of this formula is that T (as well as N_1 and N_2) rotates about $\boldsymbol{\omega}(s)$ with speed $|\boldsymbol{\omega}(s)|$ instantaneously. To see this, we recall that the equation for rigid-body rotation is

$$\dot{\mathbf{r}}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t).$$

From $\frac{d}{dt}\|\mathbf{r}(t)\|^2 = 2\dot{\mathbf{r}} \cdot \mathbf{r} = 0$, we get that $\|\mathbf{r}(t)\|^2 \equiv \textit{constant}$. Thus, the motion of $\mathbf{r}(t)$ is a rotation. In the case when ω is a constant, it is a rotation with fixed axis. Its solution has the following representations.

1. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ and define the rotational generator $\boldsymbol{\Omega}$ associated with $\boldsymbol{\omega}$ by

$$\boldsymbol{\Omega} := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

   Then the solution to $\dot{\mathbf{r}}(t) = \boldsymbol{\omega} \times \mathbf{r}(t)$ is given by

$$\mathbf{r}(t) = e^{\boldsymbol{\Omega} t}\mathbf{r}(0).$$

   Here,

$$e^{\boldsymbol{\Omega} t} := \sum_{n=0}^{\infty} \frac{\boldsymbol{\Omega}^n t^n}{n!}.$$

2. We can decompose $\mathbf{r}$ into $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^\perp$:

$$\mathbf{r}(0) = a(t)\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} + \mathbf{r}^\perp,$$

   where $\mathbf{r}^\perp \in \boldsymbol{\omega}^\perp$. Then the solution is given by

$$a(t)(t) = a(0)$$

$$\mathbf{r}^\perp(t) = e^{J\omega t}\mathbf{r}^\perp(0),$$

   where J is 90° rotation on $\boldsymbol{\omega}^\perp$ and $\omega = |\boldsymbol{\omega}|$. If $\mathbf{r}(0) \perp \boldsymbol{\omega}$, then the curve $\mathbf{r}(t)$ rotates along a great circle, a geodesic curve on the sphere.

3. **Tangent hodograph** The tangent hodograph of a space curve $\boldsymbol{\gamma}(s)$ is the trajectory $T(s)$ on the unit sphere. From

$$T'(s) = (\omega T - \psi_2 N_1 + \psi_1 N_2) \times T = (-\psi_2 N_1 + \psi_1 N_2) \times T := \boldsymbol{\omega}_1 \times T,$$

   we see that T rotates on the unit sphere about $\boldsymbol{\omega}_1$ instantaneously. Since $T \perp \boldsymbol{\omega}_1$, T must rotate along a great circle on the unit sphere instantaneously. Note that if $T' \neq 0$, then $\boldsymbol{\omega}_1(s) \parallel B(s)$.

1.3.6 Complex structure on the Normal Plane

Gauge transformation on normal plane The complex curvature and the twist defined above depend on the choice of gauge $N_1(s)$. If we choose another gauge $\tilde{N}_1(s)$, we would like to see how the corresponding complex curvature $\tilde{\psi}$ and twist $\tilde{\omega}$ change. Since both $N_1(s)$ and $\tilde{N}_1(s)$ are unit vectors on the normal plane $\mathcal{N}(s)$, the vector $\tilde{N}_1(s)$ must be a rotation of $N_1(s)$ by some angle $\alpha(s)$. Before stating the result, let us define rotation on the normal plane as the follows.

Complex structure on the normal plane For any vector $Y \in \mathbb{R}^3$, define

$$JY := T \times Y.$$

Note that $JT = 0$. The operator $J : \mathcal{N} \to \mathcal{N}$ has the properties:

$$J^2 Y = T \times (T \times Y) = -Y \text{ for } Y \in \mathcal{N}.$$

Thus, $J^2 = -I$ on the normal plane $\mathcal{N}$. It is a 90° rotation on the normal plane. We can further define

$$e^{\alpha J} := \sum_{n=0}^{\infty} \frac{\alpha^n J^n}{n!}, \quad \alpha \in \mathbb{R}.$$

Then $e^{\alpha J} : \mathcal{N} \to \mathcal{N}$ is a rotation on $\mathcal{N}$ by an angle α.

Given a gauge $N_1(s) \in \mathcal{N}(s)$, we express $N_2(s) = JN_1(s)$. Any vector $Y \in \mathcal{N}$ can be expressed as

$$Y = a_1 N_1 + a_2 N_2 = (a_1 + Ja_2) N_1.$$

This makes a one-to-one correspondence between $Y \in \mathcal{N}$ and the complex number $a_1 + ia_2$. This correspondence induces multiplication on the normal plane:

$$c_1 + Jc_2 = (a_1 + Ja_2) \circ (b_1 + Jb_2) \Leftrightarrow c_1 + ic_2 = (a_1 + ia_2) \cdot (b_1 + ib_2).$$

Change of complex structure $J(s)$ along the curve $\boldsymbol{\gamma}$. Along $\boldsymbol{\gamma}(s)$, for any $Y(s) \in \mathcal{N}(s)$,

$$(JY)' = (T \times Y)' = T' \times Y + T \times Y' = T' \times Y + JY'.$$

We thus define $J'Y := T' \times Y$ in order to have the formal Libniz rule: $(JY)' = J'Y + JY'$ hold. Since $T' \in \mathcal{N}$, we have $J'Y = T' \times Y \parallel T$ for any $Y \in \mathcal{N}$.

Now, we can state the gauge transformation theorem.

Theorem 1.6. *Given a gauge N_1 with the corresponding complex curvature ψ and twist ω. Let $\tilde{N}_1(s) = e^{J\theta(s)} N_1(s)$, a rotation of $N_1(s)$ on the normal plane $\mathcal{N}(s)$ by an angle $\theta(s)$. Then the corresponding complex curvature $\tilde{\psi}$ and twist $\tilde{\omega}$ satisfy*

$$\boxed{\tilde{\psi} = e^{-i\theta} \psi, \quad \tilde{\omega} = \omega + \theta'.} \tag{1.8}$$

Proof. We have

$$\begin{aligned} T' &= \psi_1 N_1 + \psi_2 N_2 \\ &= (\psi_1 + \psi_2 J) N_1 \\ &= (\psi_1 + \psi_2 J) e^{-J\theta} \tilde{N}_1 \\ &= (\tilde{\psi}_1 + J\tilde{\psi}_2) \tilde{N}_1 \end{aligned}$$

The last two lines lead to

$$\tilde{\psi}_1 + i\tilde{\psi}_2 = (\psi_1 + i\psi_2) e^{-i\theta}.$$

For $\tilde{\omega}$, we have

$$\begin{aligned} \tilde{\omega} = \langle \tilde{N}_1', \tilde{N}_2 \rangle &= \langle (e^{J\theta} N_1)', e^{J\theta} N_2 \rangle \\ &= \langle (J\theta' + J'\theta) e^{J\theta} N_1 + e^{J\theta} N_1', e^{J\theta} N_2 \rangle \\ &= \langle \theta' e^{J\theta} N_2 + e^{J\theta} N_1', e^{J\theta} N_2 \rangle = \theta' + \omega. \end{aligned}$$

Here, we have used

$$\langle e^{J\theta} Y_1, e^{J\theta} Y_2 \rangle = \langle Y_1, Y_2 \rangle$$
$$\langle J' Y_1, Y_2 \rangle = 0,$$

for any $Y_1, Y_2 \in \mathcal{N}$. The latter follows from $J' Y_1 \parallel T$. $\square$

Remark The transformation $\boldsymbol{\gamma} \mapsto \psi_P$ is called Hashimoto transformation. It plays important role in the soliton theory for the binormal equation $\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}' \times \boldsymbol{\gamma}''$.

There are two special frames of particular interest:

- **Frenet frame:** It is the case when $T' \neq 0$ and with $N_1 = T'/|T'| = N$. It is equivalent to $Re(\psi) = \kappa$, $Im(\psi) = 0$ and $\omega = \tau$.
- **Parallel frame or Bishop frame:** It is the case when $\omega = 0$.

Remark From gauge transformation formula, we note that $|\psi|$ is gauge invariant. For Frenet frame, $\psi = \kappa$. Thus, we obtain

$$|\psi(s)| = \kappa(s) \tag{1.9}$$

for any frame.

1.3.7 Parallel frame: $\omega = 0$

- **Construction of parallel frame through Frenet frame.** Let us denote the complex curvature for parallel frame by ψ_P. We can express Ψ_P in terms of κ and τ by the following gauge transformation. We look for angle $\theta(s)$ such that

$$N_1 = e^{-J\theta} N$$

with N_1 being the parallel frame gauge and N the Frenet frame gauge. From the gauge transformation formula, we get

$$\psi_P = \kappa e^{i\theta}, \quad 0 = \tau - \theta'.$$

Thus, we can construct a parallel frame with a starting $N_1(0)$ from a Frenet frame (T, N, S) through the formula $N_1(s) = e^{-J\theta(s)}$ with $\theta(s) = \theta(0) + \int_0^s \tau(s)\, ds$. This N_1 is a gauge of a parallel frame with

$$\psi_P(s) = \kappa(s) e^{i\theta_0 + i\int_0^s \tau(s)\, ds}.$$

- **Construction of parallel frame without using Frenet frame.** The above construction of parallel frame requires $T'(s) \neq 0$ ($\because$ existence of N). We can also construct a parallel frame for a regular curve without this assumption. Indeed, given $T(s)$, we can solve the following equation

$$\frac{dN_1(s)}{ds} = \left(T(s) \times \frac{dT(s)}{ds} \right) \times N_1(s). \tag{1.10}$$

From $(A \times B) \times C = \langle A, C \rangle B - \langle B, C \rangle A$, we get that the above equation is equivalent to

$$\frac{dN_1(s)}{ds} = -\langle \frac{dT(s)}{ds}, N_1(s) \rangle T(s).$$

Thus, $\omega := \langle \frac{dN_1}{ds}, N_2(s) \rangle = 0$.

- Note that if $T'(s) \neq 0$, then (1.10) is

$$N_1'(s) = (T(s) \times T'(s)) \times N_1(s) = \kappa(s) B(s) \times N_1(s). \tag{1.11}$$

That is, N_1 rotates about $B(s)$ with angular speed κ instantaneously. Comparing (1.5) and (1.11), we find

$$\boldsymbol{\omega}_1 := -\psi_2 N_1 + \psi_1 N_2 = \kappa B. \tag{1.12}$$

The rotation

$$\begin{cases} N_1' = \boldsymbol{\omega} \times N_1 \\ N_2 = \boldsymbol{\omega} \times N_2 \end{cases}$$

with $\boldsymbol{\omega} = \boldsymbol{\omega}_1$ is a parallel transport.

1.3.8 Quaternion Representation for Framed Curves

Quaternion is an algebraic tool for calculating rotation in $\mathbb{R}^3$. It was invented by Hamilton. See Wiki, Quaternion and Spatial Rotation `https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation`.

Algebra of quaternion

A quaternion has the form

$$q = a + \mathbf{v}, \quad a \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^3.$$

We call a the real part of q and $\mathbf{v}$ the imaginary part of q. We express them by

$$a = Re(q), \quad \mathbf{v} = Im(q).$$

The quaternion space is

$$\mathbb{H} = \{a + \mathbf{v} | a \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^3\} = \mathbb{R} \oplus \mathbb{R}^3.$$

Multiplication The multiplication rule of quaternions is defined by

$$(a + \mathbf{v})(b + \mathbf{w}) = ab + a\mathbf{w} + b\mathbf{v} + \mathbf{v}\mathbf{w},$$

$$\mathbf{v}\mathbf{w} := -\langle \mathbf{v}, \mathbf{w} \rangle + \mathbf{v} \times \mathbf{w}.$$

If we introduce orthonormal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$ and express quaternion

$$q = a + \mathbf{v} = a + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

then the above multiplication rule for the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are

$$\mathbf{i}^2 = -1, \quad \mathbf{j}^2 = -1, \quad \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, .$$

These rule for the basis together with the *distributive rule* are equivalent to the orignal multiplication rule.

Associative rule

Proposition 1.1. *For $p, q, r \in \mathbb{H}$, one has*

$$(pq)r = p(qr).$$

Proof. You can check this associative rule is valid for the basis: $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. This together with the distributive rule imply that the associative rule is also valid for general quaternions. Alternatively, there is a connection between the basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and the Pauli matrices. Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

The Span$\{I, X, Y, Z\}$ is a closed subalgebra of $\mathbb{C}^{2\times 2}$. One can check

$$X^2 = -I, \quad Y^2 = -I, \quad Z^2 = -I,$$

$$XY = -YX = Z, \quad YZ = -ZY = X, \quad ZX = -XZ = -Y.$$

The mapping

$$\mathbb{H} \to \text{Span}\{I, X, Y, Z\} \subset \mathbb{C}^{2\times 2}$$

$$a + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \mapsto aI + v_1X + v_2Y + v_3Z$$

is an algebra isomorphism. It carries the associative rule of matrix multiplication to quaternion multiplication. □

Conjugate If $q = a + \mathbf{v}$, define its conjugate

$$\bar{q} = a - \mathbf{v},$$

and the norm

$$|q|^2 = |a|^2 + |\mathbf{v}|^2.$$

For unit quaternion, we have $\bar{q} = q^{-1}$. We have the following properties for conjugacy and absolute value.

- $|q|^2 = q\bar{q} = \bar{q}q$,
- $|pq| = |p|\,|q|$,
- $\overline{pq} = \bar{q}\bar{p}$,
- $q^{-1} = \frac{\bar{q}}{|q|^2}$.

homework Check the above properties.

Geometry of Quaternions

Unit quaternion as a rotation Fact: any $q \in \mathbb{H}$ with $|q| = 1$ can be represented as

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{v}, \text{ with } |\mathbf{v}| = 1,$$

and vice versa.

Theorem 1.7. *Let $q = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\mathbf{v}$ with $|\mathbf{v}| = 1$. Let $\mathbf{y} \in \mathbb{R}^3$. Then $q\mathbf{y}\bar{q} \in \mathbb{R}^3$ and the mapping*

$$F : \mathbf{y} \mapsto q\mathbf{y}\bar{q}$$

is a rotation of $\mathbf{y}$ around $\mathbf{v}$ by the angle θ.

Proof.

1. To show $p = q\mathbf{y}\bar{q} \in \mathbb{R}^3$, it is equivalent to show $\bar{p} = -p$. We check

   $$\overline{q\mathbf{y}\bar{q}} = \bar{\bar{q}}\bar{\mathbf{y}}\bar{q} = q(-\mathbf{y})\bar{q} = -q\mathbf{y}\bar{q}.$$

   Hence $q\mathbf{y}\bar{q} \in \mathbb{R}^3$.

2. $F(\mathbf{y})$ is linear and $|F(\mathbf{y})| = |q\mathbf{y}\bar{q}| = |\mathbf{y}|$, because $|q| = 1$. Thus, F is either a rotation, or a mirrored-rotation, depending $\det(F) = 1$ or -1, respectively.

3. F is a rotation. This is because F is a continuous function in q, thus in θ. Since F is the identity map when $\theta = 0$, we get $\det(F) = 1$.

4. $F(\mathbf{v}) = \mathbf{v}$. (Check by yourself.) F is a rotation on $\mathbf{v}^\perp$ by the angle θ. Let $\mathbf{w} \in \mathbf{v}^\perp$, i.e. $\langle \mathbf{w}, \mathbf{v} \rangle = 0$. We have $\mathbf{wv} = \mathbf{w} \times \mathbf{v} = -\mathbf{vw}$.

$$\begin{aligned} q\mathbf{w}\bar{q} &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}\right) \mathbf{w} \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{v}\right) \\ &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}\right) \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}\right) \mathbf{w} \\ &= \left(\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{v} - \sin^2\frac{\theta}{2}\right) \mathbf{w} \\ &= (\cos\theta + \sin\theta\mathbf{v})\,\mathbf{w} \\ &= \cos\theta\mathbf{w} + \sin\theta(\mathbf{v} \times \mathbf{w}). \end{aligned}$$

Viewed in the orthogonal frame $(\mathbf{w}, \mathbf{v} \times \mathbf{w}, \mathbf{v})$, the vector $q\mathbf{w}\bar{q}$ is the rotation of $\mathbf{w}$ about $\mathbf{v}$ by the angle θ.

□

Composition of rotations Suppose $F(\mathbf{y}) = p\mathbf{y}\bar{p}$, $G(\mathbf{z}) = q\mathbf{z}\bar{q}$ be two rotations. Then the composition of the two rotations is

$$(F \circ G)(\mathbf{z}) = p(q\mathbf{z}\bar{q})\bar{p} = (pq)\mathbf{z}\overline{(pq)},$$

which is the rotation corresponding to the product of p and q..

Dihedral Given two unit vector $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, one can find a unit quaternion q to rotate $\mathbf{x}$ to $\mathbf{y}$. The rotation axis is $\mathbf{v} = \mathbf{x} \times \mathbf{y}/|\mathbf{x} \times \mathbf{y}|$. The angle is $\cos\theta = \langle \mathbf{x}, \mathbf{y} \rangle$. Thus,

$$q = c + s\mathbf{v}, \quad c = \sqrt{\frac{1 + \langle \mathbf{x}, \mathbf{y} \rangle}{2}}, \quad s = \sqrt{\frac{1 - \langle \mathbf{x}, \mathbf{y} \rangle}{2}}$$

We denote $q = \text{dihedral}(\mathbf{x}, \mathbf{y})$.

Representation of a framed curve by quaternion

Proposition 1.2. *Given a framed curve $(T(s), N_1(s), N_2(s))$, there exists a unique unit quaternion $q(s)$ which rotates $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to $(T(s), N_1(s), N_2(s))$:*

$$T = q\mathbf{i}\bar{q}, \quad N_1 = q\mathbf{j}\bar{q}, \quad N_2 = q\mathbf{k}\bar{q}.$$

Proof. We can first find q_1, which rotates $\mathbf{i}$ to T. Since $\mathbf{j} \perp \mathbf{i}$, we have $q_1\mathbf{j}\bar{q}_1 \perp q_1\mathbf{i}\bar{q}_1 = T$. Next, we find q_2 which rotates $q_1\mathbf{j}\bar{q}_1$ to N_1. Since both $q_1\mathbf{j}\bar{q}_1 \perp T$ and $N_1 \perp T$, the rotation q_2 must leave T unchanged. Thus, the resulting rotation $q = q_2 q_1$ rotates $\mathbf{i}$ to T and $\mathbf{j}$ to N_1. To conclude, if

$$q = q_2 q_1, \quad q_1 = \text{dihedral}(\mathbf{i}, T), \quad q_2 = \text{dihedral}(q_1\mathbf{j}\bar{q}_1, N_1)$$

then

$$T = q\mathbf{i}\bar{q}, \quad N_1 = q\mathbf{j}\bar{q}, \quad N_2 = q\mathbf{k}\bar{q}.$$

□

Frame equation Given a complex curvature $\psi(s)$ and a twist $\omega(s)$, we can determine a framed curve $(T(s), N_1(s), N_2(s))$ by using the frame equation

$$\begin{aligned} T'(s) &= \boldsymbol{\omega}(s) \times T(s), \\ N_1'(s) &= \boldsymbol{\omega}(s) \times N_1(s) \\ N_2'(s) &= \boldsymbol{\omega}(s) \times N_2(s), \end{aligned}$$

where $\boldsymbol{\omega} = \omega T - \psi_2 N_1 + \psi_1 N_2$. Since $(T(s), N_1(s), N_2(s))$ can also be represented from $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ by a quaternion $q(s)$. We would like to find a similar ODE that $q(\cdot)$ should satisfy.

Proposition 1.3. *Given complex curvature $\psi(s)$ and twist $\omega(s)$, let $(T(s), N_1(s), N_2(s))$ be the associated framed curve and $q(s)$ be the associated quaternion which rotates $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to $(T(s), N_1(s), N_2(s))$. Then $q(s)$ satisfies the frame equation:*

$$q'(s) = \mathbf{v}q = q\mathbf{w},$$

where

$$\mathbf{v} = \frac{1}{2}\left(\omega T - \psi_2 N_1 + \psi_1 N_2\right), \quad \mathbf{w} = \frac{1}{2}\left(\omega \mathbf{i} - \psi_2 \mathbf{j} + \psi_1 \mathbf{k}\right).$$

Proof.

1. We claim $q'\bar{q} \in \mathbb{R}^3$ for $q(s)$ with $|q(s)| \equiv 1$. We differentiate $q\bar{q} = 1$ to get $q'\bar{q} + q\bar{q}' = 0$. Then $q'\bar{q} + \overline{q'\bar{q}} = q'\bar{q} + q\bar{q}' = 0$. Thus, $q'\bar{q}$ is pure imaginary, which means $q'\bar{q} \in \mathbb{R}^3$.

2. Let us denote $q'\bar{q} = \mathbf{v}$. We claim that if $q(s)$ represents the framed curve $(T(s), N_1(s), N_2(s))$, then
$$\mathbf{v} = \frac{1}{2}\boldsymbol{\omega} = \frac{1}{2}\left(\omega T - \psi_2 N_1 + \psi_1 N_2\right).$$
With this, multiplying $q'\bar{q} = \mathbf{v}$ by q from right and using $q\bar{q} = 1$, we get $q' = \mathbf{v}q$. To show this claim, let us compute
$$\begin{aligned} T' &= (q\mathbf{i}\bar{q})' = q'\mathbf{i}\bar{q} + q\mathbf{i}\bar{q}' = q'(\bar{q}q)\mathbf{i}\bar{q} + q\mathbf{i}(\bar{q}q)\bar{q}' \\ &= (q'\bar{q})(q\mathbf{i}\bar{q}) + (q\mathbf{i}\bar{q})(q\bar{q}') = \mathbf{v}T + T\bar{\mathbf{v}} = \mathbf{v}T - T\mathbf{v} = 2\mathbf{v} \times T. \end{aligned}$$
Similarly, we have
$$N_1' = 2\mathbf{v} \times N_1, \quad N_2' = 2\mathbf{v} \times N_2.$$
Comparing with Theorem (1.5), we get
$$\mathbf{v} = \frac{1}{2}\boldsymbol{\omega} := \frac{1}{2}\left(\omega T - \psi_2 N_1 + \psi_1 N_2\right).$$

3. We can have another representation: $q' = q\mathbf{w}$:
$$\begin{aligned} q' = \mathbf{v}q &= \frac{1}{2}\left(\omega T - \psi_2 N_1 + \psi_1 N_2\right) q = \frac{1}{2}\left(\omega q\mathbf{i}\bar{q} - \psi_2 q\mathbf{j}\bar{q} + \psi_1 q\mathbf{k}\bar{q}\right) q \\ &= q\left(\frac{\omega}{2}\mathbf{i} - \frac{\psi_2}{2}\mathbf{j} + \frac{\psi_1}{2}\mathbf{k}\right) := q\mathbf{w}. \end{aligned}$$

□

Remark This means that the frame (T, N_1, N_2) rotates around $\mathbf{v}$ with angular speed (angle change per unit arclength ds) $|2\mathbf{v}|$.

1.4 Geometric Quantities

A geometric quantity is a quantity which is independent of gauge. From the above gauge transformation formula, we see that $|\psi|$ is independent of the choice of gauge. In particular, $\psi = \kappa$ when we choose $N_1 = N$. Thus $|\psi| = \kappa$ for any gauge N_1. It is a geometric quantity.

Geometric invariants A geometric quantity should be independent from how we parameterize the curve, and how we set up a coordinate to describe the curve. The physics associated with the curve should only depend on such geometric quantities. For the curve, these quantities are the curvature and the torsion. For the framed curve, they are the complex curvature and the twist. For physical applications, the following geometric quantities are important.

- Total length: $\int_a^b |\boldsymbol{\gamma}_\xi|\, d\xi$.
- Total curvature: $\int_0^L \kappa(s)\, ds$.
- Total torsion: $\int_0^L \tau(s)\, ds$.
- Bending energy: $\int_0^L |\kappa(s)|^2\, ds$.
- Torsion energy: $\int_0^L |\tau(s)|^2\, ds$.

Summary

- Frame equations:
  - In rotation matrix form:

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} & \psi_1(s) & \psi_2(s) \\ -\psi_1(s) & & \omega(s) \\ -\psi_2(s) & -\omega(s) & \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}.$$

  - In rotation vector form

$$\begin{cases} T'(s) &= \boldsymbol{\omega}(s) \times T(s), \\ N_1'(s) &= \boldsymbol{\omega}(s) \times N_1(s) \\ N_2'(s) &= \boldsymbol{\omega}(s) \times N_2(s) \end{cases}$$

    where

$$\boldsymbol{\omega}(s) = \omega T - \psi_2 N_1 + \psi_1 N_2.$$

- In terms of quaternion

$$q' = q\mathbf{v}, \quad \mathbf{v} = \frac{1}{2}\boldsymbol{\omega}.$$

- $T(s)$ rotates instantaneously about

$$\boldsymbol{\omega}_1 = -\psi_2 N_1 + \psi_1 N_2 = \kappa B \text{ if } T'(s) \neq 0$$

along a great circle on the unit sphere with angular speed κ. For parallel frame, N_1 and N_2 also rotates instantaneously about $\boldsymbol{\omega}_1 (= \kappa B)$.

- Gauge transformation formula:

$$\tilde{\psi} = e^{-i\theta}\psi, \quad \tilde{\omega} = \omega + \theta'.$$

1.5 Discrete framed curves

1.5.1 Discrete curve

We shall consider arclength-parameterized discrete curve: $\boldsymbol{\gamma} : \{0, 1, ..., n-1\} \to \mathbb{R}^3$. We define the following discrete geometric quantities

- The tangent $T_i := \boldsymbol{\gamma}_{i+1} - \boldsymbol{\gamma}_i$. T_i lives on edge $[\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_{i+1})$.
- The averaged tangent $\bar{T}_i :=$ normalize$(T_{i-1} + T_i)$. It lives at $\boldsymbol{\gamma}_i$.
- The binormal B_i the normal of the osculating plane.

$$B_i := \frac{T_{i-1} \times T_i}{|T_{i-1} \times T_i|}.$$

  The binormal B_i lives at the vertex $\boldsymbol{\gamma}_i$.

- Normal:

$$N_i := \text{normalize}\left(B_i \times \bar{T}_i)\right).$$

- curvature $\kappa_i^a := \alpha_i =$ the angle from T_{i-1} to T_i:

$$\alpha_i := \cos^{-1}\langle T_{i-1}, T_i\rangle.$$

1.5.2 Construction of parallel gauge for a discrete curve

Given a discrete curve $\boldsymbol{\gamma}_i$, $i = 1, ..., n-1$, we can construct a parallel gauge along the curve. This is to construct $N_{1,i}$, $i = 1, ..., n-2$. $N_{1,i}$ is defined on the edges. We start from $N_{1,i-1}$ to construct $N_{1,i}$. It is simply by the dihedral rotation

$$R_{Dh}(T_{i-1}, T_i)$$

which rotates T_{i-1} to T_i along a great circle on the unit sphere. It also plays the role to parallel transport any vector on $T_{i-1}^{\perp}$ to $T_i^{\perp}$, because

$$N_1' = \boldsymbol{\omega}_1 \times N_1, \quad \boldsymbol{\omega}_1 = -\psi_2 N_1 + \psi_1 N_2.$$

has $\omega = 0$. Thus, we define

$$N_{1,i} = R_{Dh}(T_{i-1}, T_i) N_{1,i-1}.$$

In terms of quaternion, it is equivalent to define q_i, $i = 1, ..., n-2$

$$q_{i+1} = \text{dihedral}(T_i, T_{i+1}) q_i.$$

Then $N_{1,i} = q_i \mathbf{j} \bar{q}_i$ is the parallel gauge.

Discrete twist Given a discrete framed curve $(\gamma_i, N_{1,i})$, $i = 1, ..., n-1$. We define its twist

$$\omega_i := \angle(R_{Dh}(T_{i-1}, T_i)(N_{1,i-1}), N_{1,i})$$

1.5.3 Discrete complex curvature

At $\boldsymbol{\gamma}_i$, we draw a unit sphere. Let $p_- := -T_{i-1}$ be the south pole and T_{i-1} be the north pole. Consider the polar stereographic projection which maps $S^2 \setminus \{-T_{i-1}\}$ to a plane $T_{i-1}^{\perp}$. The plane is tangent to S^2 at the north pole. The projection is given by

$$\text{StereoProj}\left(xN_{i-1} + yp_+ \times N_{i-1} + zp_-)\right) = \frac{2x}{1-z} + \frac{2y}{1-z} i.$$

The complex curvature ψ_i is defined to be

$$\psi_i := \text{StereoProj}_{T_{i-1}}(T_i). \tag{1.13}$$

One can check that

$$|\psi_i| = 2 \tan\left(\frac{\alpha_i}{2}\right), \tag{1.14}$$

$$\arg(\psi_i) = \text{ the angle from } N_{i-1} \text{ to } -\frac{T_{i-1} \times B_i}{|T_{i-1} \times B_i|}. \tag{1.15}$$

Homework Verify the above two formulae (1.14), (1.15).

Theorem 1.8. *Under a gauge transform $\tilde{N}_k = e^{i\phi_k} N_k$, the corresponding complex curvature and twist satisfy*

$$\tilde{\psi}_k = e^{i\phi_k} \psi_k, \quad \tilde{\omega}_k = [\omega_k + \phi_i - \phi_{i-1}]_{(-\pi,\pi]}$$

1.5.4 Hodograph of a discrete curve

- Given a closed discrete curve $\boldsymbol{\gamma} : V = \{0, 1, ..., n-1\} to \mathbb{R}^3$ with equal edge length ℓ. Its tangent $T_{i-1,i} := |\boldsymbol{\gamma}_i - \boldsymbol{\gamma}_{i-1})/\ell$ sits in S^2. Let us connect $T_{i-1,i}$ and $T_{i,i+1}$ by a great circle for $i = 1, ..., n-1$ and form a spherical polygon. This is the hodograph of the discrete curve $\boldsymbol{\gamma}$.

- The discrete curvature is the change from $T_{i-1,i}$ to $T_{i,i+1}$, which is the angle α_i from $T_{i-1,i}$ to $T_{i,i+1}$.

- The discrete binormal is defined to satisfy $\alpha_i B_i = (T_{i-1,i} \times T_{i,i+1})$.

- The discrete torsion $\tau_{i,i+1}$ is the angle-change from B_i to B_{i+1}, which is the angle-change from the great circle $\overline{T_{i-1,i}, T_{i,i+1}}$ to the great circle $\overline{T_{i,i+1}, T_{i+1,i+2}}$, or the exterior angle of the spherical polygon at $T_{i,i+1}$.

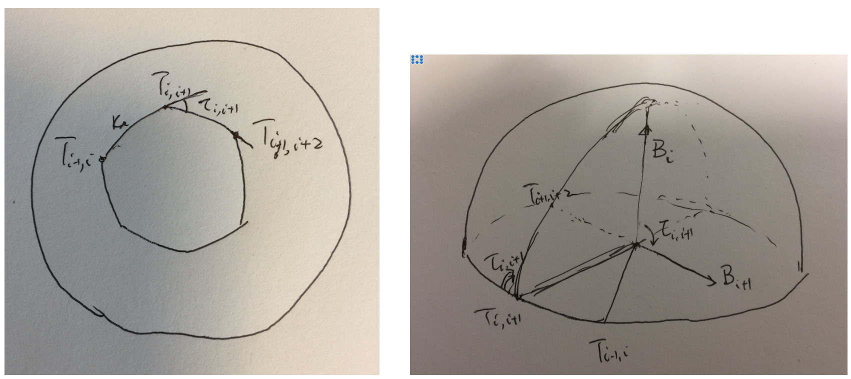


Figure 1.4: Hodograph of a discrete curve

1.5.5 Summary

Given a discrete curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$.

- Tangent $T_{i-1,i} = \boldsymbol{\gamma}_i - \boldsymbol{\gamma}_{i-1}$

- Normal plane: $N_{i-1,i} = T_{i-1,i}^{\perp}$

- Binormal $B_i = \frac{T_{i-1,i} \times T_{i,i+1}}{|T_{i-1,i} \times T_{i,i+1}|}$

- Curvature: κ_i^a is turning angle from $T_{i-1,i}$ to $T_{i,i+1}$

- Parallel transport of gauge N_1:

$$N_{1,i,i+1} = R_{Dh}(T_{i-1,i}, T_{i,i+1})(N_{1,i-1,i})$$

- Twist:
$$\omega_i := \angle(R_{Dh}(T_{i-1}, T_i)(N_{1,i-1}), N_{1,i})$$

- Discrete complex curvature
$$\psi_i := \text{StereoProj}_{T_{i-1,i}}(T_{i,i+1}).$$

- Discrete torsion: $\tau_{i,i+1}$ is the angle-change from B_i to B_{i+1}, which is the angle-change from the great circle $\overline{T_{i-1,i}, T_{i,i+1}}$ to the great circle $\overline{T_{i,i+1}, T_{i+1,i+2}}$, or the exterior angle of the spherical polygon at $T_{i,i+1}$.

Chapter 2

Combinatorial Surfaces

This chapter is based on Chapter 1 of Crane's note on Discrete Differential Geometry.

2.1 Abstract simplicial complex

Discrete surfaces consist of connected vertices. There are many different ways to describe the connectivity of a discrete surface. What we shall described here is so-called simplicial complex, which encodes dimensionality information.

Simplex

- An abstract k-simplex is a set of $(k+1)$ distinct vertices, denoted by $\{v_1, ..., v_{k+1}\}$, or abbreviated by $\{1, ..., k+1\}$. For example: consider vertices $\{v_1, ..., v_4\}$ in $\mathbb{R}^3$, a 0-simplex $\{v_1\}$ is a single vertex; a 1-simplex $\{v_1, v_2\}$ is an edge, a 2-simplex $\{v_1, v_2, v_3\}$ is a triangle, and a 3-simplex $\{v_1, v_2, v_3, v_4\}$ is a tetrahedron. In general, the vertices are not necessary in $\mathbb{R}^3$, we treat them in abstract way. We call k the degree of a k-simplex.
- Faces of a simplex: Any nonempty subset of a simplex is another simplex, which we call a face; a strict subset is called a proper face. For example, if $\sigma = \{1, 2, 3\}$ is a 2-simplex, then $\{1, 2\}, \{2, 3\}, \{1, 3\}$, $\{1\}, \{2\}, \{3\}$ and $\{1, 2, 3\}$ are faces of σ.

Note that an abstract simplicial complex specifies how vertices are connected, but not where they are in space.

Simplicial Complex

Definition 2.1. A collection of simplices $\mathcal{K}$ is called a *simplicial complex* if for every simplex $\sigma \in \mathcal{K}$, every face $\sigma' \subset \sigma$ is also in $\mathcal{K}$.

For example, $\mathcal{K} = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1\}, \{2\}, \{3\}\}$ is a simplicial complex, while the collection $\{\{1, 2, 3\}, \{1, 2\}, \{1\}, \{2\}\}$ is not.

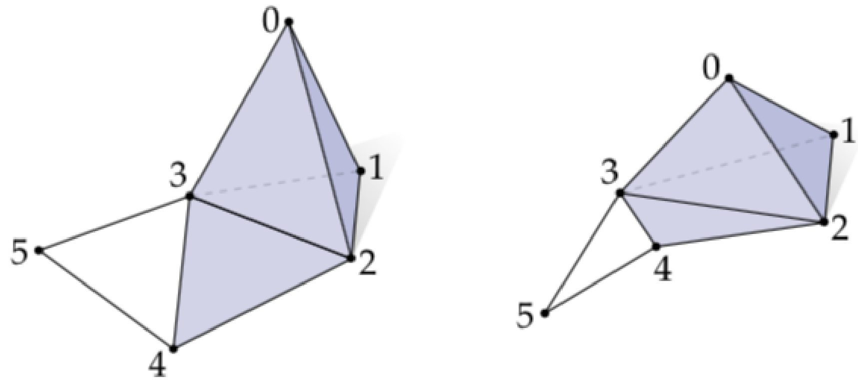


Figure 2.1: Examples of simplicial complex. But they are not pure 2-simplicial complex. It describes how vertices are connected, but not where they are in the space. Copied from Crane's note.

Definition 2.2. subcomplex, pure k-simplicial complex skeleton

1. A *subcomplex* $\mathcal{K}'$ of a simplicial complex $\mathcal{K}$ is a subset that is also a simplicial complex.
2. A complex $\mathcal{K}$ is a *pure k-simplicial complex* if every simplex $\sigma' \in \mathcal{K}$ of degree $l < k$ is contained in some simplex of degree k.
3. The k-skeleton of K is the collection of all simplices of K of dimension at most k. We denote it by $\mathcal{K}^{(k)}$.

Thus, a simplicial complex $\mathcal{K}$ is the union of its skeletons:

$$\mathcal{K} = \cup_k \mathcal{K}^{(k)},$$

Examples

- A tetrahedron can be represented by the following simplicial complex: Let $V = \{1,2,3,4\}$, define $\mathcal{K} = \mathcal{K}^{(3)} \cup \mathcal{K}^{(2)} \cup \mathcal{K}^{(1)} \cup \mathcal{K}^{(0)}$ with

$$\begin{aligned}
\mathcal{K}^{(3)} &= \{\{1,2,3,4\}\},\\
\mathcal{K}^{(2)} &= \{\{1,2,3\},\{1,2,4\},\{2,3,4\},\{3,1,4\}\},\\
\mathcal{K}^{(1)} &= \{\{1,2\},\{2,3\},\{3,1\},\{1,4\},\{2,4\},\{3,4\}\},\\
\mathcal{K}^{(0)} &= \{\{1\},\{2\},\{3\},\{4\}\}.
\end{aligned}$$

- In Figure 1.1,

$$\mathcal{K}^{(3)} = \{\{0,1,2,3\}\}$$

$$\mathcal{K}^{(2)} = \{\{0,1,2\},\{0,2,3\},\{0,1,3\},\{1,2,3\},\{2,3,4\}\}$$
$$\mathcal{K}^{(1)} = \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\}$$
$$\mathcal{K}^{(0)} = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\}\}.$$

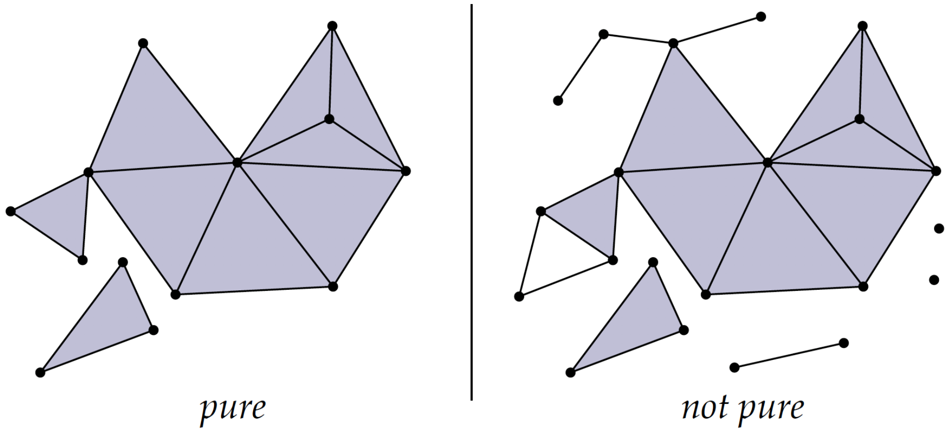


Figure 2.2: Example of pure simplex and non-pure simplex. Copied from Crane's note.

Summary An abstract simplicial complex is just a subset of the integers, closed under the operation of taking subsets.

2.2 Basic topological operation of simplicial complex

Given a simplicial complex $\mathcal{K}$, we define the following topological operators:

- **Star**: Let i be a vertex. $St(i) = \cup\{\sigma \in \mathcal{K} \,|\, i \in \sigma\}$. In example below (see Figure 2.3), vertex i is surrounded by vertices $\{1, ..., 6\}$. We have

$$St(i) = \{i\} \cup \{\{i,1\}, ..., \{i,6\}\} \cup \{\{i,1,2\},\{i,2,3\}, ..., \{i,5,6\},\{i,6,1\}\}$$

- **Closure**: The closure of a set, denoted by $Cl(S)$, is the smallest subcomplex that contains S. For example, if S is the above $St(i)$, then

$$Cl(S) = \{\{i\},\{1\},\{2\}, ..., \{6\}\} \cup St(i) \cup \{\{i,1\},\{i,2\}..., \{i,6\},\{1,2\},\{2,3\}, ..., \{6,1\}\}$$

- **Link**: $Lk(i) := Cl(St(i)) \setminus St(i)$. In the example above (see Fig. 2.4)

$$Lk(i) = \{\{1\},\{2\}, ..., \{6\}\} \cup \{\{1,2\},\{2,3\}, ..., \{6,1\}\}.$$

- **Boundary**: Suppose $\mathcal{K}'$ is a pure k-subcomplex. Its boundary is the closure of the set of all simplices σ that are proper faces of exactly one simplex of $\mathcal{K}'$.
- **Interior**: The interior $int(\mathcal{K}') = \mathcal{K}' - bd(\mathcal{K}')$ is then everything but the boundary.

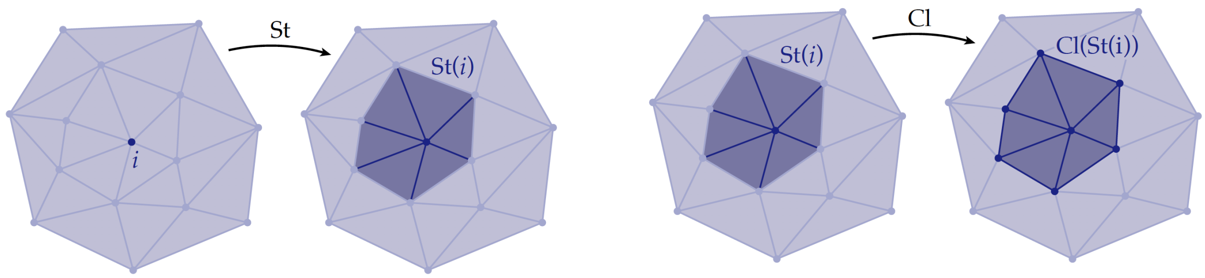


Figure 2.3: Left is the star operation of a vertex i. The surrounding vertices of i are labeled by $\{1, 2, ..., 6\}$. Right subfigure is the closure operation. Copied from Crane's note.

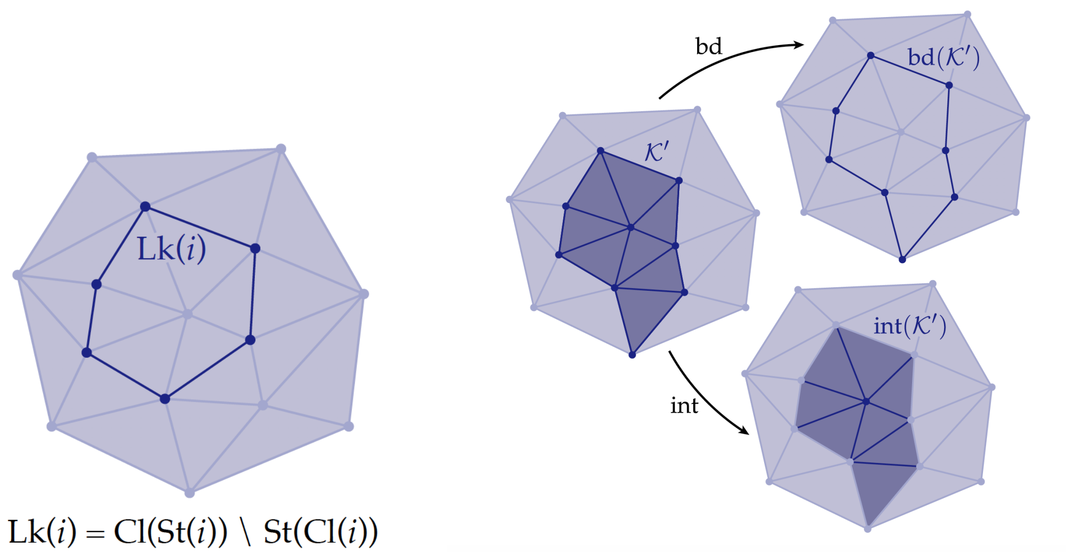


Figure 2.4: Left is the link of the vertex i. Right is the boundary operation of a subcomplex $\mathcal{K}'$. It consists of σ' such that σ' is a proper face of *exactly one* σ with $\sigma \in \mathcal{K}'$. Copied from Crane's note

2.3 Orientation

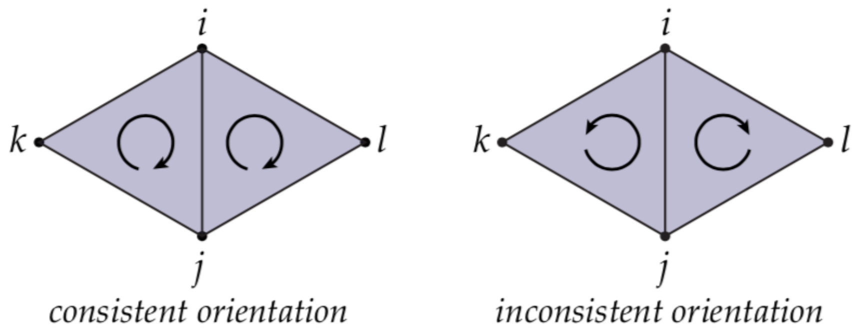


Figure 2.5: Consistent orientation of two adjacent oriented simplices, copied from Crane's note.

For each k-simplex, we define its orientation as the follows.

Orientation

- For 1-simplex, the orientation of $\{i, j\}$ is either from i pointing to j, or the opposite direction. We denote it by $[i, j]$ for the formal one, and $[j, i]$ for the latter one. Graphically, we draw an arrow from i to j to represent the orientation $[i, j]$.

- For 2-simplex $\{i, j, k\}$, we can loop them in the order of (i, j, k) or (j, i, k). We treat (i, j, k), (j, k, i) and (k, i, j) the same orientation. Thus, a 2-simplex has two orientations, denoted by $[i, j, k]$ and $[j, i, k]$, respectively. Note that we can obtain (j, i, k) from(i, j, k) by one permutation $i \leftrightarrow j$, and obtain (j, k, i) by two permutations, first $i \leftrightarrow j$, then $i \leftrightarrow k$.

- For a k-simplex $\{0, ..., k\}$, we consider permutation between the order set $[i_0, ..., i_k]$, where $0 \leq i_\alpha \leq k$. For instance $[0, 1, 2, ..., k] \mapsto [1, 0, 2, ..., k]$ is a permutation, which permutes $0 \leftrightarrow 1$. $[0, 1, 2, ..., k] \mapsto [1, 2, 0, 3, ..., k]$ is a composition of two permutations: $0 \leftrightarrow 1$, $1 \leftrightarrow 2$. There are two equivalent classes of permutations, even or odd numbers of permutations from the order set $[0, 1, ..., k]$. This defines two orientations. One can be permuted to $[0, 1, ..., k]$ by even numbers of permutations. The other class can be permuted to $[0, 1, ..., k]$ by odd numbers of permutations. Thus, every k-simplex with $k \geq 1$ can be associated with one of the two orientations, either $[0, 1, 2, ..., k]$, or $[1, 0, 2, ..., k]$, and denote $[1, 0, 2, ..., k]$ by $-[0, 1, 2, ..., k]$. For instance, $[1, 0] = -[0, 1]$, $[0, 2, 1] = -[0, 1, 2]$, $[1, 2, 0] = [0, 1, 2]$, $[1, 2, 3, 4] = -[2, 3, 4, 1]$.

Consistent Orientation of Adjacent Simplices

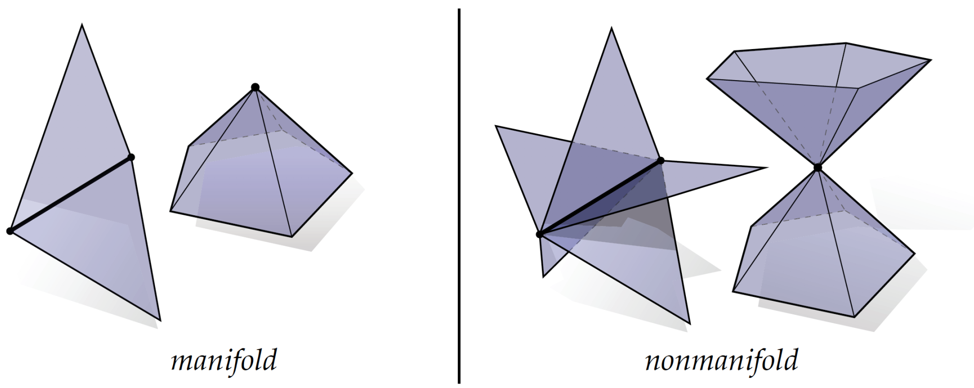


Figure 2.6: Manifold and non-manifold. Copied from Crane's note.

- **Two adjacent 1-simplices** The following two adjacent oriented 1-simplices $[i, j]$, $[j, k]$ are said to have consistent orientation, while $[j, i]$ and $[j, k]$ are inconsistent.

- **Two adjacent 2-simplices** The following two adjacent oriented 2-simplices $[i, j, k]$ and $[k, j, \ell]$ are oriented consistently, while $[i, j, k]$ and $[j, k, \ell]$ are not. Note that the common edge $[j, k]$ the same order as that in $[j, k, i]$, which is equivalent to $[i, j, k]$. On the other hand, $[j, k]$ has opposite order as that in $[k, j, \ell]$. See Fig. 2.5.

2.4 Simplicial Surfaces

Abstract simplicial surface An abstract simplicial surface is

- a pure simplicial 2-complex,

- the link of every vertex is a single loop of edges, or equivalently, where the star of every vertex is a combinatorial disk made of triangles.

The fact that every vertex has a disk-like neighborhood captures the basic idea of a topological surface; we therefore say that such a complex is a manifold. See Fig. 2.6.

Oriented simplicial surface An oriented simplicial surface is an abstract simplicial surface where we can assign a consistent orientation to every triangle.

Examples

- A tetrahedra $[0, 1, 2, 3]$ is oriented. Its boundaries are the oriented faces: $[0, 1, 2]$, $-[1, 2, 3]$, $[0, 2, 3]$ and $-[0, 1, 3]$, which are consistent to the orientation of $[0, 1, 2, 3]$.

We define the boundary operator for an oriented tetrahedra by

$$\partial[v_0, v_1, v_2, v_3] = \sum_{i=0}^{3}(-1)^i[v_0, .., \hat{v}_i, ..., v_3].$$

where $\hat{v}_i$ means v_i is dropped from the list.

- Möbius band is not orientable, see Fig. 2.7.
- The n dimensional ball S^n is orientable.

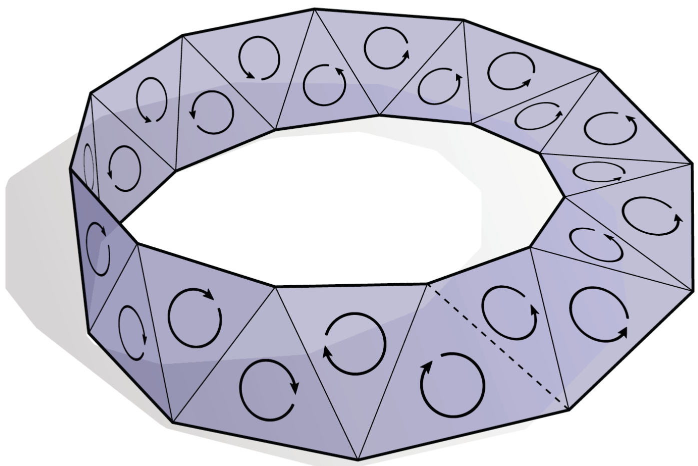


Figure 2.7: Möbius band is not orientable. Copied from Crane's note

Oriented discrete surface

- A polygon f is an ordered set $[i_0, i_1, ..., i_{n-1}]$ with $i_0 = i_n$. The set $\{i_0, ..., i_{n-1}\}$ are its vertices, and the set $\{[i_k, i_{k+1}]|k = 0, ..., n\}$ are its edges. Here we identify $i_n = i_0$. The edges can be ordered consistently. The polygon can also be oriented consistently to its edges. Such an oriented polygon is an oriented face. We say this oriented edge is incident to this oriented face, and vertex i is incident to the edges $[i, j]$ and $[j, i]$.
- Two polygons are adjacent if they have a common edge. They have consistent orientation if their common edge has opposite orientation.
- An oriented discrete surface $S = \{V, E, F\}$ is composed of consistent polygons and satisfies
  - each edge $e \in E$ is incident to one or two faces;

– for each index $i \in V$, the faces that i is incident to form a closed fan or an open fan.

- red Boundary of an oriented discrete surface. Given an oriented surface $S = (V, E, F)$, the boundary operator $\partial_{21} : F \to E$ with $\partial_{21}(f) = \sum e$ if e is incident to f. The boundary operator $\partial_{10} : E \to V$ with $\partial_{10}(e) = \sum v$ if v is incident to e.

2.5 Adjacent Matrices

Given a 2-simplex with vertices V, edges E and triangles F. We index vertices by $(0, ..., |V| - 1)$, edges by $(0, ..., |E| - 1)$ and triangles by $(0, ..., |F| - 1)$.

Adjacent matrix A_k encodes composition of $(k+1)$-simplicies in k-simplicies.

- A_0: row (edge index), column (vertex index).

$$(A_0)_{ij} = \begin{cases} 1 & \text{if } i\text{th edge contains vextex } j; \\ 0 & \text{otherwise.} \end{cases}$$

- A_1: row (face index), column (edge index).

$$(A_1)_{ij} = \begin{cases} 1 & \text{if } i\text{th face contains edge } j; \\ 0 & \text{otherwise.} \end{cases}$$

Signed adjacent matrix for an oriented simplicial surface We first index oriented edges and triangles.

- A_0: row (oriented edge index), column (vertex index) .

$$(A_0)_{ij} = \begin{cases} 1 & \text{if } i\text{th edge contains } j \text{ as an end vextex;} \\ -1 & \text{if } i\text{th edge contains } j \text{ as a starting vextex;} \\ 0 & \text{otherwise.} \end{cases}$$

- A_1: row (oriented edge index), column (oriented triangle index).

$$(A_1)_{ij} = \begin{cases} 1 & \text{if } i\text{th face contains } j\text{th edge with consistent orientation;} \\ -1 & \text{if } i\text{th face contains } j\text{th edge with inconsistent orientation;} \\ 0 & \text{otherwise.} \end{cases}$$

Example (see Crane's note, pp.16, 17.)

2.6 Half-edge Data Structure

Half-edges We want to create a data structure to describe oriented discrete surface. Let $S = \{V, E, F\}$ be a discrete surface with vertices V, edges E and faces F.

- **Helf-edge:** For each edge, there are two orientations. We call them a pair of half-edge. For instance, $[i, j]$ and $[j, i]$ are a pair of half-edge. The half-edge set H contains all these pairs of edges. Thus, $|H| = 2|E|$.
- On H, we define two pointers:
  - src: $H \to V$ maps the oriented edge to its start vertex;
  - face: $H \to F$ maps the half-edge to the face it belongs to;
- On H we can define two operations: flip and next.
  - **flip**: $\eta : H \to H \cup \{\text{NULL}\}$.

$$\eta([i,j]) = \begin{cases} [j,i] & \text{if } [i,j] \text{ is an interior edge} \\ \text{NULL} & \text{if } [i,j] \text{ is a boundary edge} \end{cases}$$

  - **next**: $\rho : H \to H$

$$\rho([i,j]) = [j,k] \text{ if both } [i,j] \text{ and } [j,k] \text{ belong to the same face.}$$

- From these two operations, we can easily produce topological information.
  - Get face from a half-edge $[i, j]$. We perform $\rho[i, j] = [j, k]$, $\rho[j, k] = [k, i]$ and $\rho[k, i] = [i, j]$. We obtain the face $[i, j, k]$ that contains $[i, j]$ as its boundary. Thus, the faces are orbits of ρ.
  - Get edges with common vertex i or the vertices adjacent to i. We start from $[i, j]$. Apply flip η to get $[ji]$, then apply ρ to get $[i, k]$. Apply this $\rho \circ \eta$ repeatedly until we recover $[i, j]$. From this process, we get all edges connecting to i. We also obtain the vertices that are adjacent to i. Thus, surround vertices of a given vertex are orbits of $\rho \circ \eta$.

We can make the following index table for flip and next operations.

half-edge	twin half-edge	start vertex	face	next half-edge

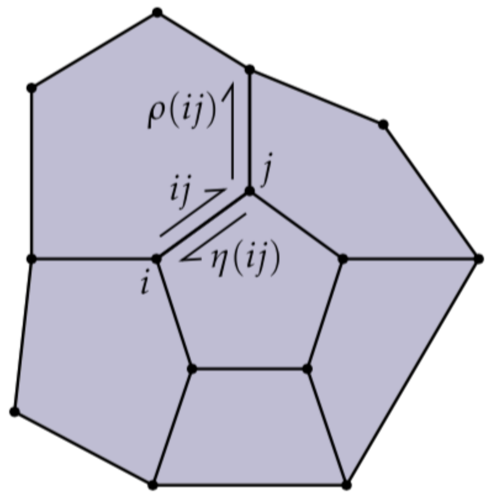


Figure 2.8: The *flip* and *next* operations on half-edges. Copied from Crane's note.

2.7 Topological Invariants

Definition

- **Topological disk** A topological disk is, roughly speaking, any shape you can get by deforming the unit disk in the plane without tearing it, puncturing it, or gluing its edges together. Some examples of shapes that are disks include a flag, a leaf, and a glove. Some examples of shapes that are not disks include a circle (i.e., a disk without its interior), a ball, a sphere, a donut, a washer, and a teapot.

- **Polygonal disk** A polygonal disk is any topological disk constructed out of simple polygons.

- **Topological sphere** A topological sphere is any shape resembling the standard sphere, and a polyhedron is a sphere made of polygons.

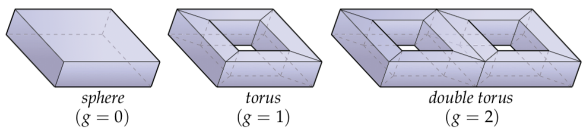
- **Piecewise linear surface** A piecewise linear surface is any surface made by gluing together polygons along their edges.

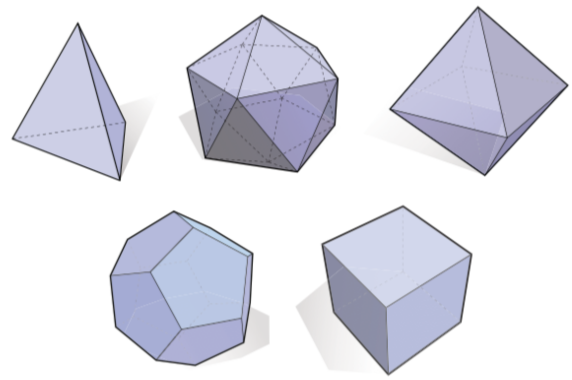
- **Simplicial surface** A simplicial surface is a special case of a piecewise linear surface where all the faces are triangles.

- **Polygonal sphere** A polygonal sphere is a sphere made of polygons, i.e. a polyhedron.

- **Genus** Numbers of handles. Sphere has no handle. Torus has one handle.

- **Valence of a vertex** is the number of edges that contains that vertex. A vertex of a simplicial surface is said to be *regular* if its valence is 6.





Theorem 2.1 (Euler's polyhedra formula). *Any polygonal disk with V vertices, E edges and F faces satisfies*

$$V - E + F = 1.$$

Any polygonal sphere satisfies

$$V - E + F = 2.$$

Proof. Using induction by reducing number of edges. When you cut off one edge, both E and F are reduced by 1, but V is unchanged. Thus the Euler characteristic remains the same.

Exercise 1. Complete the proof. □

In general, we have the Euler-Poincaré formula:

Theorem 2.2 (Euler-Poincaré). *A compact oriented polygonal surface with genus g satisfies*

$$V - E + F = 2 - 2g.$$

Proof. See Eppstein, David. "Twenty Proofs of Euler's Formula: V-E+F=2" (2013). For general formula in high dimensions, see Petr Hliněný, "A Short Proof of Euler–Poincaré Formula" (2017). □

Exercise 3. The only simplicial surface for which every vertex is regular (i.e. its valence is 6) is torus.
Hint: use Euler-Poincaré characteristic formula.

Exercise 4. The minimum irregular valence in a simplicial complex $\mathcal{K}$ is

$$m(\mathcal{K}) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2. \end{cases}$$

Exercise 5. Show that the mean valence approaches 6 as the number of vertices in a simplicial surface goes to infinity.

Exercises 6-7

Exercises 8-12, 14-15

2.8 Coding exercise

Please see

- matlab introduction on `http://math.ntnu.edu.tw/~yueh/courses/MATLAB_Introduction.html` and
- construction of adjacent matrices on `http://math.ntnu.edu.tw/~yueh/courses/MATLAB_MeshDataStructure.html`

provided by Mei-Heng Yueh.

Chapter 3

Discrete Surfaces

3.1 Discrete surfaces

3.1.1 Basic notions

- **Polygon mesh:** A polygon mesh is a triple (V, E, F), which are vertices, edges and faces.

- A face is a polygon, described by $f = \{\{i_1, i_2\}, \{i_2, i_2\}, ..., \{i_n, i_1\}\}$, or simply a cyclically ordered list $[i_1, ..., i_n]$.

- **Incident relation:** We say e is incident to f (denoted by $e \prec f$) if e is an edge of f. If v is a vertex of an edge e, we say $v \prec e$. The incident relation is transitive, that means, if $v \prec e$ and $e \prec f$, we say $v \prec f$.

- **Discrete surfaces** are those polygonal mesh (V, E, F) satisfying the following manifold conditions:

  - each $e \in E$ is incident to one or two faces;
  - for each vertex $v \in V$, the faces incident to v form a closed fan or an open fan.

- **Triangulated surfaces** are those discrete surfaces with only triangle faces. Triangulated surfaces are also called simplicial surfaces.

- **Discrete metric** on a triangulated surface $M = (V, E, F)$ is a set of positive edge lengths $\ell : E \to \mathbb{R}^+$ satisfying the triangle inequality

$$\ell_{ij} + \ell_{jk} > \ell_{ki} \quad \text{for all } [ijk] \in F.$$

- **Realization of a discrete surface** $M = (V, E, F)$ **in 3D** by a map

$$f : V \to \mathbb{R}^3$$

with f being linear on F by linear interpolation. Such a surface in $\mathbb{R}^3$ is called a piecewise linear surface. The mapping f is called an immersion if (i) the image of each face has nonzero area; (ii) the faces incident to each vertex do not intersect. It is called embedding if the image of M has no intersection.

3.1.2 Geometric measurements for triangulated surfaces

- **edge vector** $v_{ij} := f_j - f_i$ for $[ij] \in H$
- **edge length** $\ell_{ij} := |v_{ij}|$
- **area** For an oriented triangle $[ijk]$, A_{ijk} is its signed area;
- **normal** $N_{ijk} := \frac{v_{ij} \times v_{ik}}{|v_{ij} \times v_{ik}|}$
- **interior angle** $\theta^i_{jk} := \cos\left(\langle \hat{v}_{ij}, \hat{v}_{ik} \rangle\right)$
- **bending angle** For an interior edge $\{j, k\}$ incident to $[ijk]$ and $\ell kj]$, the bending angle

$$\alpha_{jk} := \sin^{-1}\left(\langle \hat{v}_{jk}, N_{ijk} \times N_{\ell kj} \rangle\right).$$

3.1.3 Intrinsic quantities

- Let $M = (V, E, F)$ be a triangulated surface with a metric ℓ. A quantity on M is called intrinsic if it only depends on the metric, not on how M is realized in $\mathbb{R}^3$.
- **Area A is intrinsic.**

$$A_{ijk} = \frac{1}{4}\sqrt{\left(\ell_{ij} + \ell_{jk} + \ell_{ki}\right)\left(-\ell_{ij} + \ell_{jk} + \ell_{ki}\right)\left(\ell_{ij} - \ell_{jk} + \ell_{ki}\right)\left(\ell_{ij} + \ell_{jk} - \ell_{ki}\right)}$$

  Proof. Let us call the length of the triangle by a, b, c. By sine law,

$$A^2 = \frac{1}{4}a^2b^2\sin^2 C = \frac{1}{4}a^2b^2\left(1 - \cos^2 C\right) = \frac{1}{4}a^2b^2\left(1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2\right).$$

  You can complete the rest. □

- **Interior angle θ^i_{jk} is intrinsic:**

$$\theta^i_{jk} = \cos\left(\frac{\ell_{ij}^2 + \ell_{ki}^2 - \ell_{jk}^2}{2\ell_{ij}\ell_{ki}}\right).$$

- **The angle defect $d(i)$ is intrinsic.** For an interior vertex i, we define the angle defect (or the discrete Gaussian curvature form) of M at i by

$$d(i) := 2\pi - \sum_{i \prec [ijk]} \theta^i_{jk} \tag{3.1}$$

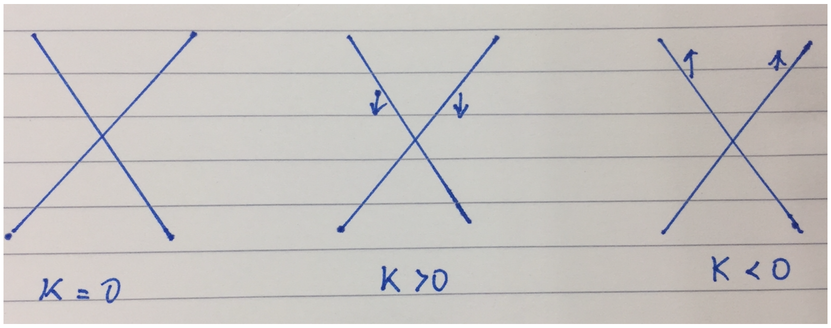


Figure 3.1: In this figure, we cut the lines to get four triangles. In the second subfigure, we move down the upper left and upper right edges. These left/right angles decrease. We then paste the to triangle and get a convex vertex. In the third subfigure, we increase the two angles on the two wings, then paste the upper triangle. In this case, $K < 0$.

Theorem 3.1 (Descartes Theorem)**.** *Consider a discrete surface M without boundary with simplicial complex (V, E, F). Then*

$$\sum_{v \in V} d(v) = 2\pi\chi(M) = 2\pi(|V| - |E| + |F|).$$

Proof. Let $v \in V$ be a vertex. By definition, the defect angle

$$d(v) = 2\pi - \sum_{\substack{i \\ \{iv\} \in E}} \alpha_i = 2\pi - \sum_{\substack{i \\ \{iv\} \in E}} (\pi - \beta_i),$$

where β_i is the supplementary (exterior) angle of α_i. Thus,

$$\begin{aligned}\sum_{v \in V} d(v) &= \sum_{v \in V} 2\pi - \sum_{v \in V} \sum_{\substack{i \\ \{iv\} \in E}} (\pi - \beta_i) \\ &= 2\pi|V| - 2\pi|E| + \sum_{v \in V} \sum_{\substack{i \\ \{iv\} \in E}} \beta_i\end{aligned}$$

For the last term, we can sum it over faces.

$$\begin{aligned}\sum_{v \in V} \sum_{\substack{i \\ \{iv\} \in E}} \beta_i &= \sum_{v \in V} \sum_{\substack{i \\ \{iv\} \in E}} (\pi - \alpha_i) = \sum_{v \in V} \sum_{\substack{f \in F \\ v \prec f}} (\pi - \alpha_f) \\ &= \sum_{f \in F} \sum_{\substack{f \in F \\ v \prec f}} (\pi - \alpha_f) = \sum_{f \in F} (3\pi - \pi) = 2\pi|F|.\end{aligned}$$

Thus,

$$\sum_{v \in V} d(v) = 2\pi\chi(M).$$

□

- **Parallel transport is intrinsic.** Let $M = (V, E, F)$ be a triangulated surface. Let $\boldsymbol{\gamma}$ be a curve on M. We assume $\boldsymbol{\gamma}$ does not hit any vertex. Let X be a vector on the tangent plane of M at $p_1 \in \boldsymbol{\gamma}$. The parallel transport of X along $\boldsymbol{\gamma}$ is by the following:
  - In a single triangle, parallel transport of X is the same in the Euclidean space (plane);
  - As $\boldsymbol{\gamma}$ crawls over an edge, flatten the two triangles and parallel transport X as that on the flan plane.

  Under this definition, the parallel transport is independent how the trangle bent across an edge. Thus, it is intrinsic.

- **Holonomy angle is intrinsic.** Let $\boldsymbol{\gamma}$ be a closed curve that winds around a vertex $i \in V$ counterclockwise once. Let X be a vector field parallel transported along $\boldsymbol{\gamma}$. The angle it rotates after traveling along $\boldsymbol{\gamma}$ once is called the holonomy angle of i.

Theorem 3.2 (Discrete Local Gauss-Bonnet Theorem (Holonomy))**.** *Let $M = (V, E, F)$ be a discrete surface. Let $i \in V$ be a vertex. The holonomy $h(i)$ equals the angle defect $d(i)$.*

Proof.

1. Let us choose a special curve $\boldsymbol{\gamma}$ which is constructed as the follows. Let $[ijk]$ be a triangle incident to i. Let B and C are the points of the edge $[ij]$ and ik with equal distances to i. We draw two lines from B and C perpendicular to $[ij]$ and $[ik]$ respectively. These two lines intersect at D. The angle $\angle BDC$ is the supplementary angle of θ^i_{jk}. By construction, the curve $\boldsymbol{\gamma}$ is a closed curve.
2. Let X be a vector field parallel transported along $\boldsymbol{\gamma}$. There is no angle change as it craws across an edges. In the triangle $[ijk]$, the angle changes by θ^i_{jk} as it passes by the intersection point D. Thus, the total angle change in one loop is $\sum_{[ijk]\in F} \theta^i_{jk}$.
3. For general $\boldsymbol{\gamma}$, we get the same angle change in a triangle because it is flat and no angle change across edges from the definition of parallel transport.

□

- **Geodesic curvature is intrinsic.**. Let $\boldsymbol{\gamma} : I \to M$ be a regular curve. Then by laying the triangles flat locally, we have a plan curve. The curvature of this plane curve is called the geodesic curvature, denote by κ_g. From this definition, it is easy to see that $\int_{\boldsymbol{\gamma}} \kappa_g \, ds = 0$ for those closed curve $\boldsymbol{\gamma}$ that does not wind around any vertex.

  Theorem 3.3 (Discrete Local Gauss-Bonnet Theorem (Turning Angle))**.** *Let $\boldsymbol{\gamma}$ be a closed curve on M that winds around a vertex i counterclockwise once, then the total winding number*

  $$\int_{\boldsymbol{\gamma}} \kappa_g \, ds = \sum_{[ijk]\succ i} \theta^i_{jk}$$

  In other words,

  $$d(i) + \int_{\boldsymbol{\gamma}} \kappa_g \, ds = 2\pi. \tag{3.2}$$

Proof. The geodesic curvature is the angle change rate. The integral $\int_{\boldsymbol{\gamma}} \kappa_g \, ds$ is the total angle change along $\boldsymbol{\gamma}$, which is exactly the holonomy at i. □

3.1.4 Discrete Gaussian curvature

Consider vertex p_i surrounded by the triangles t_{ijk}. We would like to define the Gaussian curvature at p_i. Let c_{ijk} be the circumcenter of the triangle f_{ijk} (the image $f(t_{ijk})$), c_{jk} the center of the edge u_{jk}, and U_i the dual cell that contains f_i with boundary which is one-loop geodesic curves that connects c_{jk}, c_{ijk}. Let us compute $\int_{U_i} K$. We use the formula

$$d(p_i) = \text{Area}(N(U_i)).$$

The image $N(U_i)$ is a spherical polygon on unit sphere. It consists of vertices $N_{ijk} \in S^2$ with $p_i \prec t_{ijk}$. The interior angle at N_{ijk} is β_{ijk}. We claim that $\beta_{ijk} + \alpha_{ijk} = \pi$, where α_{ijk} is the interior angle of the triangle $f(t_{ijk})$ at vertex f_i. This can be shown by investigating interior angles of the quadrilateral $[f_i, c_{ik}, c_{ijk}, c_{ij}]$. Its proof is left for exercise.

Lemma 3.1. *The area of a spherical triangle on S^2 with interior angles β_1, β_2 and β_3 is*

$$A = \beta_1 + \beta_2 + \beta_3 - \pi.$$

The area of a spherical polygon on S^2 with interior angles $\beta_1, ..., \beta_n$ is

$$A = \sum_{i=1}^{n} \beta_i + (2 - n)\pi.$$

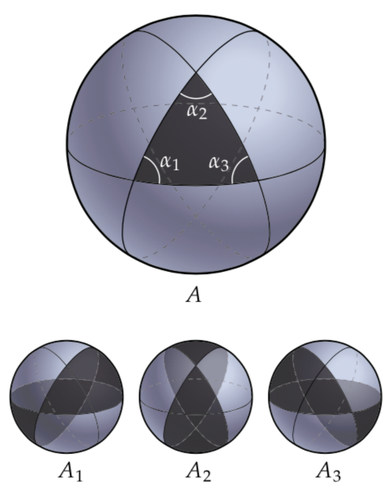


Figure 3.2: Copied from Crane's note. The angle α in this Figure is the β in this note.

By using this lemma, we get

$$\boxed{d(p_i) = \sum_{p_i \prec t_{ijk}} \beta_{ijk} + (2-n)\pi = 2\pi - \sum_{p_i \prec t_{ijk}} \alpha_{ijk}.}$$

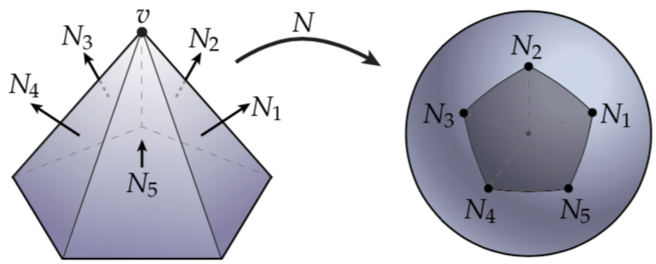


Figure 3.3: Copied from Crane's note.

Chapter 4

Surfaces

4.1 Basic notions of differential geometry

- Charts, Atlas, abstract definition of manifolds, parametrization
- Realization of abstract surfaces, immersion, embedding
- Tangent and normal.
- Metric, Riemannian manifolds

4.1.1 Abstract differential manifolds

Definition of n-dimensional differential manifold

- A parametrized manifold is a set M endowed with an atlas $(U_\alpha, \mathbf{x}_\alpha)$ such that

  (i) $U_\alpha \subset M$ are open and $\cup_\alpha U_\alpha = M$

  (ii) $\mathbf{x}_\alpha : U_\alpha \to \mathbb{R}^n$ such that on each overlapping region $U_\alpha \cap U_\beta$, the function

  $$\varphi_{\alpha,\beta} := \mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1} : \mathbf{x}_\alpha(U_\alpha \cap U_\beta) \to \mathbf{x}_\beta(U_\alpha \cap U_\beta)$$

  is a diffeomorphism (i.e. $\varphi_{\alpha,\beta}$ is 1-1, onto, and both $\varphi_{\alpha,\beta}$ and $\varphi_{\alpha,\beta}^{-1}$ are continuously differentiable). Here, $\mathbf{x}$ is called a coordinate. Its inverse $\mathbf{x}^{-1} : \mathbf{x}(U) \to U$ is called a parametrization.

- Two atlases $(U_\alpha, \mathbf{x}_\alpha)_\alpha$ $(U_\beta, \mathbf{x}_\beta)_\beta$ are called equivalent if their union forms a consistent new atlas.
- The equivalent class of atlased manifold is called a *differential manifold*.

Orientable manifold

- Suppose M is a manifold with atlas $(U_\alpha, \mathbf{x}_\alpha)_\alpha$. Two overlapping charts $(U_\alpha, \mathbf{x}_\alpha)$, $(U_\beta, \mathbf{x}_\beta)$ are called orientation consistent if $\det(d\varphi_{\alpha,\beta}) > 0$ in $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$.

- If M owns an orientation consistent atlas, we call M is orientable.

- The Möbius band is not orientable. The projective space CP^2 is not orientable.

4.1.2 Tangent spaces

- **Tangent** : Given a point $p \in M$. Let $(U, \mathbf{x})$ be a coordinate chart covering p. The region U is described by the coordinate $\mathbf{x}$. The tangent space at $\mathbf{x}(p) \in \mathbf{x}(U) \subset \mathbb{R}^n$ is $\mathbb{R}^n$. Thus, it is natural to identify a vector in R^n as a tangent vector of M at p. This definition should be consistent in the following sense: if $(U_\alpha, \mathbf{x}_\alpha)$, $(U_\beta, \mathbf{x}_\beta)$ are two charts covering p and $\varphi_{\alpha,\beta}$ be coordinate transformation from $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$ to $\mathbf{x}_\beta(U_\alpha \cap U_\beta)$, then the tangent vectors $\mathbf{v}_\alpha \in T_{\mathbf{x}_\alpha(p)}\mathbb{R}^n$ and $\mathbf{v}_\beta \in T_{\mathbf{x}_\beta(p)}\mathbb{R}^n$ satisfy

$$\mathbf{v}_\beta = d\varphi_{\alpha,\beta}|_{\mathbf{x}_\alpha(p)}(\mathbf{v}_\alpha).$$

  Let us denote $T_p(M)$ the tangent space of M at p.

- Let $\gamma : I \to M$ be a parametrized curve with $\gamma(0) = p$. We can identify $\gamma'(0)$ as a tangent vector at p. Conversely, for any $X \in T_p(M)$, there exists a parametrized curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(0) = X$.

4.1.3 Realization of abstract surfaces

- **Immersion** Let M be a two dimensional manifold. A differential

$$f : M \to \mathbb{R}^3$$

  is called an immersion of M in $\mathbb{R}^3$ if for any chart $(U, \mathbf{x})$, the Jacobian of the map $f \circ \mathbf{x}^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$ is of full rank.

- **Embedding** It is called an embedding if f is 1-1.

- **Representation of tangent space** Let M be a surface, $p \in M$. Suppose $(U, (u, v))$ is a coordinate chart containing $p \in M$. The tangent plane is spanned by ∂_u and ∂_v (i.e. $\partial/\partial u$ and $\partial/\partial v$). Their images on $\mathbb{R}^3$ are $df_p(\partial_u) = f_u$ and $df_p(\partial_v) = f_v$. The matrix representation of df_p is called the Jacobian of f, which is $J := (f_u, f_v)_{3\times 2}$.

4.1.4 Examples of surfaces

Let us study a surface imbedded in $\mathbb{R}^3$. The surface is parameterized by $f : M(\subset \mathbb{R}^2) \to \mathbb{R}^3$. Several concrete examples are

- Sphere:
$$f(u,v) = (\cos u \cos v, \sin u \cos v, \sin v), \quad 0 \le u \le 2\pi, -\pi/2 \le v \le \pi/2.$$
- Ellipsoid:
$$f(u,v) = (a\cos u \cos v, b\sin u \cos v, c\sin v), \quad 0 \le u \le 2\pi, -\pi/2 \le v \le \pi/2.$$
- Hyperboloid:
$$f(u,v) = (a\cos u \cosh v, b\sin u \cosh v, c\sinh v), \quad 0 \le u \le 2\pi, -\infty < v < \infty.$$
- Cylinder: $f(u,z) = (\cos u, \sin u, z)$, $0 \le u \le 2\pi$, $0 \le z \le 1$.
- Torus:
$$\begin{aligned} x(\theta,\varphi) &= (R + r\cos\theta)\cos\varphi \\ y(\theta,\varphi) &= (R + r\cos\theta)\sin\varphi \\ z(\theta,\varphi) &= (R + r\sin\theta) \\ 0 \le \theta &< 2\pi, \quad 0 \le \varphi < 2\pi. \end{aligned}$$
- Möbius band:
$$\begin{aligned} x(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u \\ x(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u \\ z(u,v) &= \frac{v}{2}\sin\frac{u}{2}. \\ 0 \le u &< 2\pi, \quad -1 \le 1 \le 1. \end{aligned}$$
- Projective plane: is the set of all 1D subspace in $\mathbb{R}^3$. That is, we define an equivalent relation $\sim$ in $\mathbb{R}^3 - \{0\}$ by $x \sim y$ if there exist a number a such that $y = ax$. The projective plane is defined as
$$\mathbb{R}P^2 := \left(\mathbb{R}^3 - \{0\}\right) / \sim .$$
We can define the following open sets:
$$U_i = \{(x^0, x^1, x^2) \in \mathbb{R}^3 - \{0\} | x^i \ne 0\}, \quad i = 0,1,2.$$
U_i is an open set in $\mathbb{R}^3$. U_0 (in general, U_i) can be parameterized by
$$\xi_0^1 = x^1/x^0, \quad \xi_0^2 := x^2/x^0.$$
Then (U_i, ξ_i) is a coordinate chart of $\mathbb{R}P^2$.

We call M the abstract manifold, and the image $f(M)$ the embedded manifold in $\mathbb{R}^3$. We shall investigate surface curvature.

4.2 Surface measurements

4.2.1 Intrinsic measurement

- **Metric (inner product structure):** The metric of M is induced by the metric in $\mathbb{R}^3$ through f: for any two tangent vectors $X, Y \in T_p(M)$, define their inner product to be
$$g_p(X,Y) := \langle X, Y\rangle_p := \langle df_p(X), df_p(Y)\rangle_{\mathbb{R}^3},$$
which satisfies
  - * bilinear: $\langle aX + bY, Z\rangle = a\langle X, Z\rangle + b\langle Y, Z\rangle$,
  - * symmetry: $\langle X, Y\rangle = \langle Y, X\rangle$,
  - * positive definite: $\langle X, X\rangle \geq 0$, $\langle X, X\rangle = 0$ if and only if $X = 0$.

  Definition 4.1. (1) A metric on M is a smooth inner product structure $g_p : T_pM \times T_pM \to \mathbb{R}$ at every point p.

  (2) A manifold M equipped with a smooth inner product structure g_p is called a *Riemannian manifold*.

- **Length**: the length of a tangent vector X is defined to be $\|X\| = \sqrt{\langle X, X\rangle}$. The length of a C^1 curve $\boldsymbol{\gamma} : I \to M$ is defined to be
$$\int_I \|\boldsymbol{\gamma}'(t)\|\, dt.$$

- **Angle**: The angle of two tangent vectors $X, Y \in T_pM$ is defined to be
$$\theta := \cos^{-1}\left(\frac{\langle X, Y\rangle}{\|X\|\|Y\|}\right).$$

- **Area:** Given two tangents $X, Y \in T_pM$. The signed area of the parallelogram spanned by X, Y is given by
$$\det(X,Y) = \|X\|\|Y\|\sin(\theta).$$
It can also be expressed in terms of inner product as
$$\det(X,Y)^2 = \begin{vmatrix} \langle X, X\rangle & \langle X, Y\rangle \\ \langle X, Y\rangle & \langle Y, Y\rangle \end{vmatrix}.$$
The (unsigned) area of a region $U \subset M$ with coordinate $\mathbf{x} = (u, v)$ can be computed by
$$\mathcal{A}(U) = \int_{\mathbf{x}(U)} |\det(\partial_u, \partial_v)|\, du\, dv.$$
Such definition is independent of choice of coordinate chart. The term $\sigma := \det(\partial_u, \partial_v)\, du \wedge dv$ is called a signed area element of M. You can show it is independent of choice of coordinate chart.

- **Orthogonal coordinate chart** Given a coordinate chart $(U, (u, v))$ at $p \in M$, we can locally reparametrize it as (u, w) in a small neighborhood W of p such that $\partial_u \perp \partial_w$. In other words, $(W, (u, w))$ is an orthogonal coordinate system. *

4.2.2 Extrinsic approach through embedding

Gauss Map

- **Gauss map**: The normal of the surface at $p \in M$ is the unit vector that is normal to the tangent plane at p. In terms of coordinate system

$$N(p) = N(u, v) = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

The map $N : M \to S^2$ is called the Gauss map. The Gauss map can reflect how the surface is curved in $\mathbb{R}^3$.

Shape Operator

- **Definition of Shape operator**: Given any $X \in T_pM$. By differentiating $\langle N, N \rangle = 1$ in the direction X, we get $dN_p(X) \perp N_p = df_p(T_pM)^\perp$. Thus, $dN_p(X) \in df_p(T_pM)$. In other words, for any $X \in T_p(M)$, there exists a unique $Y \in T_p(M)$ such that

$$dN_p(X) = df_p(Y).$$

The mapping $S : X \mapsto Y$ is called the shape operator on T_pM. That is

$$\boxed{dN_p = df_p \circ S.} \tag{4.1}$$

The shape operator is linear because both dN_p and df_p are linear.

- **Coordinate representation of the shape operator** Let $(U, (u, v))$ be a coordinate chart containing $p \in M$. The corresponding tangent (∂_u, ∂_v) forms a basis in T_pM. We have

$$\begin{bmatrix} \langle S\partial_u, \partial_u \rangle & \langle S\partial_u, \partial_v \rangle \\ \langle S\partial_u, \partial_v \rangle & S\langle \partial_v, \partial_v \rangle \end{bmatrix} = - \begin{bmatrix} f_{uu} \cdot N & f_{uv} \cdot N \\ f_{uv} \cdot N & f_{vv} \cdot N \end{bmatrix}.$$

This is because

$$\langle S\partial_u, \partial_u \rangle = \langle dN_p(\partial_u), df_p(\partial_u) \rangle = \langle N_u, f_u \rangle = -\langle N, f_{uu} \rangle.$$

*We look for a function $w(u, v)$ such that $(w_u\partial_u + w_v\partial_v) \perp \partial_u$ using the metric induced by f. That is,

$$\langle w_u\partial_u + w_v\partial_v, \partial_u \rangle = 0.$$

This is a linear PDE for w which can be solved by method of characteristics locally.

- **Shape operator is self-adjoint:**

$$\langle SX, Y\rangle = \langle X, SY\rangle$$

This is easily followed from the coordinate representation of S and the symmetry property of the Hessian of f.

- **The shape operator S as a quadratic form** The quadratic form $\langle SX, X\rangle = \langle dN_p(X), df_p(X)\rangle$ measures how much deviation of dN from X as N travels along X.

- **Shape operator in coordinate**: We introduce the second fundamental form. We take the variation of N in direction X, then project the resulting vector to $df_p(Y)$. This gives
$$\langle dN_p(X), df_p(Y)\rangle = \langle df_p(SX), df_p(Y)\rangle = \langle SX, Y\rangle.$$
We define
$$I\!I(X, Y) := \langle dN_p(X), df_p(Y)\rangle = \langle SX, Y\rangle.$$
This is called the second fundamental form of M.

- $I\!I(\cdot,\cdot)$ is symmetric. The symmetry of $I\!I$ can be read from the coordinate representation of $I\!I(\cdot,\cdot)$. Note that $\langle N, df(\partial_u)\rangle = \langle N, f_u\rangle = 0$ and $\langle N, f_v\rangle = 0$. We differentiate these two equations in u and v to get

$$I\!I(\partial_u, \partial_v) = \langle N_u, f_v\rangle = -\langle N, f_{vu}\rangle = -\langle N, f_{uv}\rangle = I\!I(\partial_v, \partial_u).$$

Thus, the coordinate representation of the second fundamental form is

$$\begin{bmatrix} I\!I(\partial_u, \partial_u) & I\!I(\partial_u, \partial_v) \\ I\!I(\partial_u, \partial_v) & I\!I(\partial_v, \partial_v) \end{bmatrix} = -\begin{bmatrix} f_{uu}\cdot N & f_{uv}\cdot N \\ f_{uv}\cdot N & f_{vv}\cdot N \end{bmatrix}$$

which is symmetric. Since the bilinear form $I\!I(\cdot,\cdot)$ is symmetric for the basis ∂_u and ∂_v, we get $I\!I(X, Y) = I\!I(Y, X)$ for arbitrary two tangent vectors X and Y.

- The shape operator $S : T_p(M) \to T_p(M)$ is self-adjoint w.r.t. the metric $\langle\cdot,\cdot\rangle$. This is because
$$\langle SX, Y\rangle = I\!I(X, Y) = I\!I(Y, X) = \langle SY, X\rangle = \langle X, SY\rangle.$$

- Coordinate representation of shape operator. Let us use (u, v) as our coordinate on M. We want to represent S in terms of the basis (∂_u, ∂_v). We write $S(\partial_u) = a\partial_u + b\partial_v$ and $S(\partial_v) = c\partial_u + d\partial_v$. From $dN(X) = df(S(X))$ with $X = \partial_u$ and ∂_v respectively, we get
$$dN(\partial_u) = df(S(\partial_u)), \quad dN(\partial_v) = df(S(\partial_v))$$
This leads to
$$N_u = af_u + bf_v, \quad N_v = cf_u + df_v. \tag{4.2}$$

Note that

$$\langle N, f_u \rangle = 0, \quad \langle N, f_v \rangle = 0$$

which lead to

$$\langle N_u, f_u \rangle = -\langle N, f_{uu} \rangle, \quad \langle N_v, f_v \rangle = -\langle N, f_{vv} \rangle,$$
$$\langle N_v, f_u \rangle = -\langle N, f_{uv} \rangle = \langle N_u, f_v \rangle.$$

Thus, by taking inner product of (4.2) with f_u and f_v, we get

$$SI = II,$$

where

$$I := \begin{bmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_u, f_v \rangle & \langle f_v, f_v \rangle \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$
$$II := -\begin{bmatrix} f_{uu} \cdot N & f_{uv} \cdot N \\ f_{uv} \cdot N & f_{vv} \cdot N \end{bmatrix} = -\begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

are matrix representation of the first and second fundamental forms, respectively. And the matrix

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Thus,

$$\boxed{S = II \cdot I^{-1} = \frac{1}{F^2 - EG} \begin{bmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{bmatrix}.} \tag{4.3}$$

Curvatures

- **Normal curvature.** Given $X \in T_pM$, the quantity $dN_p(X)$ measures how the normal N varies along direction X. Thus, we define the normal curvature

$$\kappa^N(X) := \frac{\langle dN_p(X), df_p(X) \rangle}{\|X\|^2} = \frac{\langle SX, X \rangle}{\|X\|^2}.$$

  It measure how the normal N changes in direction X.

- **Principle curvature** The eigenvalues κ_1 and κ_2 of S w.r.t. $\langle \cdot, \cdot \rangle$ are called the principal curvatures. The corresponding eigenvectors are called the principal curvature directions. They are orthogonal to each other because S is self-adjoint. The principle curvatures are the extremal values of the quadratic form $\langle SX, X \rangle$:

$$\kappa_1(p) = \min_{\substack{X \in T_pM \\ \|X\|=1}} \langle S_pX, X \rangle,$$
$$\kappa_2(p) = \max_{\substack{X \in T_pM \\ \|X\|=1}} \langle S_pX, X \rangle.$$

- **Mean curvature and Gaussian curvature**

  * The two invariants of S are

    Mean curvature: $H := (\kappa_1 + \kappa_2)/2 = \frac{1}{2}Tr(S)$,
    Gaussian curvature: $K := \kappa_1\kappa_2 = \det S$.

  * Curvature formula in terms of coordinate: using (4.3), we have

$$H = \frac{1}{2}\frac{eG + gE - 2fF}{F^2 - EG}.$$

$$K = \det S = \frac{f^2 - eg}{EG - F^2}.$$

  * **Properties of Mean curvature** We have

$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa^N(X_\theta)\, d\theta, \tag{4.4}$$

    where $X_\theta = \cos\theta X_1 + \sin\theta X_2$. That is, *the mean curvature is the average of the normal curvatures in all directions.* Formula (4.4) follows from

$$\kappa^N(X_\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

  * **Property of Gaussian curvature** The meaning of the Gaussian curvature can be read from the formula

$$dN_p = df_p \circ S$$

    by the following procedure. Let us draw a small disk $B_\varepsilon(p)$ about p on M. The image of this disk by the Gauss map $N : M \to S^2$ is $N(B_\varepsilon(p))$. The image of this disk by the embedding map $f : M \to \mathbb{R}^3$ is $f(B_\varepsilon(p))$. From $dN = df \circ S$, The ratio of these two areas is the determinant of S_p:

$$\mathcal{A}(N(B_\varepsilon(p)) \approx \mathcal{A}(f(B_\varepsilon(p)) \cdot \det(S_p) = \mathcal{A}(f(B_\varepsilon(p)) \cdot K_p.$$

    Thus,

$$K_p = \lim_{\varepsilon\to 0} \frac{\mathcal{A}(N(B_\varepsilon(p))}{\mathcal{A}(f(B_\varepsilon(p))}. \tag{4.5}$$

- **Geodesic curvature** Let γ be a curve on M and $\boldsymbol{\gamma} = f \circ \gamma$ be the curve on the manifold $f(M)$. From curve theory, $\boldsymbol{\gamma}' = T$, $\boldsymbol{\gamma}'' = T' \perp T$. There are two components of $\boldsymbol{\gamma}''$: one is on N, the normal of $f(M)$, the other one is on the tangent plane. Let us construct a local frame: $e_1 = T$, $e_3 = N$ and $e_2 = e_3 \times e_1$. Then,

$$\boxed{\boldsymbol{\gamma}'' = \kappa_g e_2 - \kappa_N e_3.} \tag{4.6}$$

  Here,

$$-\kappa_N := \langle \boldsymbol{\gamma}'', e_3\rangle = -\langle \boldsymbol{\gamma}', e_3'\rangle = -\langle T, N'\rangle.$$

$$\kappa_g := \langle \boldsymbol{\gamma}'', e_2 \rangle = -\langle \boldsymbol{\gamma}', e_2' \rangle = -\langle e_1, e_2' \rangle.$$

The projection of $\boldsymbol{\gamma}''$ onto the tangent plane T_pM is called the *geodesic curvature*. Recall in plane curve, $-\langle T, N' \rangle = \kappa$, where N is the plane normal and κ is the plane curvature, which measure angle-change of T along $\boldsymbol{\gamma}$ on the plane. On the surface M, the term e_2 is the plane normal of the curve $\boldsymbol{\gamma}$. Thus, $\kappa_g := \langle T, e_2' \rangle$ *measures the angle-change of* T *along* $\boldsymbol{\gamma}$ *on the surface* M.

Remark Recall the framed curve $(T(s), e_2(s), N(s))$ along $\boldsymbol{\gamma}(s)$, we have the following frame equation

$$\begin{bmatrix} T' \\ e_2' \\ N' \end{bmatrix} = \begin{bmatrix} & \kappa_g & -\kappa_N \\ -\kappa_g & & \omega \\ \kappa_N & -\omega & \end{bmatrix} \begin{bmatrix} T \\ e_2 \\ N \end{bmatrix}$$

Here, $\omega = -\langle dN(T), e_2 \rangle$ measures how N is twisted on the normal plane of the curve $\boldsymbol{\gamma}$. We shall come back to this frame approach in the next subsection.

Examples

- The sphere of radius r. $\kappa_1 = \kappa_2 = 1/r$. $H = 2/r$ and $K = r^2$.
- Cylinder: $S^1_r \times \mathbb{R}$. $\kappa_1 = 1/r$, $\kappa_2 = 0$. $H = 1/r$ and $K = 0$.
- Torus: $S^1_{r_1} \times S^1_{r_2}$.
- Hyperblic surface $f : (x, y) \mapsto (x, y, x^2 - y^2)$.
- Ruled surfaces

4.3 Intrinsic Surface Structure – Connection approach

We have seen that the variation of the normal vector field on a surface M reflects how M curves. From which we define normal curvature, principle curvatures, etc. In this section, we would like to study concept of curvature derived intrinsically. This means that we shall investigate how surface curved through the variation of tangent vector fields, the covariant derivative of vector fields. From which, we shall see that the geodesic curvature, the Gaussian curvature are intrinsic. The contents include

- Covariant derivative for vector fields in $\mathbb{R}^2$ with curvilinear coordinate systems
- Frame in $\mathbb{R}^3$
- Covariant derivatives for vector fields in $\mathbb{R}^3$ with frame
- Frame on surface
- Covariant derivatives for vector fields on a surface
- Gauss-Bonnet Theorem

4.3.1 Frame and covariant derivatives in Euclidean spaces

Covariant derivative in curvilinear coordinate systems Let us take the polar coordinate as an example to illustrate the concept of covariant derivatives of vector fields in $\mathbb{R}^2$ ion curvilinear coordinate system.
Suppose we have a vector field V in $\mathbb{R}^2$ with polar coordinate system. We express

$$V = V^r e_r + V^\theta e_\theta,$$

where the unit tangent vectors

$$e_r = (\cos\theta, \sin\theta) = \frac{\partial}{\partial r}, \quad e_\theta = (-\sin\theta, \cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}.$$

The covariant derivatives for vector field V in the directions of e_r and e_θ are defined to be

$$\begin{aligned}
\nabla_{e_r} V &:= \frac{\partial V}{\partial r} = \frac{\partial V^r}{\partial r} e_r + \frac{\partial V^\theta}{\partial r} e_\theta + V^r \frac{\partial e_r}{\partial r} + V^\theta \frac{\partial e_\theta}{\partial r} \\
&= \frac{\partial V^r}{\partial r} e_r + \frac{\partial V^\theta}{\partial r} e_\theta, \\
\nabla_{e_\theta} V &:= \frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{r}\left(\frac{\partial V^r}{\partial \theta} e_r + \frac{\partial V^\theta}{\partial \theta} e_\theta + V^r \frac{\partial e_r}{\partial \theta} + V^\theta \frac{\partial e_\theta}{\partial \theta}\right) \\
&= (\nabla_{e_\theta} V^r) e_r + (\nabla_{e_\theta} V^\theta) e_\theta + V^r \nabla_{e_\theta} e_r + V^\theta \nabla_{e_\theta} e_\theta.
\end{aligned}$$

For general vector field X on $\mathbb{R}^2$, X can be expressed as

$$X = X^e e_r + X^\theta e_\theta.$$

The covariant derivative of V in the direction of X is defined as

$$\nabla_X V := X^r \nabla_{e_r} V + X^\theta \nabla_{e_\theta} V.$$

The terms $\nabla_{e_\theta} e_r$ and $\nabla_{e_\theta} e_\theta$ measure how e_r and e_θ change in the direction of e_θ. Indeed,

$$\begin{aligned}
&\nabla_{e_r} e_r = 0, \quad \nabla_{e_r} e_\theta = 0 \\
&\nabla_{e_\theta} e_r = \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta, \sin\theta) = \frac{1}{r}(-\sin\theta, \cos\theta) = \frac{1}{r} e_\theta, \\
&\nabla_{e_\theta} e_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}(-\sin\theta, \cos\theta) = -\frac{1}{r}(\cos\theta, \sin\theta) = -\frac{1}{r} e_r.
\end{aligned}$$

Covariant Derivative in Euclidean space

- **Covariant Derivative for vector fields in** $\mathbb{R}^3$ Let V and Y be vector fields in $\mathbb{R}^3$. Define the covariant derivative of Y at $p \in \mathbb{R}^3$ in the direction of V by

$$\nabla_V Y := \lim_{\Delta t \to 0} \frac{Y(p + \Delta t V(p)) - Y(p)}{\Delta t}. \tag{4.7}$$

Note that $\nabla_V Y(p)$ depends on $V(p)$ and Y in a neighborhood of p. Let $\boldsymbol{\gamma}$ be the curve with $\boldsymbol{\gamma}(0) = p$ and $\dot{\boldsymbol{\gamma}}(0) = V(p)$. Then

$$\nabla_V Y = \frac{d}{dt}|_{t=0} Y(\boldsymbol{\gamma}(t)). \tag{4.8}$$

In other words, the covariant derivative $\nabla_V Y$ depends on $V(p)$ and $Y(q)$ with $q \in \boldsymbol{\gamma}$ in a neighborhood of p.

- **Covariant derivative for scalar field f**

$$\nabla_V f := Vf(p) := \lim_{\Delta t \to 0} \frac{f(p + tV(p)) - f(p)}{\Delta t}.$$

- **Properties of covariant derivative.**

Proposition 4.1. *Let V, W, Y, Z be vector fields, $a, b \in \mathbb{R}$, f, g scalar field. The covariant derivative ∇_V satisfies*

(a) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$

(b) $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_V Y$

(c) $\nabla_V(fY) = (\nabla_V f)Y + f\nabla_V Y$

(d) $\nabla_V(Y \cdot Z) = (\nabla_V Y) \cdot Z + Y \cdot (\nabla_V Z)$.

Frame and Dual frame

- **Frame** A frame $\{e_1, e_2, e_3\}$ in $\Omega \subset \mathbb{R}^3$ is a triple of three orthonormal smooth vector fields defined on Ω.
- **Dual Frame** Let $\{\theta^1, \theta^2, \theta^3\}$ be the 1-form dual to the frame $\{e_1, e_2, e_3\}$. That is, $\theta^i(e_j) = \delta^i_j$.
- **Example** The domain $\Omega = \mathbb{R}^3 \setminus \{0\}$ with spherical coordinate system

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\cos\theta\sin\phi \\ z = r\sin\theta \end{cases} \quad 0 \le \phi < 2\pi, -\frac{\pi}{2} < \theta \le \frac{\pi}{2}.$$

The spherical frame (e_r, e_ϕ, e_θ) are

$$e_r = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) = \frac{\partial}{\partial r}$$
$$e_\phi = (-\sin\phi, \cos\phi, 0) = \frac{1}{r\cos\theta}\frac{\partial}{\partial\phi}$$
$$e_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}$$

- **Connection** Given a frame $\{e_1, e_2, e_3\}$, we define the *connection* ω_{ij} to be the 1-form
$$\omega_{ij}(V) := (\nabla_V e_i) \cdot e_j.$$
That is,
$$\nabla_V e_i = \sum_{j=1}^{3} \omega_{ij}(V) e_j.$$
Thus, $\nabla_V e_i$ measure the variation of e_i in direction V, and $\omega_{ij}(V)$ is the projection on e_j.

- Since $e_i \cdot e_j = \delta_{ij}$, we take covariant differentiation on this formula to get
$$(\nabla_V e_i) \cdot e_j + e_i \cdot (\nabla_V e_j) = 0.$$
That is
$$\omega_{ij} = -\omega_{ji}.$$
Thus, we get frame equation
$$\nabla_V \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12}(V) & \omega_{13}(V) \\ -\omega_{12}(V) & 0 & \omega_{23}(V) \\ -\omega_{13}(V) & -\omega_{23}(V) & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

- **Representation of frame and connection in Euclidean coordinate.** Let $\{\partial_{x^i}\}_{i=1}^3$ be the Euclidean basis and $\{dx^i\}_{i=1}^3$ be their dual. We express
$$e_i = \sum_j a_{ij}(x) \partial_{x^j}.$$
Then we have
$$\begin{aligned} A^T A &= A A^T = Id \\ \theta^i &= \sum_k a_{ik} dx^k \\ \omega_{ij} &= (da_{ik}) a_{jk}. \end{aligned}$$

Proof. 1. Since both $\{\partial_{x_i}\}$ and $\{e_i\}$ are orthonormal, we have $A^T A = A A^T = Id$ and $e_i = \sum_j a_{ij}(x) \partial_{x_j}$.

2. Let V be a vector in $\mathbb{R}^3$. We have
$$\begin{aligned} \omega_{ij}(V) = (\nabla_V e_i) \cdot e_j &= \left(\nabla_V \sum_k a_{ik} \partial_{x_k} \right) \cdot (\sum_\ell a_{j\ell} \partial_{x_\ell}) \\ &= \sum_k (\nabla_V a_{ik}) \partial_{x_k} \cdot (\sum_\ell a_{j\ell} \partial_{x_\ell}) = \sum_k \sum_\ell (\nabla_V a_{ik}) a_{j\ell} \delta_{k\ell} \\ &= \sum_k (\nabla_V a_{ik}) a_{jk} = \sum_k (da_{ik}(V)) a_{jk}. \end{aligned}$$

□

Homework Compute $(\nabla_{e_k} e_i) \cdot e_j$ for spherical frame.

Structure Equations

Theorem 4.1 (Cartan's structure equations)**.** *Let* $\{e_1, e_2, e_3\}$ *be a frame in* $\Omega \subset \mathbb{R}^3$ *and* $\{\theta_1, \theta_2, \theta_3\}$ *be its dual frame, and* $\{\omega_{ij}\}$ *be the corresponding connection* 1*-form. Then, they satisfy the following structure equation:*

(a) First structure equation

$$\boxed{d\theta_i = \sum_j \omega_{ij} \wedge \theta_j}$$

(b) Second structure equation

$$\boxed{d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}}$$

Proof. 1. (a)

$$\begin{aligned} d\theta_i &= d\left(\sum_k a_{ik} dx^k\right) = \sum_k da_{ik} \wedge dx^k \\ &= \sum_k da_{ik} \sum_j (A^{-1})_{kj} \theta_j = \sum_j \sum_k da_{ik} a_{jk} \theta_j \\ &= \sum_j \omega_{ij} \wedge \theta_j \end{aligned}$$

2. (b)

$$\begin{aligned} d\omega_{ij} &= d\left(\sum_k da_{ik} a_{jk}\right) = -\sum_k da_{ik} \wedge da_{jk} \\ &= -\sum_k da_{ik} \sum_\ell \delta_{k\ell} \wedge da_{j\ell} \\ &= -\sum_k \sum_\ell da_{ik} \sum_m a_{mk} a_{m\ell} \wedge da_{j\ell} \\ &= -\sum_m \omega_{im} \wedge \omega_{jm} = \sum_k \omega_{ik} \wedge \omega_{kj} \end{aligned}$$

□

Remarks

1. $d\theta_i(e_k, e_\ell)$ measures the difference of changes of e_i along e_k and e_ℓ.

$$\begin{aligned} d\theta_i(e_k, e_\ell) &= \sum_j \omega_{ij}(e_k)\theta_j(e_\ell) - \omega_{ij}(e_\ell)\theta_j(e_k) \\ &= \omega_{i\ell}(e_k) - \omega_{ik}(e_\ell) \end{aligned}$$

2. The second structure equation means that the differential of ω can be expressed in terms of ω again. This means there is no need to go higher differentiation. The geometry of the frame is completely determined by $d\theta$ and $d\omega$. Later, we will see that surface curvature is in $d\omega$. We don't need to go higher derivatives of curvature to determine the surface.

3. *The frame is completely determined by the connection* 1*-form* $\{\omega_{ij}\}$ *through the first structure equation. However, the connection* $\{\omega_{ij}\}$ *should satisfy the compatibility condition, which is the second structure.*

4.3.2 Frame on a surface

Let M be a surface in $\mathbb{R}^3$.

- A frame on M is a frame $\{e_1, e_2, e_3\}$ defined on M with e_3 being the normal and e_1, e_2 the tangents of M.

- There exists a frame in $D \subset M$ if and only if D is orientable and there exists a non-vanishing tangent vector field on D.

- Given a vector field Y defined on M and a tangent vector $V \in T_p(M)$, we can find a curve $\boldsymbol{\gamma}$ on M with $\boldsymbol{\gamma}(0) = p$ and $\dot{\boldsymbol{\gamma}}(0) = V(p)$. Then the covariant derivative

$$\nabla_V Y = \frac{d}{dt}|_{t=0} Y(\boldsymbol{\gamma}(t)).$$

  is the same as (4.8).

- Suppose $\{e_1, e_2, e_3\}$ is a frame on M. Let V be a tangent vector field on M. The connection ω_{ij} is a 1-form on M defined by

$$\omega_{ij}(V) = (\nabla_V e_i) \cdot e_j$$

  We have

$$\omega_{ij} = -\omega_{ji}$$

  and for $V \in T_pM$,

$$\nabla_V \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} & \omega_{12}(V) & \omega_{13}(V) \\ -\omega_{12}(V) & & \omega_{23}(V) \\ -\omega_{13}(V) & -\omega_{23}(V) & \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

- Let $\{\theta_1, \theta_2, \theta_3\}$ be the dual of $\{e_1, e_2, e_3\}$ on M. This means that for any $V = \sum_{i=1}^3 v_i e_i$, $\theta_j(V) = v_j$ for $j = 1, 2, 3$. This can be defined only for points on M. In particular, for $p \in M$, we have
  (a) $\theta_3(V) = 0$ for any $V \in T_pM$ and $\theta_3(e_3) = 1$
  (b) $\theta_1 \wedge \theta_2$ is the area form of M.
- The Cartan's structure formula is still valid
$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j, \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$$
on M. In particular, if we restrict θ_j on T_pM, then
$$\theta_3 = 0.$$
Cartan's structure formulae become
$$\begin{cases} d\theta_1 = \omega_{12} \wedge \theta_2, \quad d\theta_2 = -\omega_{12} \wedge \theta_1 \\ d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0. \end{cases} \tag{4.9}$$
$$\begin{cases} d\omega_{12} = \omega_{13} \wedge \omega_{32} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \\ d\omega_{13} = \omega_{12} \wedge \omega_{23}. \end{cases} \tag{4.10}$$
- The shape operator $S : T_pM \to T_pM$ is defined to be
$$SV := -\nabla_V e_3 = \omega_{13}(V)e_1 + \omega_{23}(V)e_2.$$
The matrix representation of S in the basis $\{e_1, e_2\}$ is
$$S = \begin{bmatrix} \omega_{13}(e_1) & \omega_{13}(e_2) \\ \omega_{23}(e_1) & \omega_{23}(e_2) \end{bmatrix}$$
This matrix is symmetric due to the following reason. From (4.9), we get
$$0 = \omega_{31} \wedge \theta_1(e_1, e_2) + \omega_{32} \wedge \theta_2(e_1, e_2) = -\omega_{31}(e_2) + \omega_{32}(e_1).$$
From antisymmetry of ω, we thus obtain
$$\omega_{13}(e_2) = \omega_{23}(e_1).$$
As a byproduct, this also shows that $\nabla_{e_2} e_1 - \nabla_{e_1} e_2 \in T_pM$.
- The mean curvature and Gaussian curvature are defined to be
$$H = \frac{1}{2} Tr(S), \quad K = \det(S).$$
We have

(a) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$

(b) $\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2$

(c) $d\omega_{12} = -K\theta_1 \wedge \theta_2$.

Proof. 1. From

$$2H = Tr(S) = \omega_{13}(e_1) + \omega_{23}(e_2)$$

and

$$(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23})(e_1, e_2) = \omega_{13}(e_1) + \omega_{23}(e_2)$$

we get

$$\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2.$$

2. Note that

$$\begin{aligned} K = det(S) &= \omega_{13}(e_1)\omega_{23}(e_2) - \omega_{13}(e_2)\omega_{23}(e_1) \\ &= \omega_{13} \wedge \omega_{23}(e_1, e_2) \end{aligned}$$

Thus,

$$\omega_{13} \wedge \omega_{23} = \omega_{13} \wedge \omega_{23}(e_1, e_2)\, \theta_1 \wedge \theta_2 = K\theta_1 \wedge \theta_2.$$

Further, from structure equation

$$d\omega_{12} = \omega_{13} \wedge \omega_{32},$$

we get (c).

□

Remark. Note that ω_{12} depends only on e_1, e_2. The formula

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

means that K *is an intrinsic quantity of* M*. It only depends on the metric of* M*, which implies that* K *is invariant under isometric deformation of* M*. This is called Gauss's Theorema Egregium (Gauss's remarkable theorem).*

- The eigenvalue/eigenvector of S:

$$S\hat{e}_i = \kappa_i \hat{e}_i, \quad i = 1, 2$$

Let $\hat{\omega}_{ij} = \nabla_{\hat{e}_i} \hat{e}_j$. Then the representation of S in $\{\hat{e}_i\}$ is

$$S = \begin{bmatrix} \hat{\omega}_{13}(\hat{e}_1) & \hat{\omega}_{13}(\hat{e}_2) \\ \hat{\omega}_{23}(\hat{e}_1) & \hat{\omega}_{23}(\hat{e}_2) \end{bmatrix} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$$

  – The equality $\hat{\omega}_{23}(\hat{e}_1) = 0$ means that $\hat{e}_2$ parallel transports along e_1.

- Similarly, $\hat{\omega}_{13}(\hat{e}_2) = 0$ means that $\hat{e}_1$ parallel transports along e_2.
- The quantity $\hat{\omega}_{13}(\hat{e}_1)$ measures how $\hat{e}_1$ varies along $\hat{e}_1$, or equivalently, how e_3 varies along $\hat{e}_1$. Along integral curve of $\hat{e}_1$, e_3 is the normal and $\hat{e}_2$ is the binormal.

• We have

$$\hat{\omega}_{13} = \sum_k \hat{\omega}_{13}(\hat{e}_k)\hat{\theta}_k = \kappa_1\hat{\theta}_1$$

$$\hat{\omega}_{23} = \sum_k \hat{\omega}_{23}(\hat{e}_k)\hat{\theta}_k = \kappa_2\hat{\theta}_2.$$

Thus, the structure equations are

$$\hat{\omega}_{13} \wedge \hat{\theta}_2 + \hat{\theta}_1 \wedge \hat{\omega}_{23} = 2H\hat{\theta}_1 \wedge \hat{\theta}_2.$$

$$d\hat{\omega}_{12} = \hat{\omega}_{13} \wedge \hat{\omega}_{32} = -\kappa_1\kappa_2\hat{\theta}_1 \wedge \hat{\theta}_2 = -K\hat{\theta}_1 \wedge \hat{\theta}_2.$$

Theorem 4.2 (Local Gauss-Bonnet Theorem). *Let U be a disk-like domain on a 2-manifold M with smooth boundary $\boldsymbol{\gamma}$. Then*

$$\int_{\boldsymbol{\gamma}} \kappa_g\, ds = 2\pi - \int_U K\sigma.$$

where K is the Gauss curvature and σ is the area form of M.

Proof. Let e_1, e_2, e_3 be an orthogonal frame on M with e_3 being the normal of M. We can express $\boldsymbol{\gamma}'$ as

$$\boldsymbol{\gamma}' = \cos\varphi e_1 + \sin\varphi e_2.$$

$$\boldsymbol{\gamma}'' = \varphi'(-\sin\varphi e_1 + \cos\varphi e_2) + (\cos\varphi e_1' + \sin\varphi e_2')$$

$$N \times T = -\sin\varphi e_1 + \cos\varphi e_2.$$

The geodesic curvature is

$$\kappa_g := \langle \boldsymbol{\gamma}'', N \times T\rangle = \varphi' + \langle(\cos\varphi e_1' + \sin\varphi e_2'), -\sin\varphi e_1 + \cos\varphi e_2\rangle$$

From $e_i \cdot e_i = 1$ and $e_1 \cdot e_2 = 0$, we get $e_i e_i' = 0$ and $e_1 \cdot e_2' + e_1' \cdot e_2 = 0$. Thus

$$\kappa_g = \varphi' - e_1 \cdot e_2'. \tag{4.11}$$

If $\boldsymbol{\gamma}$ is the boundary of a disk-like region U, then

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \kappa_g\, ds &= \int_{\boldsymbol{\gamma}} \varphi' ds - \int_{\boldsymbol{\gamma}} e_1 \cdot e_2'\, ds \\ &= 2\pi - \int_{\boldsymbol{\gamma}} \nabla_{\boldsymbol{\gamma}'} e_2 \cdot e_1\, ds\end{aligned}$$

$$
\begin{aligned}
&= 2\pi + \int_\gamma \omega_{12}(\boldsymbol{\gamma}') \\
&= 2\pi + \int_U d\omega_{12} \\
&= 2\pi - \int_U K\sigma
\end{aligned}
$$

Here, σ is the surface area form of M and K is the Gaussian curvature. □

Remark Formula (4.11) only involves tangent vector fields, it means that the geodesic curvature κ_g is an intrinsic quantity.

Homework Consider the orthonormal frame (e_ϕ, e_θ, e_r) on the unit sphere with the spherical coordinate, where $0 \le \phi < 2\pi$, $\pi/2 \le -\theta \le \pi/2$.

1. Derive the formulae of the shape operator S.
2. Consider the following path γ: starts from north pole $(0,0,1)$, then follows $\phi = 0$, $\pi/4 \le \theta \le \pi/2$, then $\theta = \pi/4$, $0 < \phi < \pi/2$, then go back to the north pole along $\phi = \pi/2$, $\pi/4 < \theta < \pi/2$. Find the κ_g along this curve and compute the holonomy $\int_\gamma \kappa_g \, ds$.
3. Can you find the area enclosed by γ without using Gauss-Bonnet theorem?

Chapter 5

Exterior Algebra and Calculus

In this chapter, we shall develop theory of exterior calculus which includes differential forms and exterior derivatives. It is a calculus independent of the coordinate system we set up to investigate the underlying geometry. Coordinate system is a representation for calculation. But geometry and physics should not depend on the coordinate we choose. Below, we develop the exterior algebra for vector spaces and inner product spaces, then on manifolds.
There are nice youtube lectures on *geometric algebra* and *Clifford algebra*

- A swift introduction to geometric algebra `https://www.youtube.com/watch?v=60z_hpEAtD8`
- Introduction to geometric (Clifford) algebra `https://www.youtube.com/watch?v=mz3tk4LRJjc`

5.1 Exterior Algebra for Vector Space

5.1.1 Vector space and Dual Space

n-dimensional vector space

- **Vector space** A vector space V over $\mathbb{R}$ is a set V with two operations: vector addition and scalar multiplication. They satisfy: (1) $(u+v)+w = u+(v+w)$; (2) $v+w = w+v$; (3) $\exists 0 \in V$ such that $v + 0 = 0 + v$ for all $v \in V$; (4) for any $v \in V$, $\exists(-v) \in V$ such that $v + (-v) = 0$; (5) for any $a, b \in \mathbb{R}$, any $v, w \in V$, it holds $(a + b)v = av + bv$, $a(v + w) = av + aw$, $(ab)v = a(bv)$; (6) $1v = v$.

- **Dimension** Let V be vector space over $\mathbb{R}$. It is called an n dimensional space if it can be spanned by n independent elements $\{e_1, ..., e_n\}$. Such a set is called a basis of V. Any two bases of V contains same number of elements. (why?) This number is called the dimension of V. Any vector $v \in V$ can be represented uniquely as

$$v = \sum_{i=1}^{n} v^i e_i := v^i e_i.$$

Here, we will use upper index for the coefficient v^i and lower index for the basis e_i. We use Einstein's notation: whenever same upper index and lower index appear in pair, it means that this is a summation over that index.

- **Dual space** A linear functional on V is a linear function $\alpha : V \to \mathbb{R}$. The dual space of V is defined as
$$V^* := \{\alpha : V \to \mathbb{R} \text{ linear}\}.$$
A linear functional is uniquely determined by its values on a basis $\{e_1, ..., e_n\}$. Let $e^i \in V^*$ with
$$e^i(e_j) = \delta^i_j := \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$
Then $\{e^1, ..., e^n\}$ are independent. For any $\alpha \in V^*$, it can be represented uniquely as
$$\alpha = \alpha_i e^i, \qquad \text{where } \alpha_i := \alpha(e_i).$$
Thus, $dim(V^*) = dim(V)$. The basis $\{e^1, ..., e^n\}$ is called the dual basis corresponding to $\{e_1, ..., e_n\}$. An element of V is called a vector, while an element of V^* is called a co-vector.

- $\boldsymbol{V^{**} = V}$ Any vector $v \in V$ can be viewed as an element of V^{**} by $v(\alpha) := \alpha(v)$ for any $\alpha \in V^*$. Thus, we have $V \subset V^{**}$. Since $dim(V^{**}) = dim(V^*) = dim(V)$, we get $V^{**} = V$.

- Remark. Sometimes, we express $\alpha(v)$ by $\langle\alpha|v\rangle$.

5.1.2 Tensor spaces

- Tensor product of two vectors: Let U, V be two vector spaces over $\mathbb{R}$. Let $u \in U$ band $v \in V$. The tensor product of them, denoted by $u \otimes v$ is defined as an operation satisfying linearity in both u and v, associativity, but no commutativity. The tensor product of U and V is defined as
$$U \otimes V := \text{Span}\{u \otimes v | u \in U, v \in V\}$$
An element in $U \otimes V$ is called a tensor.

- If $\{u_1, \cdots, u_m\}$ and $\{v_1, \cdots, v_n\}$ are bases of U and V respectively, then $\{u_i \otimes v_j | i = 1, ..., m, j = 1, ..., n\}$ constitutes a basis of $U \otimes V$. Thus, $dim(U \otimes V) = dim(U) \cdot dim(V)$.

- A tensor T in
$$\underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}}$$
is called a tensor in V of type (r, s).

- A tensor on V can also be viewed as a multi-linear function on V. For instance, a bilinear function on V is a function

$$A : V \times V \to \mathbb{R},$$

which is linear in both arguments. We denote the set of all bilinear linear functions on V by $\mathcal{L}(V, V)$. We claim that

$$V^* \otimes V^* = \mathcal{L}(V, V).$$

To check this claim, we first see that a tensor $T \in V^* \otimes V^*$ is a bilinear function on V. If $T = \alpha \otimes \beta$ with $\alpha, \beta \in V^*$, we define

$$\alpha \otimes \beta(u, v) := \alpha(u)\beta(v)$$

for vectors $u, v \in V$. If $T = \sum_{i=1}^{N} a_i \alpha_i \otimes \beta_i$, we define

$$T(u, v) := \sum_{i=1}^{N} a_i \alpha_i(u) \beta_i(v).$$

Thus, a tensor $T \in V^* \otimes V^*$ is a bilinear function on V. Conversely, let $\{e_1, ..., e_n\}$ and $\{e^1, ..., e^n\}$ are a pair of dual bases in V and V^* with $e^i(e_j) = \delta^i_j$. For any bilinear function A, we can define a tensor T_A as

$$T_A := \sum_{i=1}^{n} \sum_{j=1}^{n} A(e_i, e_j) e^i \otimes e^j.$$

Then one can check that

$$T_A(u, v) = A(u, v)$$

for any $u, v \in V$. The mapping: $A \mapsto T_A$ is linear. The two spaces $\mathcal{L}(V, V)$ and $V^* \otimes V^*$ have the same dimension. Thus, $\mathcal{L}(V, V) = V^* \otimes V^*$.

- In general, let

$$\mathcal{L}(\underbrace{V^*, \cdots, V^*}_{r \text{ times}}, \underbrace{V, \cdots, V}_{s \text{ times}})$$

be the space of all multilinear functions from $V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}$. Then

$$\mathcal{L}(\underbrace{V^*, \cdots, V^*}_{r \text{ times}}, \underbrace{V, \cdots, V}_{s \text{ times}}) = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}}.$$

5.1.3 The Exterior Algebra for Co-vectors

The k-form is a multi-linear functional on a n-dimensional vector space V. It is to measure k-dimensional volume of a parallelepiped in an n-dimensional space V.
Let $\{e_1, ..., e_n\}$ be a basis of V and $\{e^1, ..., e^n\}$ be its dual basis in V^* with $e^i(e_j) = \delta^i_j$.

- An 1-form is a linear functional on V. If α is a 1-form, then it can be represented as
$$\alpha = \alpha_i e^i.$$
For any $v \in V$ with $v = v^i e_i$, then
$$\alpha(v) = \alpha_i v^i.$$
If $\alpha = e^i$, then $e^i(v) = v^i$ is the length of the projection of v onto e^i. In mechanics, α represents force and $\alpha(v)$ is the projection of force in the direction of v.

- **2-form** A 2-form measures two-dimensional signed area of a parallelogram spanned by the projection of two vector v, w through two linear functional $\alpha, \beta \in V^*$. We define the wedge product of α and β to be a bilinear form on V by
$$\alpha \wedge \beta(v, w) := \begin{vmatrix} \alpha(v) & \beta(v) \\ \alpha(w) & \beta(w) \end{vmatrix}.$$
Let us take $V = \mathbb{R}^3$ as an example. We measure the signed area of the projection of the parallelogram onto the plane spanned by e_1, e_2, which is
$$e^1 \wedge e^2(v, w) := \begin{vmatrix} e^1(v) & e^2(v) \\ e^1(w) & e^2(w) \end{vmatrix} = \begin{vmatrix} v^1 & v^2 \\ w^1 & w^2 \end{vmatrix}.$$
From this definition, we find
$$e^1 \wedge e^2(w, v) = \begin{vmatrix} w^1 & w^2 \\ v^1 & v^2 \end{vmatrix} = -e^1 \wedge e^2(v, w),$$
$$e^2 \wedge e^1(v, w) = \begin{vmatrix} v^2 & v^1 \\ w^2 & w^1 \end{vmatrix} = -e^1 \wedge e^2(v, w).$$
We can also project the parallelogram v, w to e_2-e_3 plane by $e^2 \wedge e^3$, and to e_3-e_1 plane by $e^3 \wedge e^1$. The object $\alpha \wedge \beta$ with $\alpha, \beta \in V^*$ is called a 2-blade. You can check that $e^i \wedge e^j = 0$ if $i = j$ and $e^i \wedge e^j = -e^j \wedge e^i$. Thus, in $\mathbb{R}^3$, the only nontrivial 2-blades spanned by basis $\{e^1, e^2, e^3\}$ are $e^1 \wedge e^2$, $e^2 \wedge e^3$ and $e^3 \wedge e^1$. A general 2-form in $\mathbb{R}^3$ is a linear combination of these three basic 2-blades:
$$\omega = \omega_{12} e^1 \wedge e^2 + \omega_{23} e^2 \wedge e^3 + \omega_{31} e^3 \wedge e^1, \quad \omega_{12}, \omega_{23}, \omega_{31} \in \mathbb{R}.$$
Thus, the set of all 2-forms is the vector space spanned by $e^1 \wedge e^2$, $e^2 \wedge e^3$ and $e^3 \wedge e^1$. We denote is by $\Lambda^2(V, \mathbb{R})^*$.

Remark In $\mathbb{R}^3$, the 2-form ω measures flux. $\omega(v, w)$ is the ω-flux through the parallelogram spanned by v and w.

- **2-form expressed in terms of tensor product** For $\alpha, \beta \in V^*$, we can define their tensor product $\alpha \otimes \beta$ as a bilinear form by

$$\alpha \otimes \beta(v, w) := \alpha(v)\beta(w) \text{ for } v, w \in V.$$

The wedge product $\alpha \wedge \beta$ can also be defined to be

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha.$$

Then for any two vectors $v, w \in V$,

$$\begin{aligned} \alpha \wedge \beta(v, w) &= \alpha \otimes \beta(v, w) - \beta \otimes \alpha(v, w) \\ &= \alpha(v)\beta(w) - \beta(v)\alpha(w) \end{aligned}$$

which is the same as our earlier definition for $\alpha \wedge \beta$.

Determinant We recall the definition of determinant $\det(a_i^j)_{n\times n}$. Let $A = (a^1, ..., a^n)$, where $a^j = (a_i^j)_{n\times 1}$ are column vectors.

$$\det(A) = \det(a^1, ..., a^n) := \sum_\sigma \text{sign}(\sigma) a_1^{\sigma(1)} \cdots a_n^{\sigma(n)}$$

where $\sigma : \{1, ..., n\} \to \{1, ..., n\}$. The sign for $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ is defined as

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the determinant has the following properties.

1. $\det(A) = \sum_\sigma \text{sign}(\sigma) a_1^{\sigma(1)} \cdots a_n^{\sigma(n)} = \sum_\sigma \text{sign}(\sigma) a_{\sigma(1)}^1 \cdots a_{\sigma(n)}^n$.
   That is, $\det(A) = \det(A^T)$.

2. $\det(a^1, ..., a^n) = 0$ if $a^i = a^j$ for some $i \neq j$.

3. $\det(A) = \frac{1}{n!} \sum_{\sigma,\tau} \text{sign}(\sigma)\text{sign}(\tau) a_{\tau(1)}^{\sigma(1)} \cdots a_{\tau(n)}^{\sigma(n)}$.

4. Rank reduction property: Let A_i^j be the determinant of the $(n-1) \times (n-1)$ submatrix by eliminating ith row and jth column from A. Then for each j

$$\det(A) = \sum_i (-1)^i a_i^j A_i^j.$$

The proof for this reduction is by decomposing a permutation $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ into

$$\sigma = (i \mapsto j) \left(\sigma_i' : \{1, ..., \hat{\imath}, ..., n\} \to \{1, ..., \hat{\jmath}, ..., n\} \right)$$

where $\hat{\imath}$ means the term i is eliminated from the set. Note that with fixed j, $\text{sign}(\sigma) = (-1)^{i+j}\text{sign}(\sigma_i')$. We have

$$\det(A) = \sum_i \sum_{\sigma_i'} (-1)^{i+j} \text{sign}(\sigma_i') a_i^j \prod_{k \neq i} a_i^{\sigma_i'(k)} = \sum_i (-1)^{i+j} a_i^j \det(A_i^j).$$

k-blade and k-forms

- Let $\alpha^1, ..., \alpha^k \in V^*$. We define

$$\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, ..., v_k) := det \left(\alpha^j(v_i) \right)_{n \times n}.$$

  The tuple $\alpha_1 \wedge \cdots \wedge \alpha_k$ is called a k-blade.

  Proposition 5.1. *Let $\sigma : \{1, ..., k\} \to \{1, ..., k\}$ be a permutation. We have*

  (a) $\alpha^1 \wedge \cdots \wedge \alpha^k(v_{\sigma(1)}, ...v_{\sigma(k)}) = sign(\sigma)\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, ..., v_k)$.

  (b) $\alpha^{\sigma(1)} \wedge \cdots \wedge \alpha^{\sigma(k)} = sign(\sigma)\alpha^1 \wedge \cdots \wedge \alpha^k$.

- Let $I = \{i_1, ..., i_k\}$ be an index set with $1 \leq i_1 < ... < i_k \leq n$. We shall call it an ordered index set in $\{1, ..., n\}$ with size k. We denote

$$e^I := e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

  We have

$$\begin{aligned} e^I(e_{j_1}, ..., e_{j_k}) &= \begin{vmatrix} \langle e^{i_1}, e_{j_1} \rangle & \cdots & \langle e^{i_1}, e_{j_k} \rangle \\ \vdots & \ddots & \vdots \\ \langle e^{i_k}, e_{j_1} \rangle & \cdots & \langle e^{i_k}, e_{j_k} \rangle \end{vmatrix} \\ &= \delta^{i_1, ..., i_k}_{j_1, ..., j_k} := \delta^I_J, \end{aligned}$$

  where

$$\delta^I_J := \delta^{i_1, ..., i_k}_{j_1, ..., j_k} = \begin{cases} 1 & \text{if } i_1, ..., i_k \text{ are distinct and } J \text{ is an even permutation of } I \\ -1 & \text{if } i_1, ..., i_k \text{ are distinct and } J \text{ is an odd permutation of } I \\ 0 & \text{otherwise.} \end{cases}$$

  The set $\{e^I | I = \{i_1, ..., i_k\}$ and $1 \leq i_1 < ... < i_k \leq n\}$ are independent. There are $\binom{n}{k}$ of them.

Definition 5.1. A k-form on an n-dimensional vector space $(V, \langle \cdot, \cdot \rangle)$ is a multi-linear map

$$\omega : V \times \cdots \times V \to \mathbb{R}$$

which is also alternative in the sense that

$$\omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \text{sign}(\sigma)\omega(v_1, ..., v_k),$$

for any permutation $\sigma : \{1, ..., k\} \to \{1, ..., k\}$.

We denote the set of all k-forms on V by $\Lambda^k(V, \mathbb{R})^*$. We see that $\Lambda^1(V, \mathbb{R})^* = V^*$.

Proposition 5.2. *The set* $\{e^I | I = \{i_1, ..., i_k\}$ *and* $1 \le i_1 < ... < i_k \le n\}$ *forms a basis of* $\Lambda^k(V, \mathbb{R})^*$. *Thus, dim*$(\Lambda^k(V, \mathbb{R})^*) = \binom{n}{k}$.

Proof.

1. $\{e^I | I = \{i_1, ..., i_k\}$ and $1 \le i_1 < ... < i_k \le n\}$ are independent. If $a_I e^I = 0$, then for any $J = (j_1, ..., j_k)$,

$$0 = a_I e^I(e_{j_1}, ..., e_{j_k}) = a_I \delta^I_J = a_J.$$

   Thus, e^I are independent.

2. For any $\omega \in \Lambda^k(V, \mathbb{R})^*$, let

$$\omega_I := \omega(e_{i_1}, ..., e_{i_k}).$$

   You can show that (check)

$$\omega = \omega_I e^I.$$

□

Wedge product We have defined wedge product for two 1-forms. We have also defined k-blades. Now, we will define wedge product for general k-form and l-form as the follows.

Definition 5.2. (a) Let I and J be two ordered subindex sets of $\{1, ..., n\}$. We define

$$e^I \wedge e^J := \begin{cases} e^K & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset \end{cases}$$

where $K = (I, J)$.

(b) Let $\alpha = \alpha_I e^I$ and $\beta = \beta_J e^J$, define

$$\alpha \wedge \beta = \sum_{IJ} \alpha_I \beta_J e^I \wedge e^J.$$

Proposition 5.3. *Let α and β are k-form and ℓ-form. Then for general vectors $v_1, ..., v_{k+\ell}$, we have*

$$\alpha \wedge \beta(v_1, ..., v_{k+\ell}) = \sum_{\sigma \in Sh(k,\ell)} sign(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})\beta(v_{\sigma(k+1)}, ..., v_{\sigma(k+\ell)}), \tag{5.1}$$

where $Sh(k, \ell)$ is the set of all permutations σ satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$.

This proposition says that the definition of wedge product is independent choice of basis.

Proposition 5.4. *Suppose α, β and γ are k, l, m-forms, respectively. Let $a, b \in \mathbb{R}$. We have*

(a) $(a\alpha + b\beta) \wedge \gamma = a\alpha \wedge \gamma + b\beta \wedge \gamma$.

(b) $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$.

(c) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

Proof. (Sketch) The bilinearity of the wedge product follows from the construction.

For the associativity for the basis, it follows from definition:

$$(e^{i_1} \wedge e^{i_2}) \wedge e^{i_3} = e^{i_1} \wedge (e^{i_2} \wedge e^{i_3}) = e^{i_1} \wedge e^{i_2} \wedge e^{i_3}$$

In general, we need to show

$$(\alpha \wedge \beta) \wedge \gamma(u, v, w) = \alpha \wedge \beta \wedge \gamma(u, v, w) = \begin{vmatrix} \alpha(u) & \beta(u) & \gamma(u) \\ \alpha(v) & \beta(v) & \gamma(v) \\ \alpha(w) & \beta(w) & \gamma(w) \end{vmatrix} \tag{5.2}$$

It can be proved by using the rank reduction property of determinant.

□

Remarks

- The associativity property states that the definition of k-blade $\alpha_1 \wedge \cdots \wedge \alpha_k$ is indeed the wedge product of the 1-forms α_1,...,α_k.

Volume form The dimension of $\Lambda^n(V, \mathbb{R})^*$ is $\binom{n}{n} = 1$. For any $\alpha^1, ..., \alpha^n$ with $\alpha^j = a_i^j e^i$. We have

$$\alpha^1 \wedge \cdots \wedge \alpha^n = \det(a_i^j)_{n\times n} e^1 \wedge \cdots \wedge e^n.$$

The n-blade $\alpha^1 \wedge \cdots \wedge \alpha^n$ is called the volume form generated by $\alpha^1, ..., \alpha^n$.

5.1.4 Interior Product

The interior product contracts a k-form with a vector v to produce a $(k-1)$-form.

Definition 5.3. Let $v \in V$. The interior product i_v is an operator

$$i_v : \Lambda^k V^* \to \Lambda^{k-1} V^*$$

defined by

$$(i_v\alpha)(v_1, ..., v_{k-1}) = \alpha(v, v_1, ..., v_{k-1}). \tag{5.3}$$

Theorem 5.1. *Let $v \in V$. The interior product i_v has the following properties:*

(i) For 1-form α,

$$i_v(\alpha) = \alpha(v).$$

(ii) $i_v \circ i_v = 0$

(iii) Leibniz rule: Let α and β are k-form and ℓ-form respectively. Then

$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta). \tag{5.4}$$

Proof. 1. (ii) can be proven for basis e^I by:

$$e^{i_1} \wedge \cdots \wedge e^{i_k}(v, v, v_1, ..., v_{k-2}) = 0.$$

2. proof of (iii) is left for exercise.

□

Homework

1. Show (5.2).

2. Show (5.1).

3. Show the Leibniz rule for the interior product.

5.1.5 The Exterior Algebra for Vectors

Exterior algebra handles k-dimensional volumes in n dimensional space. Let V be an n-dimensional vector space. Let $\{e_1, ..., e_n\}$ be a basis of V.

k-vectors An 1-vector is just another name of vectors in V. We denote the set of all 1-vectors by $\Lambda^1(V, \mathbb{R})$, which is V.

2-vector Let $\{e_1, ..., e_n\}$ be a basis of V. The 2-vectors $e_i \wedge e_j$ represents a signed area of the parallelogram spanned by e_i and e_j. We define $e_j \wedge e_i = -e_i \wedge e_j$. This implies $e_i \wedge e_i = 0$. With these $e_i \wedge e_j$, $1 \leq i < j \leq n$, we define the set of 2-vectors $\Lambda^2(V, R)$ to be the linear span of all such $e_i \wedge e_j$. Its dimension is $\binom{n}{2}$.

Example Suppose $V = \mathbb{R}^3$. Let $u = e_1 + e_2$ and $v = e_2 - e_3$. The wedge product $u \wedge v$ can be viewed as the parallelogram spanned by u and v. It also has the following expression

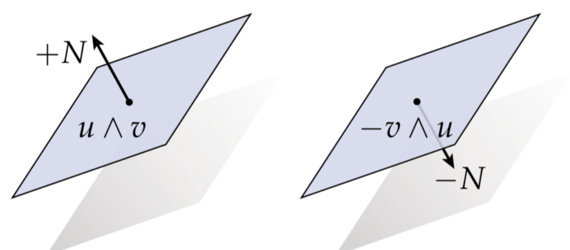
$$u \wedge v = (e_1 + e_2) \wedge (e_2 - e_3) = e_1 \wedge e_2 - e_2 \wedge e_3 + e_3 \wedge e_1.$$

We can also take the cross product of u and v to get

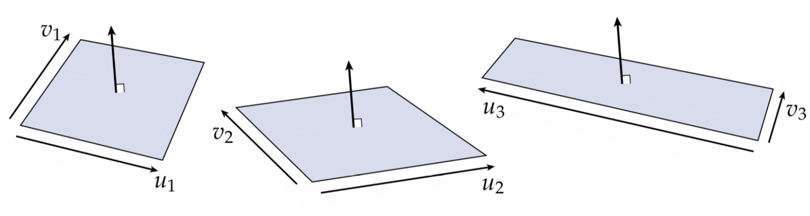
$$u \times v = -\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We find a similarity between $u \wedge v$ and $u \times v$ through the correspondence

$$e_2 \wedge e_3 \leftrightarrow \mathbf{i}, \quad e_3 \wedge e_1 \leftrightarrow \mathbf{j}, \quad e_1 \wedge e_2 \leftrightarrow \mathbf{k}.$$



3-vector We first define $e_{i_1} \wedge e_{i_2} \wedge e_{i_3}$ as an volume element spanned by the vector $e_{i_1}, e_{i_2}, e_{i_3}$. Let $\Lambda^3(V, \mathbb{R})$ be the linear space spanned by these basic 3-vectors. The space of $\Lambda^3(V, \mathbb{R})$ has dimension $\binom{n}{3}$.



$u_1 \wedge v_1 = u_2 \wedge v_2 = u_3 \wedge v_3$

Example Suppose $V = \mathbb{R}^3$. Let us consider

$$e_1 \wedge e_2 \wedge (e_3 + e_1) = e_1 \wedge e_2 \wedge e_3.$$

This is a (sheared) parallelepiped. Indeed, the wedge product of any three vectors in $\mathbb{R}^3$ has the form

$$v_1 \wedge v_2 \wedge v_3 = det(v_1, v_2, v_3) e_1 \wedge e_2 \wedge e_3.$$

The dimension of $\Lambda^3(\mathbb{R}^3) = 1$.

Wedge product The wedge product $\wedge$ applies to any k-vector u and l-vector v. It satisfies

- Anti-symmetry $u \wedge v = (-1)^{kl} v \wedge u$
- Bilinearity: $(au + bv) \wedge w = au \wedge w + bv \wedge w$
- Associativity: $(u \wedge v) \wedge w = u \wedge (v \wedge w)$.

Remark

- The exterior algebra we define here is through construction. The construction relies on the choice of bases $\{e_1, ..., e_n\}$ and its dual $\{e^1, ..., e^n\}$. However, the vector space operations and the wedge product for the k-vectors and k-covectors are independent of the choice of basis.

5.2 Inner Product and Hodge $\star$

5.2.1 Inner product space and representation of inner product

- A vector space V endowed with an inner product structure $\langle \cdot, \cdot \rangle$ is called an inner product space. Inner product is used to measure length of a vector by $\|v\| := \sqrt{\langle v, v \rangle}$ and the angle $\cos\theta = \langle u, v \rangle / (\|u\| \|v\|)$ between two vectors.
- Suppose $\{e_1, ..., e_n\}$ is a basis of V. Let $g_{ij} := \langle e_i, e_j \rangle$. Then (g_{ij}) is a symmetric positive definite matrix. Moreover, for $v = v^i e_i$ and $w = w^i e_i$, their inner product has the following formula
$$\langle v, w \rangle = g_{ij} v^i w^j.$$
- If $(g_{ij}) = (\delta_{ij})$, then the corresponding basis $\{e_1, ..., e_n\}$ is called an orthonormal basis (ONB).
- Given an n-dimensional inner product space V, we can always find an orthonormal basis (ONB) $\{e_1, ..., e_n\}$. They can be constructed through Gram-Schmidt process.
- Note that the inner product of v, w is independent to the choice of basis.

5.2.2 Inner product in the dual space V^*

- **Music isomorphism** The inner product $\langle\cdot,\cdot\rangle$ in V induces a natural isomorphism between V and V^* called music isomorphism.

  (a) $V \xrightarrow{\flat} V^*$: For any $w \in V$, it induces a natural linear functional $v \mapsto \langle w, v\rangle$. We denote this linear map by $w^\flat$.

  (b) $V^* \xrightarrow{\sharp} V$: For any $\alpha \in V^*$, there exists a unique vector w such that $\alpha(v) = \langle w, v\rangle$ for all $v \in V$. This statement is called the Riesz representation theorem. We denote w by $\alpha^\sharp$.

- **Inner product in V^* induced by** $(V, \langle\cdot,\cdot\rangle)$ For $\alpha, \beta \in V^*$, define
$$\langle \alpha, \beta\rangle := \langle \alpha^\sharp, \beta^\sharp\rangle.$$

- **Inner product representation in basis.** Suppose $\{e_1, .., e_n\}$ is a basis of V with $\langle e_i, e_j\rangle = g_{ij}$. Let $\{e^1, ..., e^n\}$ be its dual basis in V^* with $e^i(e_j) = \delta^i_j$. From definition of the flat map $\flat$,
$$e_i^\flat(e_j) = \langle e_i, e_j\rangle = g_{ij}.$$
That is,
$$e_i^\flat = g_{ij}e^j.$$
Thus, under the basis $\{e_1, ..., e_n\}$ in the domain V and $\{e^1, ..., e^n\}$ in the range V^*, the flat map $\flat$ has the following matrix representation: for any $v \in V$,
$$v^\flat = v^i e_i^\flat = g_{ij}v^i e^j.$$

- The sharp $\sharp$ map from V^* to V is the inverse of flat $\flat$. Let
$$(g^{ij})_{n\times n} = (g_{ij})^{-1}_{n\times n}.$$
It is the matrix representation of the sharp map under the bases $\{e^1, .., e^n\}$ and $\{e_1, ..., e_n\}$. That is,
$$\alpha^\sharp = \alpha_i e^{i\sharp} = g^{ij}\alpha_i e_j.$$

- The induced inner product $\langle\cdot,\cdot\rangle$ in V^* has the following representation:
$$\langle e^i, e^j\rangle = \langle e^{i\sharp}, e^{j\sharp}\rangle = \langle g^{ik}e_k, g^{jl}e_l\rangle = g^{ik}g^{jl}g_{kl} = g^{ij}.$$
Thus,
$$\langle \alpha, \beta\rangle = \langle a_i e^i, b_j e^j\rangle = g^{ij}a_i b_j.$$

- If $\{e_1, ..., e_n\}$ is an ONB of V, then $\{e^i := e_i^\flat | i = 1, ..., n\}$ is an ONB of V^*.

- Suppose $\{e_1, ..., e_n\}$ and $\{e^1, ..., e^n\}$ are ONB in V and V^* with $\langle e_i, e^j\rangle = \delta_i^j$. Then
  - For $v = v^i e_i$, then $v^\flat = v_i e^i$ with $v_i = v^i$.
  - For $\alpha = \alpha_i e^i$, then $\alpha^\sharp = \alpha^i e_i$ with $\alpha^i = \alpha_i$.

5.2.3 Inner product structure for k-vectors and k-covectors

- **Inner product for** 2**-forms** Let $\{e^1, ..., e^n\}$ be a basis of V^* with $\langle e^i, e^j\rangle = \delta^{ij}$. We recall that 2-forms can be expressed in terms of 2-tensors as

$$e^{i_1} \wedge e^{i_2} = e^{i_1} \otimes e^{i_2} - e^{i_2} \otimes e^{i_1}.$$

We define the inner product for 2-tensor by

$$\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2\rangle := \frac{1}{2}\langle \alpha_1, \beta_1\rangle \cdot \langle \alpha_2, \beta_2\rangle.$$

The factor 1/2 is a dimension factor. We can choose any constant. With this definition of inner product for tensors, we can define inner product for 2-forms by

$$\begin{aligned}
\langle e^{i_1} \wedge e^{i_2}, e^{j_1} \wedge e^{j_2}\rangle &= \langle (e^{i_1} \otimes e^{i_2} - e^{i_2} \otimes e^{i_1}), (e^{j_1} \otimes e^{j_2} - e^{j_2} \otimes e^{j_1})\rangle \\
&= \big(\langle e^{i_1} \otimes e^{i_2}, e^{j_1} \otimes e^{j_2}\rangle + \langle e^{i_2} \otimes e^{i_1}, e^{j_2} \otimes e^{j_1}\rangle\big) \\
&- \big(\langle e^{i_1} \otimes e^{i_2}, e^{j_2} \otimes e^{j_1}\rangle + \langle e^{i_2} \otimes e^{i_1}, e^{j_1} \otimes e^{j_2}\rangle\big) \\
&= \big(\delta^{i_1,j_1}\delta^{i_2,j_2} - \delta^{i_1,j_2}\delta^{i_2,j_1}\big) \\
&= \begin{vmatrix} \delta^{i_1,j_1} & \delta^{i_1,j_2} \\ \delta^{i_2,j_1} & \delta^{i_2,j_2} \end{vmatrix} \\
&= \delta^{i_1 i_2}_{j_1 j_2}
\end{aligned}$$

In general, if $\langle e^i, e^j\rangle = g^{ij}$, then

$$\begin{aligned}
\langle e^{i_1} \wedge e^{i_2}, e^{j_1} \wedge e^{j_2}\rangle &= \langle (e^{i_1} \otimes e^{i_2} - e^{i_2} \otimes e^{i_1}), (e^{j_1} \otimes e^{j_2} - e^{j_2} \otimes e^{j_1})\rangle \\
&= \big(g^{i_1,j_1}g^{i_2,j_2} - g^{i_1,j_2}g^{i_2,j_1}\big) \\
&= \begin{vmatrix} g^{i_1,j_1} & g^{i_1,j_2} \\ g^{i_2,j_1} & g^{i_2,j_2} \end{vmatrix}
\end{aligned}$$

- **Inner product for k-forms** Let $I = \{i_1, ..., i_r\}$ and $J = \{j_1, ..., j_r\}$ be two index sets in $\{1, ..., n\}$. Let $e^I = e^{i_1} \wedge \cdots \wedge e^{i_r}$, $e^I = e^{j_1} \wedge \cdots \wedge e^{j_r}$. The k-blade e^I can be expressed in terms of tensor product:

$$e^I = \sum_\sigma \text{sign}(\sigma) e^{i_{\sigma(1)}} \otimes \cdots \otimes e^{i_{\sigma(r)}} := \sum_\sigma \text{sign}(\sigma) e_\otimes^{I_\sigma}$$

The inner product for two k-tensors is defined by

$$\langle \alpha_\otimes^I, \beta_\otimes^J\rangle := \langle \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}, \beta^{j_1} \otimes \cdots \otimes \beta^{j_k}\rangle = \frac{1}{k!}\langle \alpha^{i_1}, \beta^{j_1}\rangle \cdots \langle \alpha^{i_k}, \beta^{j_k}\rangle.$$

This leads to an inner product structure for k-forms:

$$\langle e^I, e^J\rangle = \langle \sum_\sigma \text{sign}(\sigma) e_\otimes^{I_\sigma}, \sum_\tau \text{sign}(\tau) e_\otimes^{J_\tau}\rangle$$

$$
\begin{aligned}
&= k! \sum_{\tau} \mathrm{sign}(\tau) \langle e^{I}_{\otimes}, e^{J_\tau}_{\otimes} \rangle \\
&= \sum_{\tau} \mathrm{sign}(\tau) g^{i_1, j_{\tau(1)}} \cdots g^{i_k, j_{\tau(k)}} \\
&= det\left((g^{ij})_{i \in I, j \in J} \right)_{k \times k} \\
&= g^{IJ}
\end{aligned}
$$

Suppose $\{e_1, ..., e_n\}$ be an ONB of V and $\{e^1, ..., e^n\}$ be the corresponding ONB in V^*. Then $g^{ij} = \delta^{ij}$. We have

$$
\langle e^I, e^J \rangle = \begin{vmatrix} \delta^{i_1, j_1} & \dots & \delta^{i_1, j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k, j_1} & \dots & \delta^{i_k, j_k} \end{vmatrix} = \delta^{IJ} := \delta^{j_1, \dots, j_k}_{i_1, \dots, i_k}
$$

Thus, $\{e^I | I \text{ is ordered index subset of } \{1, ..., n\}, |I| = k\}$ is ONB of $\Lambda^k(V, \mathbb{R})^*$. The inner product of $\omega = \omega(e_I)e^I$ and $\eta = \eta(e_I)e^I$ in $\Lambda^k(V, \mathbb{R})^*$ is

$$
\langle \omega, \eta \rangle := \sum_{I} \omega(e_I) \eta(e_I),
$$

where $\omega(e_I) := \omega(e_{i_1}, ..., e_{i_k})$.

- **General inner product representation for k-vectors**. Suppose $\{e_1, ..., e_n\}$ is a basis in V. Let $g_{ij} = \langle e_i, e_j \rangle$. Let $I = \{i_1, ..., i_k\}$ and $J = \{j_1, ..., j_k\}$ be two ordered index set in $\{1, ..., n\}$. Let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $e_I = e_{j_1} \wedge \cdots \wedge e_{j_k}$. Following the same process for covectors, we have

$$
\begin{aligned}
\langle e_I, e_J \rangle &= \langle \sum_{\sigma} \mathrm{sign}(\sigma) e^{\otimes}_{I_\sigma}, \sum_{\tau} \mathrm{sign}(\tau) e^{\otimes}_{J_\tau} \rangle \\
&= k! \sum_{\tau} \mathrm{sign}(\tau) \langle e^{\otimes}_{I}, e^{\otimes}_{J_\tau} \rangle \\
&= \sum_{\tau} \mathrm{sign}(\tau) g_{i_1, j_{\tau(1)}} \cdots g_{i_k, j_{\tau(k)}} \\
&= det\left((g_{ij})_{i \in I, j \in J} \right)_{k \times k} \\
&:= g_{IJ}
\end{aligned}
$$

Then the inner product of two k-vectors $u = u^I e_I$ and $v = v^I e_I$ is

$$
\langle u, v \rangle = g_{IJ} u^I v^I.
$$

5.2.4 Hodge $\star$ for Vectors and Forms

Unit Volume

- If $\{e_1, ..., e_n\}$ is an ONB of V, then we define the unit volume $\bar{\mu} = e_1 \wedge \cdots \wedge e_n$. This definition is independent of choice of ONB. If $\{\hat{e}_1, ..., \hat{e}_n\}$ be another ONB with $\hat{e}_i = R_i^j e_j$. Since both bases are ONB, we have $R^T R = I$. This implies $det R = \pm 1$. Note that
$$\hat{e}_1 \wedge \cdots \wedge \hat{e}_n = det(R) e_1 \wedge \cdots \wedge e_n = \pm e_1 \wedge \cdots \wedge e_n.$$
If $det R = 1$, we say that $\{\hat{e}_1, ..., \hat{e}_n\}$ and $\{e_1, ..., e_n\}$ have same orientation.

- Suppose $\{e_1, ..., e_n\}$ is a basis with $\langle e_i, e_j \rangle = g_{ij}$. Let $\{\hat{e}_1, ..., \hat{e}_n\}$ be an ONB. We can express $e_i = a_i^j \hat{e}_j$. Then we have
$$G = (g_{ij}) = (\langle e_i, e_j \rangle) = (\langle a_i^k \hat{e}_k, a_j^\ell \hat{e}_\ell \rangle) = (\sum_k a_i^k a_j^k) = AA^T.$$
This gives $det(A) = \sqrt{det(G)}$. The n-volume
$$e_1 \wedge \cdots \wedge e_n = det(A) \hat{e}_1 \wedge \cdots \wedge \hat{e}_n = \sqrt{det(G)} \hat{e}_1 \wedge \cdots \wedge \hat{e}_n = \sqrt{det(G)}\, \bar{\mu}.$$

Hodge $\star$ for Vectors The Hodge star $\star_k$ maps a k-vector to an $(n-k)$-vector which is its orthogonal complement. Sometimes, we abbreviate $\star_k$ by $\star$ when its object v is already known as a k-vector. We start from a definition through an ONB first. In the examples below, $\{e_1, ..., e_n\}$ is an ONB of V.

- **Example: 2D**
  1. $\star e_1 = e_2, \quad \star e_2 = -e_1, \quad \star 1 = e_1 \wedge e_2.$
  2. $\star(2e_1 + e_2) = 2e_2 - e_1.$

- **Example** in $\mathbb{R}^3$,
  1. $\star(e_1 \wedge e_2) = e_3, \quad \star(e_2 \wedge e_3) = e_1, \quad \star(e_3 \wedge e_1) = e_2.$
  2. $\star e_1 = e_2 \wedge e_3, \quad \star e_2 = e_3 \wedge e_1, \quad \star e_3 = e_1 \wedge e_2.$
  3. $\star 1 = e_1 \wedge e_2 \wedge e_3, \quad \star(e_1 \wedge e_2 \wedge e_3) = 1.$
  4. Let $u = e_1 + e_2$, $v = e_2 - e_3$. We have seen that
  $$u \wedge v = e_1 \wedge e_2 - e_2 \wedge e_3 + e_3 \wedge e_1.$$
  Thus,
  $$\star(u \wedge v) = e_3 - e_1 + e_2.$$
  This is exactly $u \times v$. Thus, $u \wedge v$ is the parallelogram spanned by u and v, while $\star(u \wedge v)$ is $u \times v$, which is perpendicular to u and v.

- **Examples in $\mathbb{R}^4$**

  1. $\star(e_1 \wedge e_2) = e_3 \wedge e_4, \quad \star(e_2 \wedge e_3) = e_4 \wedge e_1, \quad \star(e_1 \wedge e_3) = -e_2 \wedge e_4.$
  2. $\star e_1 = e_2 \wedge e_3 \wedge e_4, \quad \star e_2 = -e_1 \wedge e_3 \wedge e_4.$

  Definition 5.4. Let $\{e_1, ..., e_n\}$ be an ONB in V. The Hodge $\star_k : \Lambda^k(V, \mathbb{R}) \to \Lambda^{n-k}(V, \mathbb{R})$ is defined by

  (a) $\star_k$ is linear,

  (b) $\star_k e_I := \text{sign}(\sigma) e_{\hat{I}}$, where $\hat{I} = \{1, ..., n\} \setminus I$ and σ is the permutation between $(I, \hat{I})$ and $\{1, ..., n\}$. In fact, one can show that

  $$\text{sign}(\sigma) = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k}.$$

- The definition is equivalent to the following general definition.

  Proposition 5.5. *If $v, w \in \Lambda^k(V, \mathbb{R})$, then*

  $$\boxed{v \wedge (\star w) = \langle v, w \rangle \bar{\mu}}$$

  where $\bar{\mu}$ is the unit volume.

  Proof. Let $v = v^I e_I$ and $w = w^I e_I$. Then

  $$\begin{aligned} v \wedge (\star w) &= \sum_{IJ} v^I w^J e_I \wedge (\star e_J) \\ &= \sum_I v^I w^I e_I \wedge (\star e_I) \\ &= \langle v, w \rangle e_1 \wedge \cdots \wedge e_n. \end{aligned}$$

  □

- Now suppose $\langle e_i, e_j \rangle = g_{ij}$. Let $I = \{i_1, ..., i_k\}$ be an order index set of $\{1, ..., n\}$, $\hat{I}$ be the order index set in $\{1, ..., n\}$ complement to I, and σ be the permutation between $(I, \hat{I})$ and $\{1, ..., n\}$. From the definition, we have

  $$e_I \wedge (\star e_I) = \|e_I\|^2 \mu$$

  We recall that

  $$e_1 \wedge \cdots \wedge e_n = \sqrt{g}\mu, \quad g = \det((g_{ij})_{n \times n}),$$
  $$\|e_I\|^2 = \langle e_I, e_I \rangle = g_{II}, \quad g_{II} = \det((g_{ij})_{i \in I, j \in I}).$$

  Hence

  $$\star e_I = \frac{g_{II}}{\sqrt{g}} \text{sign}(\sigma) e_{\hat{I}}.$$

Volume form Suppose $\{e^1, ..., e^n\}$ is a basis in V with $\langle e^i, e^j \rangle = g^{ij}$, where $(g^{ij}) = ((g_{ij})_{n\times n})^{-1}$. Let $\hat{e}^1, ..., \hat{e}^n$ be an ONB in V^*. We define the unit volume form in V by

$$\mu = \hat{e}^1 \wedge \cdots \wedge \hat{e}^n.$$

Then

$$e^1 \wedge \cdots \wedge e^n = \sqrt{det(g^{ij})}\, \mu.$$

Exercise

1. Let $I = (i_1, ..., i_k)$ and σ be the permutation between $(I, \hat{I})$ and $\{1, ..., n\}$. Show that

$$\text{sign}(\sigma) = (-1)^{i_1+\cdots+i_k+1+\cdots+k}.$$

2. Show that $\star_{n-k}\star_k = (-1)^{(n-k)k}$.

3. Show that $\langle \star v, \star w \rangle = \langle v, w \rangle$.

Hodge $\star$ for k-forms

Definition 5.5. Suppose $\{e^1, ..., e^n\}$ be a basis in V^* with $\langle e^i, e^j \rangle = g^{ij}$. The Hodge $\star$ for k form is a map

$$\star_k : \Lambda^k(V, \mathbb{R})^* \to \Lambda^{n-k}(V, \mathbb{R})^*$$

satisfying

(a) $\star_k$ is linear

(b) for basis e^I, where the order index $I \subset \{1, ..., n\}$ and $|I| = k$,

$$\star_k e^I := \frac{g^{II}}{\sqrt{g^{-1}}}\text{sign}(\sigma)e^{\hat{I}},$$

where $g = det(g_{ij})$ and $g^{-1} = det(g^{ij})$, and $\hat{I}$ is the order index set in $\{1, ..., n\}$ complement to I, and σ is the permutation between $(I, \hat{I})$ and $\{1, ..., n\}$.

Proposition 5.6. *If $\omega, \eta \in \Lambda^k(V, \mathbb{R})^*$, then*

$$\boxed{\omega \wedge \star\eta = \langle \omega, \eta \rangle \mu}$$

where μ is the unit volume form.

5.3 Differential Forms on Manifolds

5.3.1 Manifold

Definition 5.6. A differential manifold of dimension n is a set M with a family of injective (i.e. 1-1) mappings $\mathbf{x}_\alpha : U_\alpha(\subset M) \to V_\alpha(\text{open} \subset \mathbb{R}^n)$ such that:

1. $\cup_\alpha U_\alpha = M$;
2. For any α, β, with $U_\alpha \cap U_\beta = W \neq \emptyset$, the sets $\mathbf{x}_\alpha(W)$, $\mathbf{x}_\beta(W)$ are open and the mapping $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1} : \mathbf{x}_\alpha(W) \to \mathbf{x}_\beta(W)$ is differentiable.

Remarks.

- $\mathbf{x}_\alpha$ is called a parameterization (or system of coordinate) of M.
- The couple $(U_\alpha, \mathbf{x}_\alpha)$ is called an atlas or a coordinate chart of M. The family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is called a differentiable structure of M.
- This definition is intrinsic, it tells nothing where M is in the space.

Examples

- Consider a graph of a function in $\mathbb{R}^n$: $z = z(x^1, ..., x^n)$ is a smooth function with $\mathbf{x} \in V(\text{open } \subset \mathbb{R}^n)$. Let
$$M = \{(\mathbf{x}, z(\mathbf{x}))|\mathbf{x} \in V\}.$$
Such M is an n-manifold. The mapping: $(\mathbf{x}, z(\mathbf{x})) \mapsto \mathbf{x}$ is from M to $V \subset \mathbb{R}^n$, and $\mathbf{x}$ is a parameterization of M.
- Disk: $\{(x, y)|x^2 + y^2 \leq 1\}$. We can use polar coordinate as our coordinate chart:
$$x = r\cos\theta, y = r\sin\theta, \quad 0 \leq r \leq 1, 0 \leq \theta < 2\pi.$$
- Sphere: Consider the spherical coordinate
$$x = \cos\phi\sin\theta, y = \sin\phi\sin\theta, z = \cos\theta, \quad \phi \in [0, 2\pi], \theta \in [-\pi/2, \pi/2]$$
The spherical coordinate (θ, ϕ) can serve as a local coordinate on sphere.

Function defined on *M*

- A function $f : M \to \mathbb{R}$ is said to be differentiable on a chart $(U, \mathbf{x})$ if $f \circ \mathbf{x}^{-1}$, which maps $\mathbf{x}(U) \subset \mathbb{R}^n \to \mathbb{R}$, is differentiable. A function $f : M \to \mathbb{R}$ is said to be differentiable on M if it is differentiable on every chart of M.

5.3.2 Tangent space and cotangent space

Tangent vector and cotangent vectors Let M be an n-dimensional manifold. Consider a chart $(U, \mathbf{x})$ of M. That is, $U \subset M$ is locally parameterized by the map $\mathbf{x} : U \to \mathbb{R}^n$. Let $p \in U$.

- A function f defined on U can be treated as a function defined on $\mathbf{x}(U) \subset \mathbb{R}^n$, and is denoted by $f(x^1, ..., x^n)$ in this chart.

- The variation of f at $p \in U$ in the direction x^i is defined to be the partial derivative:

$$\left.\frac{\partial}{\partial x^i}\right|_p f \text{ or } f_{x^i}(p).$$

  Let $v = v^i e_i \in \mathbb{R}^n$. The variation of f in the direction of v at p is defined to be

$$\langle df_p | v\rangle := df_p(v) := \frac{\partial f}{\partial x^i}(p) v^i.$$

  From this expression, it is natural to define the tangent space of M at p to be $\mathbb{R}^n$. It is denoted by T_pM. The coordinate vector $e_i \in \mathbb{R}^n$ will be denoted as $\frac{\partial}{\partial x^i}|_p \in T_pM$. A tangent vector $v \in T_pM$ is expressed as

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

  And the the differential of f at p, that is, df_p is in T_p^*M.

- If $(V, \mathbf{y})$ is another coordinate chart containg p, then by the differential structure of M, $\mathbf{x} \leftrightarrow \mathbf{y}$ is diffeomorphism, and we have

$$\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}.$$

  From this, one can show that the definition of $df_p : T_pM \to \mathbb{R}$ is independent of choice of coordinate.

- The coordinate function x^i is a smooth function in U. At each point $p \in U$, $dx^i_p \in T_p(M)$ and

$$\langle dx^i | \frac{\partial}{\partial x^j}\rangle|_p = \delta^i_j \text{ for every } p \in U.$$

  Thus, $\{dx^1, ..., dx^n\}$ is the dual basis of $\{\partial/\partial x^1, ..., \partial/\partial x^n\}$. The differential df in U can be expressed as

$$df = f_{x^i} dx^i.$$

- The tangent bundle $TM := \cup_{p\in M} T_pM$, the cotangent bundle $T^*M := \cup_{p\in M} T_p^*M$.

5.3.3 Functions defined on manifolds

Let M be an n-dimensional manifold with a family of charts $\{(U_\alpha, \mathbf{x}_\alpha)\}$.

- Let M and N are smooth manifolds. A function $\varphi : M \to N$ is called smooth if for every $p \in M$, there exist compatible coordinate charts $(U, \mathbf{x})$ in M with $p \in U$ and $(V, \mathbf{y})$ in N with $\varphi(p) \in V$ such that the mapping

$$\mathbf{y} \circ \varphi \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbf{y}(V)$$

  is smooth. We express this function in terms of $(x^1, ..., x^n)$ and $(y^1, ..., y^m)$ by $(\varphi^1(\mathbf{x}), ..., \varphi^m(\mathbf{x}))$.

- The differential $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is defined to be

$$d\varphi_p \left(\frac{\partial}{\partial x^i} \right) = \left. \frac{\partial \varphi^j}{\partial x^i} \right|_p \frac{\partial}{\partial y^j}.$$

  One can show that the definition of $d\varphi$ is independent to the choice of coordinate charts.

- Push forward $\varphi_* : T_pM \to T_{\varphi(p)}N$. The differential $d\varphi$ is also called push forward map, and is denoted by φ_*.

- Pull back $\varphi^* : T^*_{\varphi(p)}N \to T^*_pM$ is the dual map of φ_*. It is defined by

$$\langle \varphi^* dy^j | \frac{\partial}{\partial x^i} \rangle := \langle dy^j | \varphi_* \frac{\partial}{\partial x^i} \rangle = \frac{\partial \varphi^j}{\partial x^i}.$$

  That is,

$$\varphi^* dy^j = \frac{\partial \varphi^j}{\partial x^i} dx^i.$$

- Suppose M and N are smooth manifolds and $\varphi : M \to N$ smooth. If $\varphi_* : T_pM \to T_{\varphi(p)}N$ is non-degenerate (i.e. this linear map is 1-1) for all $p \in M$, then we call φ is an immersion and (M, φ) is an immersed submanifold of N. If in addition φ is also 1-1, we call φ an imbedding and (M, φ) an imbedded submanifold of N.

5.3.4 Differential forms

- **0-forms** $\Omega^0(M, \mathbb{R}) = \{f : M \to \mathbb{R} \text{ smooth}\}$.

- **1-forms**

$$\Omega^1(M, \mathbb{R}) = \{\alpha : M \to T^*M \mid \alpha(p) \in T^*_p(M) \text{ and } \alpha \text{ is smooth}\}.$$

  In a coordinate chart $(U, \mathbf{x})$, α can be represented as

$$\alpha(x) = \alpha_i(x) dx^i,$$

  with $\alpha_i(\cdot)$ being smooth in U.

- **k-forms**

  $$\Omega^k(M,\mathbb{R}) = \{\omega : M \to \cup_{p\in M}\Lambda^k(T_pM,\mathbb{R})^* \mid \omega(p) \in \Lambda^k(T_pM,\mathbb{R})^* \text{ and } \omega \text{ is smooth}\}$$

  The representation of a k-form ω in a coordinate chart $(U, \mathbf{x})$ is

  $$\omega(\mathbf{x}) = w_I(\mathbf{x})d\mathbf{x}^I.$$

  Here, $I = (i_1, ..., i_k)$, $d\mathbf{x}^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

- Differential forms are independent of parameterization.

5.3.5 Exterior Derivatives for Differential Forms

We define the *exterior derivative* $d : \Omega^k \to \Omega^{k+1}$ as the follows.

(a) d is linear;

(b) For $k = 0$, define $df := f_{x^i}dx^i$.

(c) For $\omega = w_I(x)dx^I$, define $d\left(w_I(x)dx^I\right) := dw_I(x) \wedge dx^I$.

Examples in $\mathbb{R}^3$

1. $df = f_{x^1}dx^1 + f_{x^2}dx^2 + f_{x^3}dx^3$.
2. For 1-form

   $$\begin{aligned} d(A_1dx^1 + A_2dx^2 + A_3dx^3) &= (A_{1,x^2}dx^2 + A_{1,x^3}dx^3) \wedge dx^1 \\ &\quad + (A_{2,x^1}dx^1 + A_{2,x^3}dx^3) \wedge dx^2 + (A_{3,x^1}dx^1 + A_{3,x^2}dx^2) \wedge dx^3 \\ &= (A_{3,x^2} - A_{2,x^3})dx^2 \wedge dx^3 + (A_{1,x^3} - A_{3,x^1})dx^3 \wedge dx^1 + (A_{2,x^1} - A_{1,x^2})dx^1 \wedge dx^2. \end{aligned}$$

3. For 2-form

   $$d(B_1dx^2 \wedge dx^3 + B_2dx^3 \wedge dx^1 + B_3dx^1 \wedge dx^2) = (B_{1,x^1} + B_{2,x^2} + B_{3,x^3})dx^1 \wedge dx^2 \wedge dx^3.$$

4. $d(\rho(x)dx^1 \wedge dx^2 \wedge dx^3) = 0$.
5. One can check that $d \circ d = 0$.

   $$\begin{aligned} d^2f &= d(f_{x^i}dx^i) \\ &= (f_{x^3x^2} - f_{x^2x^3})dx^2 \wedge dx^3 + (f_{x^1x^3} - f_{x^3x^1})dx^3 \wedge dx^1 + (f_{x^2x^1} - f_{x^1x^2})dx^1 \wedge dx^2 \\ &= 0. \\ d^2A &= d\left[(A_{3,x^2} - A_{2,x^3})dx^2 \wedge dx^3 + (A_{1,x^3} - A_{3,x^1})dx^3 \wedge dx^1 + (A_{2,x^1} - A_{1,x^2})dx^1 \wedge dx^2\right] \\ &= \left[(A_{3,x^2} - A_{2,x^3})_{x^1} + (A_{1,x^3} - A_{3,x^1})_{x^2} + (A_{2,x^1} - A_{1,x^2})_{x^3}\right] dx^1 \wedge dx^2 \wedge dx^3 \\ &= 0. \\ d^2B &= d\left[(B_{1,x^1} + B_{2,x^2} + B_{3,x^3})dx^1 \wedge dx^2 \wedge dx^3\right] \\ &= 0. \end{aligned}$$

Proposition 5.7. *The exterior derivative d is uniquely characterized by the following properties:*

(a) d is linear,

(b) $df = f_{x^i}dx^i$ for any $f \in \Omega^0(M, \mathbb{R})$,

(c) $d \circ d = 0$,

(d) For $\alpha \in \Omega^k(M)$, $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$.

Proof. We shall only prove (d). The rests are left for exercises. To prove (d), suppose $\alpha = \alpha_I dx^I$, $|I| = k$, and $\beta = \beta_J dx^J$. We have

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(\alpha_I \beta_J) \wedge dx^I \wedge dx^J \\
&= ((d\alpha_I)\beta_J + \alpha_I(d\beta_J)) \wedge dx^I \wedge dx^J \\
&= (d\alpha_I \wedge dx^I) \wedge \left(\beta_J dx^J\right) + (-1)^k \alpha_I dx^I \wedge d\beta_J \wedge dx^J \\
&= (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).
\end{aligned}$$

□

Remarks

1. The proposition also implies that the definition of the exterior derivative d is independent of the choice of coordinate chart.

2. If $(U, \mathbf{x})$ and $(V, \mathbf{y})$ are two charts covers a point p. Then, from definition, $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$, and for any function $f : M \to \mathbb{R}$,

$$df = \frac{\partial f}{\partial y^j} dy^j = \frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} dx^i = \frac{\partial f}{\partial x^i} dx^i.$$

5.3.6 Pullback

In multivariable calculus, we need to perform change-of-variable. This is the pullback operation of differential form.

Definition 5.7. Let $\varphi : M_0 \to M$. Let $\alpha \in \Omega^k(M)$ be a k-form. Its pullback $\varphi^*\alpha \in \Omega^k(M_0)$ is defined by

$$\varphi^*(\alpha)(v_1, ..., v_k) := \alpha(d\varphi(v_1), ..., d\varphi(v_k)). \tag{5.5}$$

Proposition 5.8. *The pullback φ^* has the following properties.*

- *$\varphi^*(f) = f \circ \varphi$ for $f \in \Omega^0(M)$*
- *$\varphi^*(\alpha \wedge \beta) = (\varphi^*\alpha) \wedge (\varphi^*\beta)$.*

- $\varphi^*(f\alpha) = (\varphi^* f)\varphi^*\alpha$.
- $\varphi^*(d\alpha) = d\varphi^*(\alpha)$.

Let $\alpha \in \Omega_k(M)$ and $\Sigma \subset M_0$ be a k-dimensional submanifold, we have

$$\int_{f(\Sigma)} \alpha = \int_\Sigma f^*\alpha.$$

5.3.7 Stokes' Theorem

Stokes' Theorem in vector calculus In vector calculus, we have three important theorems related to the fundamental theorem of calculus:

- Fundamental Theorem of Calculus: Let C be a curve expressed by $\mathbf{x}(t)$ with $t \in [0,1]$. We have
$$\int_C \nabla f(\mathbf{x}(t)) \cdot d\mathbf{x}(t) = f(\mathbf{x}(1)) - f(\mathbf{x}(0)).$$
- Kelvin-Stokes' theorem: Let Σ be a surface in $\mathbb{R}^3$, $\boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^3$. We have
$$\iint_\Sigma \nabla \times \boldsymbol{u}(\mathbf{x}) \cdot \boldsymbol{n}\, dS = \int_{\partial\Sigma} \boldsymbol{u} \cdot d\mathbf{x}.$$
- Divergence theorem: Let D be a domain in $\mathbb{R}^3$ and $\boldsymbol{q} : D \to \mathbb{R}^3$. We have
$$\iiint_D \nabla \cdot \boldsymbol{q}(\mathbf{x})\, d\mathbf{x} = \iint_{\partial D} \boldsymbol{q} \cdot \boldsymbol{n}\, dS.$$

Here, the *surface normal element* $\boldsymbol{n}\, dS$ has the following representation in terms of a parameterization. Suppose Σ is parameterized by $\mathbf{x}(u,v)$. Then the outer normal of Σ is

$$\boldsymbol{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|}.$$

The area element is

$$dS = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| du \wedge dv.$$

Thus,

$$\begin{aligned}
\boldsymbol{n}\, dS &= \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\, du \wedge dv \\
&= \begin{vmatrix} i & j & k \\ \partial_u x^1 & \partial_u x^2 & \partial_u x^3 \\ \partial_v x^1 & \partial_v x^2 & \partial_v x^3 \end{vmatrix} du \wedge dv
\end{aligned}$$

$$= \left(\frac{\partial(x^2,x^3)}{\partial(u,v)}, \frac{\partial(x^3,x^1)}{\partial(u,v)}, \frac{\partial(x^1,x^2)}{\partial(u,v)} \right) du \wedge dv$$
$$= (dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2).$$

Here,

$$\frac{\partial(x^2,x^3)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\ \frac{\partial x^3}{\partial u} & \frac{\partial x^3}{\partial v} \end{vmatrix}$$

is the Jacobian and

$$\frac{\partial(x^2,x^3)}{\partial(u,v)} du \wedge dv = dx^2 \wedge dx^3.$$

The meaning of $dx^2 \wedge dx^3$ is the following. It is a functional applied to a two dimensional surface S. It returns the infinitesimal area element of the projection of S onto the x^2-x^3 plane.

Generalized Stokes' Theorem for differential forms All of above formulae (1D, 2D, 3D) can be unified into one single formula

$$\boxed{\int_D d\omega = \int_{\partial D} \omega}$$

where ∂D denotes the boundary of a domain D. The domain D can be one, two or three dimensional. The integrand ω is

- 1D: $\omega = f$, $d\omega = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \frac{\partial f}{\partial x^3} dx^3$,
- 2D: $\omega = u_1 dx^1 + u_2 dx^2 + u_3 dx^3$,

  $$d\omega = (\partial_2 u_3 - \partial_3 u_2) dx^2 \wedge dx^3 + (\partial_3 u_1 - \partial_1 u_3) dx^3 \wedge dx^1 + (\partial_1 u_2 - \partial_2 u_1) dx^1 \wedge dx^2.$$

- 3D: $\omega = q_1 dx^2 \wedge dx^3 + q_2 dx^3 \wedge dx^1 + q_3 dx^1 \wedge dx^2$, $d\omega = (\partial_1 q_1 + \partial_2 q_2 + \partial_3 q_3)\, dx^1 \wedge dx^2 \wedge dx^3$.

Let us state this generalized Stokes Theorem below.

Theorem 5.2 (Stokes-Cartan)**.** *Let M be a smooth k-dimensional manifold with boundary ∂M. Let $\omega \in \Omega^{k-1}(M, \mathbb{R})$. We have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. 1. We partition the parameter domain into union of small cubes. We only need to prove the case when $M = [0,1]^k$ and $\omega = f_i\, du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^k$.

$$\int_{[0,1]^k} d\omega = \int_{[0,1]^{k-1}} \int_0^1 \frac{\partial f_i}{\partial u^i} du^i \wedge du^1 \wedge \cdots \widehat{du^i} \cdots \wedge du^k$$

$$= \int_{[0,1]^{k-1}} (f_i(1) - f_i(0)) du^1 \wedge \cdots \widehat{du^i} \cdots \wedge du^k$$
$$= \int_{\partial[0,1]^k} f_i du^1 \wedge \cdots \widehat{du^i} \cdots \wedge du^k$$

□

Remarks

1. The Stokes theorem in vector calculus seems depending on the inner product structure of $\mathbb{R}^3$, it is indeed independent of the inner product. It depends only on the wedge product and the exterior derivative. Later, we shall show that the Stokes theorem is purely topological, which is the deRham Theorem.

2. Sometimes we denote the integral by

$$\langle \omega | M \rangle := \int_M \omega.$$

and the Stokes-Cartan theorem can be expressed as

$$\langle d\omega | M \rangle = \langle \omega | \partial M \rangle.$$

This means that d and ∂ are dual to each other. In this setting, the concept of the vector space Ω^k is clear. However, it is not clear what is the vector space for the collection of k-dimensional sub-manifolds M. In fact, we can construct such a vector space algebraically, called simplicial complex. The integration of ω is over those k-simplices. We shall see this in later section.

5.4 Inner product structure for differential forms

5.4.1 Riemannian manifold (Inner product structure)

The vector calculus uses inner product structure in $\mathbb{R}^3$ (i.e. the $\flat, \sharp$ operations). For general manifold M, an inner product structure $G = (g_{ij})$ on M is a symmetric, positive definite, smooth bilinear function $\langle \cdot, \cdot \rangle$ on the tangent space TM. This means that if $(U, \mathbf{x})$ is a coordinate chart, the inner product of two tangent vectors $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ is defined to be

$$\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = g_{ij}.$$

with $(g_{ij}(\mathbf{x}))$ is a smooth, symmetric and positive definite matrix function. We denote the arclength

$$ds^2 = g_{ij} dx^i \otimes dx^j, \quad \text{or just} \quad ds^2 = g_{ij} dx^i dx^j$$

With the inner product on tangent space TM, it induces inner product structure g^{ij} on T^*M:

$$\langle dx^i, dx^j \rangle = g^{ij}, \quad (g^{ij}) = (g_{ij})^{-1},$$

and on $\Omega^k(M, \mathbb{R})$ by

$$\langle d\mathbf{x}^I, d\mathbf{x}^J \rangle = g^{IJ}, \quad g^{IJ} = det((g^{ij})_{i\in I, j\in J}).$$

For any $\omega, \eta \in \Omega^k(M, \mathbb{R})$, they can be represented as

$$\omega = \omega(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_k}})dx^I := \omega_I dx^I, \quad \eta = \eta_I dx^I.$$

Then

$$\langle \omega, \eta \rangle := \omega_I \eta_J g^{IJ}.$$

Example

1. Consider polar coordinate in $\mathbb{R}^2$. The coordinate chart is $(\mathbb{R}^2, (r, \theta))$. The imbedding map is $(x, y) = (r\cos\theta, r\sin\theta)$. The inner product of $(\mathbb{R}^2, (r, \theta))$ is induced by the inner product of the Euclidean plane $(\mathbb{R}^2, (x, y))$. We have

$$g_{11} = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = \|(\cos\theta, \sin\theta)\|^2 = 1,$$

$$g_{12} = g_{21} = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = \langle (\cos\theta, \sin\theta), (-r\sin\theta, r\cos\theta) \rangle = 0,$$

$$g_{22} = \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle \|(-r\sin\theta, r\cos\theta)\|^2 = r^2.$$

   The corresponding inner product in T^*M is $g^{11} = 1$, $g^{12} = 0$ and $g^{22} = 1/r^2$. The volume form $\mu = \sqrt{g}dr \wedge d\theta = rdr \wedge d\theta$.

5.4.2 Hodge $\star$ and Co-differential for Differential Forms

We are interested in the L^2 inner product structure for k-forms $\Omega^k(M, \mathbb{R})$: Suppose $\omega, \eta \in \Omega^k(M, \mathbb{R})$, we are interested in the following inner product of ω and η defined by

$$\int_M \langle \omega(p), \eta(p) \rangle \mu(p)$$

where μ is the volume form of M and $\langle \omega(p), \eta(p) \rangle$ is the inner product of two k-form in $\Lambda^k(T_pM, \mathbb{R})^*$. In performing integration-by-part, it is convenient to introduce the following Hodge star operator.

Hodge $\star$ for Differential Forms The Hodge $\star : \Omega^k(M, \mathbb{R}) \to \Omega^{n-k}(M, \mathbb{R})$ is defined as below.

Example

- Polar coordinate in $\mathbb{R}^2$: We have seen that the volume form $\mu = rdr \wedge d\theta$.

  1. $\star 1 = \mu = rdr \wedge d\theta$
  2. $\star(dr) = r\,d\theta, \quad \star(d\theta) = -\frac{1}{r}dr$.
     This follows from $d\theta \wedge (\star d\theta) = \|d\theta\|^2 rdrd\theta$, and $\|d\theta\|^2 = g^{22} = \frac{1}{r^2}$.

- **Global volume form and orientable manifolds** Given a Riemannian manifold M, we have seen that it can always a unit volume form at each point. Any two such unit volume forms is different by a sign. If the manifold has a consistent sign of a unit volume form μ, then we say the manifold is orientable. Below, to define Hodge $\star$ operator, we assume our manifold is orientable. The unit volume form is denoted by μ.

- **Hodge $\star$ on $d\mathbf{x}^I$**
  Let $I = \{i_1, ..., i_k\}$ be an ordered index in $\{1, ..., n\}$. We define

$$\star d\mathbf{x}^I := \sqrt{g} g^{II} \mathrm{sign}(\sigma) d\mathbf{x}^{\hat{I}}.$$

  where $\hat{I} = \{1, ..., n\} \setminus I$, σ is the permutation between $(I, \hat{I})$ and $\{1, ..., n\}$, $g^{II} = det((g^{ij})_{i\in I, j\in I})_{k\times k}$ and $g = det(g_{ij})$.

- **Hodge $\star$ for k-form:** For $\omega \in \Omega^k(M)$, ω can be represented as $\omega = \omega_I dx^I$, where $I = (i_1, ..., i_k)$. We define $\star\omega = \omega_I \star dx^I$. This definition is equivalent to

  Proposition 5.9. *For any $\omega, \eta \in \Omega^k(M, \mathbb{R})$, we have*

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \mu,$$

  where μ is the volume form of M.

Connection of Hodge $\star$ with Vector Calculus In vector calculus, we define grad, curl and div for vector fields in $\mathbb{R}^3$. These operators correspond to the exterior derivatives

$$d_0 \triangleq \mathrm{grad}, \quad d_1 \triangleq \mathrm{curl}, \quad d_2 \triangleq \mathrm{div}.$$

In fact,

$$\nabla f = (d_0 f)^\sharp, \quad \nabla \times \mathbf{v} = (\star d_1(\mathbf{v}^\flat))^\sharp, \quad \nabla \cdot \mathbf{v} = \star d_2 \star (\mathbf{v}^\flat). \tag{5.6}$$

The properties curl $\circ$ grad $= 0$ and div $\circ$ curl $= 0$ correspond to $d^2 = 0$.

Homeworks

1. Prove (5.6).
2. For $f \in \Omega^0(M)$, show that

$$\triangle f = \nabla \cdot (\nabla f) = \star d \star df.$$

3. For $v = (v_1, v_2, v_3) : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\alpha = v^\flat$. Show that

$$(\triangle v)^\flat = (\star d \star d - d \star d \star)\alpha.$$

Co-differential operator δ

1. In vector calculus, we use integration-by-part quite often. The formula reads

$$\int_D \nabla\phi \cdot \mathbf{v}\, dx = -\int_D \phi \nabla \cdot \mathbf{v}\, dx + \int_{\partial D} \phi \mathbf{v} \cdot \boldsymbol{n}\, dS.$$

We would like to find the formula of integration-by-part in terms of exterior derivative d. The volume integration are expressed as

$$\int_D \nabla\phi \cdot \mathbf{v}\, dx = \int_D \langle d\phi, \mathbf{v}^\flat \rangle \mu,$$

$$\int_D \phi \nabla \cdot \mathbf{v}\, dx = \int_D \langle \phi, \star d \star \mathbf{v}^\flat \rangle \mu.$$

The boundary term is

$$\int_{\partial D} \phi \mathbf{v} \cdot \boldsymbol{n}\, dS = \int_D d(\phi \mathbf{v}^\flat) \mu.$$

This $\star d \star$ is the dual of d. It is called the co-differential operator.

2. **Exercises**

   (a) Suppose $D \subset \mathbb{R}^3$. Show that

$$\int_D \nabla \times \mathbf{v} \cdot \mathbf{w} = -\int_D \mathbf{v} \cdot \nabla \times \mathbf{w}\, dx + \int_{\partial D} \langle \mathbf{v}, \mathbf{w} \rangle$$

   and to conclude that

$$\int_D \langle d\mathbf{v}^\flat, \mathbf{w}^\flat \rangle \mu = -\int_D \langle \mathbf{v}^\flat, \star d \star \mathbf{w}^\flat \rangle \mu + \int_{\partial D} \mathbf{v}^\flat \wedge \star \mathbf{w}^\flat.$$

3. To define co-differential operator for k-form, we shall define it as the dual of d_{n-k}. Let us consider the following diagram

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\delta_k} & \Omega^{k-1}(M) \\ \downarrow{\scriptstyle \star_k} & & \downarrow{\scriptstyle \star_{k-1}} \\ \Omega^{n-k}(M) & \xrightarrow{d_{n-k}} & \Omega^{n-k+1}(M) \end{array}$$

These induce a map $\delta_k : \Omega^k \to \Omega^{k-1}$ by

$$\boxed{\delta_k := (-1)^k \star_{k-1}^{-1} d_{n-k} \star_k = (-1)^{kn+n+1} \star_{n-k+1} d_{n-k} \star_k\, .}$$

Note that $\delta_0 = 0$. The operator δ_k is the dual of d_{k-1} in the following sense.

Proposition 5.10 (Integration-by-part)**.** *Let M be an orientable n-Riemannian manifold with volume form μ. For any $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$,*

$$\boxed{\langle d_{k-1}\alpha, \beta\rangle\mu - \langle \alpha, \delta_k\beta\rangle\mu = d_{n-1}(\alpha \wedge (\star\beta))}.$$

Proof. We have

$$\begin{aligned} d(\alpha \wedge (\star\beta)) &= (d_{k-1}\alpha) \wedge (\star\beta) + (-1)^{k-1}\alpha \wedge (d_{n-k} \star \beta) \\ &= \langle d\alpha, \beta\rangle\mu - \alpha \wedge \left((-1)^k \star_{k-1} \star_{k-1}^{-1} d_{n-k} \star_k \beta\right) \\ &= \langle d\alpha, \beta\rangle\mu - \alpha \wedge \star_{k-1}\delta_k\beta \\ &= \langle d\alpha, \beta\rangle\mu - \langle \alpha, \delta\beta\rangle\mu. \end{aligned}$$

□

Remark In $\Omega^k(M)$, let us define the inner product by

$$\langle\!\langle \omega, \eta\rangle\!\rangle := \int_M \langle \omega, \eta\rangle\mu = \int_M \omega \wedge \star\eta$$

for $\omega, \eta \in \Omega^k(M)$. When M has no boundary, then

$$\boxed{\langle\!\langle d_k\omega, \eta\rangle\!\rangle = \langle\!\langle \omega, \delta_{k+1}\eta\rangle\!\rangle.}$$

That is, $\delta_{k+1} : \Omega^{k+1}(M) \to \Omega^k(M)$ is the *adjoint* of $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$:

$$\Omega^k(M) \underset{\delta_{k+1}}{\overset{d_k}{\rightleftarrows}} \Omega^{k+1}(M)$$

5.4.3 Dirichlet Integral and Hodge Laplacian

For an $\alpha \in \Omega^k$, we define the Dirichlet integral

$$\begin{aligned} E[\alpha] &:= \frac{1}{2}\left(\|d_k\alpha\|^2 + \|\delta_k\alpha\|^2\right) \\ &:= \frac{1}{2}\left(\langle\!\langle d_k\alpha, d_k\alpha\rangle\!\rangle + \langle\!\langle \delta_k\alpha, \delta_k\alpha\rangle\!\rangle\right) \end{aligned}$$

which measure the roughness of α on M. Suppose M has no boundary. We take variation of E in α. That is,

$$\langle\!\langle \delta E[\alpha], \dot\alpha\rangle\!\rangle := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[\alpha + \varepsilon\dot\alpha].$$

Taking integration by part, we get

$$\langle\!\langle \delta E[\alpha], \dot\alpha\rangle\!\rangle = \int_M \left(\langle \delta_{k+1}d_k\alpha, \dot\alpha\rangle + \langle d_{k-1}\delta_k\alpha, \dot\alpha\rangle\right)\mu := \int_M \langle -\triangle\, \alpha, \dot\alpha\rangle\mu,$$

where $\triangle\alpha$ is called the Hodge Laplacian of α and is defined to be

$$\boxed{\triangle\alpha := -\left(\delta_{k+1}d_k + d_{k-1}\delta_k\right)\alpha.}$$

Note that from $\delta_0 = 0$ and $d_n = 0$, we get

$$\triangle = -\delta_1 d_0 \text{ for 0-forms}$$

$$\triangle = -d_{n-1}\delta_n \text{ for } n\text{-forms}$$

We list its expression in parameter form. Suppose M is locally parameterized by $(x^1, ..., x^n)$ with $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = g_{ij}$.

- 0-form: $f : M \to \mathbb{R}$
$$\triangle f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}f\right).$$

- k-forms

Homework

1. Derive the formula for Hodge-Laplacian in parameter form.

5.5 Hodge Decomposition

5.5.1 Helmholtz Decomposition for Vector Fields

The Helmholtz decomposition for vector fields in $\mathbb{R}^3$ was published in (1858). Such decomposition was generalized to differential forms by Hodge (1934), to general domains with boundaries by Friedrichs-Morrey (1955,1956). The theories are termed Helmholtz-Hodge decomposition and the Hodge-Morrey-Friedrichs decomposition.

Theorem 5.3. *Let M be a 3D manifold. Let $\mathbf{v} : M \to \mathbb{R}^3$ be a vector field. Then $\mathbf{v}$ can be L^2-orthogonally decomposed into*

$$\mathbf{v} = \nabla\varphi + \nabla\times\boldsymbol{\psi} + \mathbf{h}, \tag{5.7}$$

where

- $\varphi : M \to \mathbb{R}$ *(potential) satisfies*
$$\triangle\varphi = \nabla\cdot\mathbf{v}. \tag{5.8}$$
$$\varphi = 0 \text{ on } \partial M. \tag{5.9}$$

- $\boldsymbol{\psi} : M \to \mathbb{R}^3$ *(stream vector field) satisfies*
$$\nabla\times(\nabla\times\boldsymbol{\psi}) = \nabla\times\mathbf{v}, \tag{5.10}$$
$$\boldsymbol{\psi} \parallel \boldsymbol{\nu} \text{ on } \partial M \tag{5.11}$$

- $\mathbf{h} : M \to \mathbb{R}^3$ *(harmonic vector field) satisfies*

$$\nabla \cdot \mathbf{h} = 0, \quad \nabla \times \mathbf{h} = 0. \tag{5.12}$$

Proof. 1. We look for equations satisfied by φ, $\boldsymbol{\psi}$ and $\mathbf{h}$. First, By applying the divergence operator to (5.7), we get

$$\nabla \cdot \mathbf{v} = \triangle \varphi.$$

2. Next, applying the curl operator to (5.7), we obtain

$$\nabla \times \mathbf{v} = \nabla \times (\nabla \times \boldsymbol{\psi}).$$

3. Suppose we can find φ and $\boldsymbol{\psi}$ from the above two equations, then by defining $\mathbf{h} = \mathbf{v} - \nabla\varphi - \nabla \times \boldsymbol{\psi}$, we get $\nabla \cdot \mathbf{h} = 0$ and $\nabla \times \mathbf{h} = 0$.

4. By choosing proper boundary conditions for $\boldsymbol{\psi}$ and $\mathbf{h}$, we can get the orthogonality properties:

$$\int_M \langle \nabla\varphi, \nabla \times \boldsymbol{\psi} \rangle \, d\mathbf{x} = 0, \tag{5.13}$$

$$\int_M \langle \nabla\varphi, \mathbf{h} \rangle \, d\mathbf{x} = 0, \tag{5.14}$$

$$\int_M \langle \nabla \times \boldsymbol{\psi}, \mathbf{h} \rangle \, d\mathbf{x} = 0. \tag{5.15}$$

5. To fullfil (5.13), we have

$$\begin{aligned} \int_M \langle \nabla\varphi, \nabla \times \boldsymbol{\psi} \rangle \, d\mathbf{x} &= \int_M \langle \nabla \times (\nabla\varphi), \boldsymbol{\psi} \rangle \, d\mathbf{x} - \int_{\partial M} (\nabla\varphi) \times \boldsymbol{\psi} \cdot \boldsymbol{\nu} dS \\ &= \int_{\partial M} (\nabla\varphi) \times \boldsymbol{\psi} \cdot \boldsymbol{\nu} dS. \end{aligned}$$

If $\boldsymbol{\psi} \parallel \boldsymbol{\nu}$ on ∂M, then the boundary term is zero. Thus, a natural boundary condition for $\boldsymbol{\psi}$ is (5.11).

6. To satisfy (5.14), we need

$$\begin{aligned} \int_M \mathbf{h} \cdot \nabla\varphi \, d\mathbf{x} &= \int_{\partial M} \varphi \mathbf{h} \cdot \boldsymbol{\nu} dS - \int_M \varphi \nabla \cdot \mathbf{h} \, d\mathbf{x} \\ &= \int_{\partial M} \varphi \mathbf{h} \cdot \boldsymbol{\nu} dS \end{aligned}$$

Thus, boundary condition $\mathbf{h} \cdot \boldsymbol{\nu} = 0$ gives (5.14).

7. For (5.15), we have

$$\int_M \mathbf{h} \cdot (\nabla \boldsymbol{\psi})\, d\mathbf{x} = -\int_{\partial M} (\mathbf{h} \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu} dS + \int_M (\nabla \times \mathbf{h}) \cdot \boldsymbol{\psi}\, d\mathbf{x}$$
$$= -\int_{\partial M} (\mathbf{h} \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu} dS.$$

Thus, boundary condition (5.11) gives $(\mathbf{h} \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu} = 0$.

□

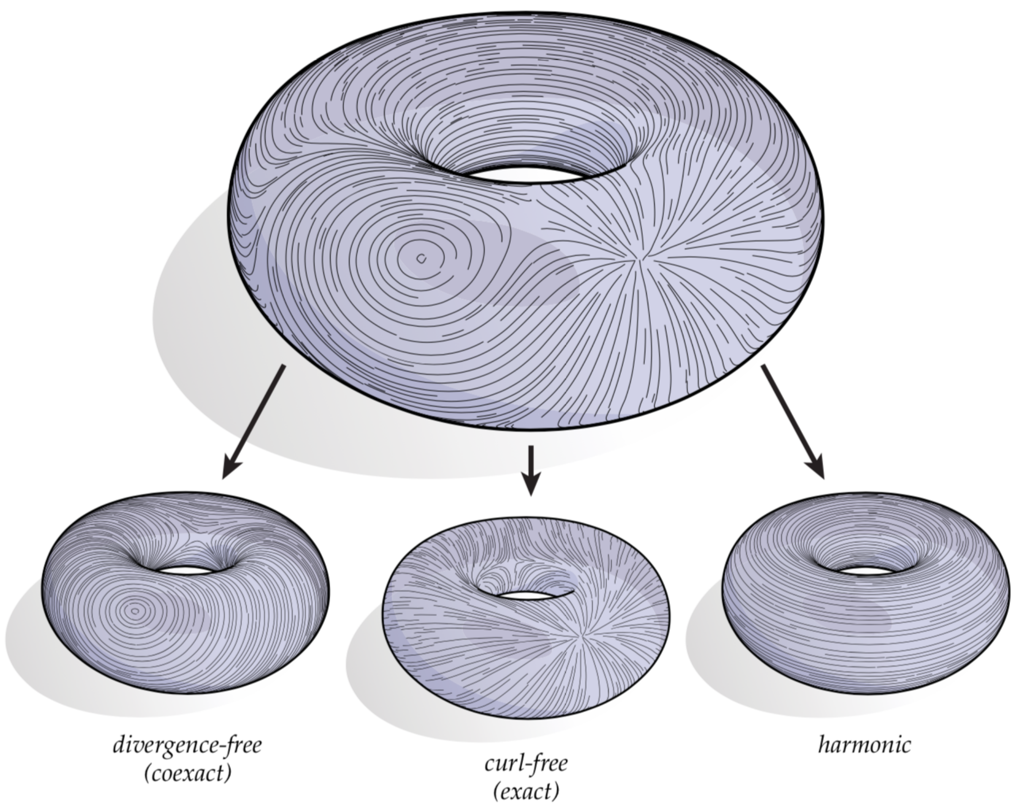


Figure 5.1: Copied from Crane's lecture note

5.5.2 Hodge decomposition for k-forms

In above decomposition, the keys are the facts

$$\text{div} \circ \text{curl} = 0, \quad \text{curl} \circ \text{grad} = 0.$$

In exterior calculus, these are unified to

$$d \circ d = 0, \quad \delta \circ \delta = 0.$$

Here, d is the differential and δ is the codifferential. We recall the space of k-forms is $\Omega^k(M)$. In $\Omega^k(M)$, we have defined the inner product

$$\langle\langle \omega, \eta \rangle\rangle := \int_M \omega \wedge \star\eta.$$

If M is a closed manifold (i.e. it has no boundary), then δ_{k+1} is the adjoint of d_k. That is,

$$\langle\langle d_k\omega, \eta \rangle\rangle = \langle\langle \omega, \delta_{k+1}\eta \rangle\rangle, \qquad \text{for any } \omega \in \Omega^k(M), \eta \in \Omega^{k+1}(M).$$

We have the following diagram

$$\Omega^{k-1}(M) \underset{\delta_k}{\overset{d_{k-1}}{\rightleftarrows}} \Omega^k(M) \underset{\delta_{k+1}}{\overset{d_k}{\rightleftarrows}} \Omega^{k+1}(M)$$

Definition 5.8. A k-form $\omega \in \Omega^k(M)$ is said to be

- *closed* if $d\omega = 0$,
- *exact* if $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}$,
- *co-closed* if $\delta\omega = 0$,
- *co-exact* if $\omega = \delta\beta$ for some $k+1$-form β,
- *harmonic* if $d\omega = 0$ and $\delta\omega = 0$.

From $d \circ d = 0$, we get that exact forms are also closed. From $\delta \circ \delta = 0$, we get that co-exact implies co-closed. Thus, $Im(d_{k-1}) \subset Ker(d_k)$ and $Im(\delta_{k+1}) \subset Ker(\delta_k)$. We now state the Hodge decomposition of k-forms.

Theorem 5.4. *Let M be a closed n-manifold, $\Omega^k(M)$ be the space of its k-forms with the following diagram*

$$\Omega^{k-1}(M) \underset{\delta_k}{\overset{d_{k-1}}{\rightleftarrows}} \Omega^k(M) \underset{\delta_{k+1}}{\overset{d_k}{\rightleftarrows}} \Omega^{k+1}(M)$$

Then

$$\boxed{\Omega^k(M) = Im(\delta_{k+1}) \oplus Ker(d_k) = Im(\delta_{k+1}) \oplus Im(d_{k-1}) \oplus \mathcal{H}^k(M).}$$

That is, for every $\omega \in \Omega^k(M)$, there exists a unique $\alpha \in \Omega^{k-1}(M)$ up to a closed form, a unique $\beta \in \Omega^{k+1}(M)$ up to a co-closed form, and a unique harmonic k-form $h \in \mathcal{H}^k(M)$, such that

$$\boxed{\omega = d\alpha + \delta\beta + h.} \tag{5.16}$$

The decomposition follows directly from the following Lemmas from Linear Algebra.

Lemma 5.1 (Four Fundamental Subspaces)**.** *Let V, W are two inner-product spaces. Let $A : V \to W$ be a linear map and A^* be its adjoint. Then*

$$(ImA^*)^\perp = kerA, \quad (ImA)^\perp = kerA^*.$$

This also means

$$V = kerA \oplus ImA^*, \quad W = kerA^* \oplus ImA.$$

Proof.

$$\begin{aligned} y \in (ImA)^\perp &\Leftrightarrow \langle y, Ax\rangle = 0 \ \forall x \in V \Leftrightarrow \langle A^*y, x\rangle = 0 \ \forall x \in V \\ &\Leftrightarrow A^*y = 0 \Leftrightarrow y \in KerA^*. \end{aligned}$$

The proof for $(ImA^*)^\perp = kerA$ is similar. □

Lemma 5.2. *Let U, V, W be inner product spaces. Let A, B be linear maps as the diagram*

$$U \xrightarrow{A} V \xrightarrow{B} W \qquad with \quad B \circ A = 0$$

Then

$$ImA \cap ImB^* = \{0\}.$$

Proof. From $B \circ A = 0$, we get $ImA \subset KerB$. From previous lemma, we have $kerB = (ImB^*)^\perp$. Thus, $ImA \subset (ImB^*)^\perp$. This implies $ImA \cap ImB^* = \{0\}$. □

Proof of Hodge decomposition

1. From $d_k : \Omega^k \to \Omega^{k+1}$ and $\delta_{k+1} = d_k^*$, we get

$$\Omega^k(M) = Ker(d_k) \oplus Im(\delta_{k+1}).$$

2. We further decompose $Ker(d_k)$ as

$$\begin{aligned} Ker(d_k) &= Im(d_{k-1}) \oplus \big((Im(d_{k-1}))^\perp \cap Ker(d_k)\big) \\ &= Im(d_{k-1}) \oplus (Ker(\delta_k) \cap Ker(d_k)) \\ &= Im(d_{k-1}) \oplus \mathcal{H}^k(M). \end{aligned}$$

□

Next we show that harmonic functions satisfy the Laplace equation.

Proposition 5.11. *On a closed manifold M (i.e. M has no boundary),*

$$\mathcal{H}^k(M) = \{h \in \Omega^k(M) | \triangle h = 0\},$$

where $\triangle = -(d_{k-1}\delta_k + \delta_{k+1}d_k)$.

Proof. 1. We show that $\triangle h = 0 \Rightarrow dh = 0$ and $\delta h = 0$. If $\triangle h = 0$, then

$$d_{k-1}\delta_k h = -\delta_{k+1} d_k h \in Im(d_{k-1}) \cap Im(\delta_{k+1}) = \{0\}.$$

Thus, $d_{k-1}\delta_k h = 0$ and $\delta_{k+1} d_k h = 0$.

2. When $d_{k-1}\delta_k h = 0$, we also have

$$0 = \langle\langle d_{k-1}\delta_k h, h\rangle\rangle = \langle\langle \delta_k h, \delta_k h\rangle\rangle$$

This leads to $\delta_k h = 0$.

3. Similarly, when $\delta_{k+1} d_k h = 0$, we have $d_k h = 0$. Thus, h is harmonic.

4. The proof of ($dh = 0$ and $\delta h = 0 \Rightarrow \triangle h = 0$) is trivial.

□

5.5.3 Solving for the Exact, Co-exact and Harmonic Components

Let M be a closed manifold. For $\omega \in \Omega^k(M)$, it can be decomposed into

$$\omega = d\alpha + \delta\beta + h. \tag{5.17}$$

Below, we will extract each components from ω.

Extracting $\alpha \in \Omega^{k-1}(M)$

1. We apply δ to the decomposition formula (5.17) to get

$$\delta d\alpha = \delta\omega. \tag{5.18}$$

2. The solvability of (5.18) is due to the following lemma (i.e. $Im(\delta_k d_{k-1}) = Im(\delta_k)$).

Lemma 5.3. *In the diagram*

$$\Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{\delta_k} \Omega^{k-1}(M)\ ,$$

it holds

$$\begin{cases} Ker(\delta_k d_{k-1}) = Ker(d_{k-1}) \\ Im(\delta_k d_{k-1}) = Im(\delta_k) \end{cases}. \tag{5.19}$$

Proof. (a) If $u \in Ker(\delta_k d_{k-1}) \subset \Omega^{k-1}(M)$, by Hodge decomposition

$$u = d\phi + \delta\psi + \chi$$

where $\chi \in \mathcal{H}^{k-1}(M)$. From $\delta du = 0$, we get

$$\delta d\delta\psi = 0.$$

This implies

$$0 = \langle\langle \delta d\delta\psi, \delta\psi \rangle\rangle = \langle\langle d\delta\psi, d\delta\psi \rangle\rangle$$

Hence, $d\delta\psi = 0$. By taking inner product with ψ, we get

$$0 = \langle\langle d\delta\psi, \psi \rangle\rangle = \langle\langle \delta\psi, \delta\psi \rangle\rangle$$

Thus, $\delta\psi = 0$. Hence $u \in Im(d_{k-2}) \oplus \mathcal{H}^{k-1}(M)$, which is $Ker(d_{k-1})$. This shows $Ker(\delta_k d_{k-1}) \subset Ker(d_{k-1})$. The other part $Ker(d_{k-1}) \subset Ker(\delta_k d_{k-1})$ is trivial.

(b) From $(\delta_k d_{k-1})^* = \delta_k d_{k-1}$, we get

$$Im(\delta_k d_{k-1}) = Ker(\delta_k d_{k-1})^{\perp}.$$

We have seen that $Ker(\delta_k d_{k-1}) = Ker(d_{k-1})$. Thus

$$Ker(\delta_k d_{k-1})^{\perp} = Ker(d_{k-1})^{\perp} = Im(d_{k-1}^*) = Im(\delta_k).$$

This shows $Im(\delta_k d_{k-1}) = Im(\delta_k)$.

□

3. The component $\alpha \in \Omega^{k-1}(M)$ in the decomposition formula (5.17) is unique up to a closed form. We apply the Hodge decomposition to α to get

$$\alpha = \chi + d\phi + \delta\psi.$$

This is an orthogonal decomposition. Hence

$$\|\alpha\|^2 = \|\chi\|^2 + \|d\phi\|^2 + \|\delta\psi\|^2.$$

The terms $\chi + d\phi \in Ker(d_{k-1})$. Thus, it makes no contribution to $d\alpha$. A canonical way to determine a unique α is to let $\alpha \perp H^{k-1}(M)$ and $\alpha \perp Im(d_{k-2})$. The latter gives $\alpha \in Im(d_{k-2})^{\perp} = Ker(d_{k-2}^*) = Ker(\delta_{k-1})$. To summarize, such unique $\alpha \in \Omega^{k-1}(M)$ satisfies

$$\begin{cases} \delta d\alpha = \delta\omega \\ \delta\alpha = 0 \\ \langle\langle \alpha, \chi \rangle\rangle = 0 \text{ for all } \chi \in H^{k-1}(M). \end{cases} \tag{5.20}$$

4. Another way to express (5.20) is

$$\begin{cases} -\triangle\,\alpha = \delta\omega \\ \langle\langle \alpha, \chi \rangle\rangle = 0 \text{ for all } \chi \in H^{k-1}(M). \end{cases} \tag{5.21}$$

Here, $-\triangle := \delta_k d_{k-1} + d_{k-2}\delta_{k-1}$. Clearly, (5.20) $\Rightarrow$ (5.21). Conversely, if $\delta d\alpha + d\delta\alpha = \delta\omega$, then $\delta d\alpha + d\delta\alpha \in Im(\delta_k)$. Hence, $d\delta\alpha = 0$ by Hodge decomposition. This implies

$$0 = \langle\langle d\delta\alpha, \alpha \rangle\rangle = \langle\langle \delta\alpha, \delta\alpha \rangle\rangle$$

This shows (5.21) $\Rightarrow$ (5.20).

5. The well-posedness of (5.20) or (5.21) follows from the Hodge decomposition theorem.

Extract β component By a similar process, one can show that $\beta \in \Omega^{k+1}(M)$ can be determined uniquely by

$$\begin{cases} d\delta\beta = d\omega \\ d\beta = 0 \\ \langle\langle \beta, \chi \rangle\rangle = 0 \text{ for all } \chi \in H^{k+1}(M). \end{cases}$$

This is equivalent to

$$\begin{cases} -\triangle \beta = d\omega \\ \langle\langle \beta, \chi \rangle\rangle = 0 \text{ for all } \chi \in H^{k+1}(M). \end{cases}$$

Harmonic Component h We simply use the decomposition formula.

$$h = \omega - d\alpha - \delta\beta.$$

to extract the harmonic component.

Chapter 6

Discrete Exterior Calculus

The discrete exterior calculus will be built on discrete (algebraic) topological structures of manifolds. There are various kinds of such discrete structures: simplicial complex, cell complex, or some other complices, depending on the mesh you use. We will use simplicial complex, which can be constructed through triangulation. This is the simplest structure and is usually called the primal complex. The simplicial complex consists of a sequence of k-simplices and a sequence of associated boundary operators. For instance, for 2-dimensional manifold M, the simplicial complex $\mathcal{K} = \{V, E, F\}$, and boundary operators $\partial_2 : F \to E$, $\partial_1 : E \to V$. Here, V, E, F, are sets of vertices, edges and faces on M, respectively.

In addition to the two main references (Crane + Chern and Schröder) of this note, the reference of this chapter also include

- Mathieu Desbrun, Anil N. Hirani, Melvin Leok, Jerrold E. Marsden, Discrete Exterior Calculus, 2005.
- D. Arnold, et al, Finite element exterior calculus, homological techniques, and applications, 2006.
- D. Arnold, R. Falk, Winther, Finite element exterior calculus: from Hodge theory to numerical stability, 2009.

6.1 Meshing

Meshing is to discretize a manifold and to build up a topological structure. For instance, triangulation is one kind of meshing. It builds up a simplicial complex on a manifold. There are other kinds of meshing, for instance, quadrilateral meshes in 2D and hexahedral meshes in 3D. We shall only discuss triangle mesh here.

6.1.1 Triangulation

Triangulation on flat domains Suppose $M \subset \mathbb{R}^m$ is a domain. We want to triangulate M. The most common way is to use Delaunay triangulation. The corresponding mesh maximizes the minimum angle of the underlying triangles. This will give nice property for the discrete Laplace-Betrami operator.

Delaunay meshes

- Given a set of points $V \subset \mathbb{R}^m$. A Delaunay triangulation $DT(V)$ is a triangulation such that no point in P is inside the circumcircle (circum-hypersphere) of any triangle (m-simplex) in $DT(V)$. This condition is called Delaunay condition. An important property of Delaunay triangulation is that it maximizes the minimum angle of all the angles of the triangles in the triangulation.

- If the points in V are in *general position*, then there exists a unique Denaunay triangulation of V. The meaning of *general position* is that the affine hull of V is m dimension and no $m+2$ points in V lie on a hypersphere whose inside contains no points in V.

- Given $V = \{\mathbf{x}_i, i = 1, ..., N\} \subset \mathbb{R}^m$, generation of $DT(V)$ can be transformed to find bottom side of convex hull of the points $\{(\mathbf{x}_i, \|\mathbf{x}_i\|^2) | i = 1, ..., N\}$ in $\mathbb{R}^{m+1}$.

- Reference:

  - Jonathan Richard Shewchuk, Lecture Notes on Delaunay Mesh Generation, 2012.
  - Per-Olof Persson and Gilbert Strang, A simple mesh generator in matlab.

Triangulation on a manifold

- A topological space X is called triangulable if there exists a simplicial complex $\mathcal{K}$ and a homeomorphism (i.e. f is 1-1, onto continuous and f^{-1} is also continuous) $f : K \to X$.

- People also study intrinsic Delaunay triangulation on a Riemannian manifold M without embedded in an Euclidean space. The edges there are geodesic paths.

- Reference: M. Fisher, B. Springborn, P. Schröder, and A. I. Bobenko, An algorithm for the construction of intrinsic delaunay triangulations with applications to digital geometry processing, Computing 2007.

6.1.2 Building a simplicial complex on a triangulated domain in $\mathbb{R}^3$

We want to build up a simplicial complex $\mathcal{K}$ on a triangulated orientable domain $M \subset \mathbb{R}^3$. This simplicial complex consists of *ordered* vertices, edges, faces and tetrahedra. $\mathcal{K} = \{V, E, F, T\}$. Here is the procedure.

1. We index the vertices in V, say $V = \{1, ..., n_0\}$.

2. We determine orientation of the edges, faces and tetrahedra. These are the sets E, F and T. The orientation of each tetrahedron should be consistent to the orientation of the domain M. Then we index E, F and T. The numbers of them are n_1, n_2 and n_3.

3. The boundary of an oriented simplex is defined to be

$$\partial_k[v_0, ..., v_k] = \sum_{i=0}^{k}(-1)^i[v_0, ..., \widehat{v_i}, ..., v_k].$$

   For example,

   - $\partial_1[v_0, v_1] = [v_1] - [v_0]$;
   - $\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$;
   - $\partial_3[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$;

4. We construct the boundary operator ∂_3, ∂_2 and ∂_1 using the example in Chern and Schröder's note for explanation. (Figure 4.5).

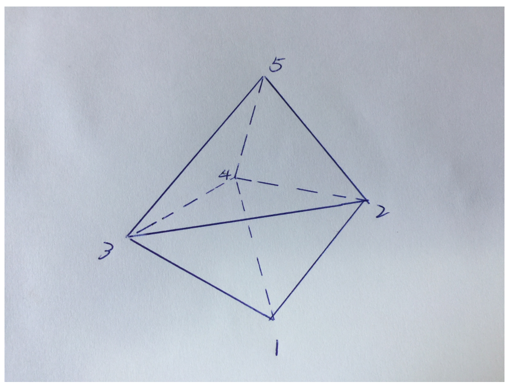


Figure 6.1: Copied from Chern and Schröder note

$$V = \begin{pmatrix}[1]\\ [2]\\ [3]\\ [4]\\ [5]\end{pmatrix} \quad E = \begin{pmatrix}[12]\\ [13]\\ [14]\\ [23]\\ [24]\\ [25]\\ [34]\\ [35]\\ [45]\end{pmatrix} \quad F = \begin{pmatrix}[123]\\ [124]\\ [134]\\ [234]\\ [235]\\ [245]\\ [345]\end{pmatrix} \quad T = \begin{pmatrix}[1234]\\ [2345]\end{pmatrix}. \tag{6.1}$$

The boundaries of the simplices are given by

$$\partial_3[1234] = [234] - [134] + [124] - [123],$$

$$\partial_3[2345] = [345] - [245] + [235] - [234].$$
$$\partial_2[123] = [23] - [13] + [12]$$

In general, the boundary operators, or the incidence matrices are given by

$$\partial_3 = \begin{pmatrix} -1 & 0 \\ +1 & 0 \\ -1 & 0 \\ +1 & -1 \\ 0 & +1 \\ 0 & -1 \\ 0 & +1 \end{pmatrix}_{|F|\times|T|} \qquad \partial_2 = \begin{pmatrix} +1 & +1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & +1 & +1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & +1 & +1 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & +1 \end{pmatrix}_{|E|\times|F|}$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & +1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & +1 & 0 & +1 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 & +1 \end{pmatrix}_{|V|\times|E|}$$

6.1.3 Building a simplicial complex on a triangulated manifold

Abstract simplicial complex

- Let $V = \{p_1, ..., p_N\}$ be N abstract vertices. A k-simplex of V is a set

  $$\sigma_k = \{v_0, ..., v_k\} \subset V.$$

  Any simplex spanned by a proper subset of $\{v_0, ..., v_k\}$ is called a face of σ_k.

- We can make this simplex abstract. Namely, consider a vertex set $V = \{1, ..., N\}$. An abstract k-simplex is $\{v_0, ..., v_k\}$, where $v_i \in V$. Any subset of a simplex is another simplex. We call it a face.

- A collection of simplices $\mathcal{K}$ is called a *simplicial complex* if for every simplex $\sigma \in \mathcal{K}$, every face $\sigma' \subset \sigma$ is also in $\mathcal{K}$. Let $\mathcal{K}^{(k)}$ be the collection of k-simplices of $\mathcal{K}$. Then $\mathcal{K} = \cup_k \mathcal{K}^{(k)}$. For simplicial complex in 3D, it is $\mathcal{K} = \{V, E, F, T\}$, vertex, edge, face and tetrahedra.

- A complex $\mathcal{K}$ is a *pure m-simplicial complex* if every simplex $\sigma' \in \mathcal{K}$ of degree $l < m$ is contained in some simplex of degree m.

- A complex is an m-dimensional simplicial manifold if

  - it is a pure m-simplicial complex

– For every $v \in V$, the star of v_i: $St(v_i) = \cup\{\sigma \in \mathcal{K} | v_i \in \sigma\}$ is composed of m-simplices.

- Given a k-simplex $\{v_0, ..., v_k\}$, there are only two types of permutations among the vertex indices, even or odd. The type of permutation determines the orientation of the simplex. An oriented k-simplex is denoted by $[v_0, ..., v_k]$. The orientation of

$$[v_{p(0)}, ..., v_{p(k)}] = \begin{cases} [v_0, ..., v_k] & \text{if } p \text{ is an even permutation,} \\ -[v_0, ..., v_k] & \text{if } p \text{ is an odd permutation.} \end{cases}$$

- **Boundary operator:** $\partial_k : \mathcal{K}^{(k)} \to \mathcal{K}^{(k-1)}$, for $\sigma_k = [v_0, ..., v_k]$, define

$$\boxed{\partial_k \sigma_k := \sum_{i=0}^{k} (-1)^i [v_0, ..., \hat{v}_i, ..., v_k].} \tag{6.2}$$

  Here, $\hat{v}_i$ means that we skip the term v_i. One can show

$$\boxed{\partial_k \partial_{k+1} = 0.}$$

  For instance,

$$\begin{aligned} \partial^2[0, 1, 2] &= ([2] - [1]) - ([2] - [0]) + ([1] - [0]) = 0. \\ \partial^2[0, 1, 2, 3] &= ([2, 3] - [1, 3] + [1, 2]) - ([2, 3] - [0, 3] + [0, 2]) \\ &\quad + ([1, 3] - [0, 3] + [0, 1]) - ([1, 2] - [0, 2] + [0, 1]) = 0. \end{aligned}$$

- We say two adjacent k-simplices $[v_0, v_1, ..., v_k]$ and $-[w_0, v_1, ..., v_k]$ have consistent orientation. This is because when we add these two simplices, their common face $[v_1, ..., v_k]$ will be cancelled.

- An oriented m-simplicial manifold is an abstract m-dimensional simplicial manifold where we can assign a consistent orientation to every m-simplex.

6.2 Chain Complex and Co-chain complex

Chain complex

- **Chain:** Given an abstract simplicial complex $\mathcal{K}$. Let $\mathcal{K}^k$ be the collection of all oriented k-simplex in $\mathcal{K}$. We label them as $\sigma_{k,i}$, $i = 1, ..., n_k$. The k-chain of $\mathcal{K}$ over a field $\mathbb{R}$ is defined to be

$$C_k(\mathcal{K}) := \{c = \sum_{i=1}^{n_k} c^i \sigma_{k,i} | c^i \in \mathbb{R}\}.$$

The k-chain $C_k(\mathcal{K})$ is a vector space of dimension n_k.
The boundary operator $\partial_k : C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K})$ is defined to be

$$\partial_k c = \partial_k \left(c^i \sigma_{k,i} \right) := c^i \partial_k \sigma_{k,i}$$

for $c = c^i \sigma_{k,i} \in C_k(\mathcal{K})$. Here, we recall

$$\partial_k [v_0, ..., v_k] := \sum_{i=0}^{k} (-1)^i [v_0, ..., \hat{v}_i, ..., v_k].$$

The boundary operator has the following properties:

- ∂_k is linear;
- $\partial_{k-1} \circ \partial_k = 0$.

For the proof of $\partial_{k-1} \circ \partial_k = 0$, we have

$$\begin{aligned}
\partial_{k-1}\partial_k[v_0, ..., v_k] &= \partial_{k-1} \sum_{i=0}^{k} (-1)^i [v_0, ..., \hat{v}_i, ..., v_k] \\
&= \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{i-1} (-1)^j [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_k] \\
&\quad + \sum_{i=0}^{k} (-1)^i \sum_{j=i+1}^{k} (-1)^{j-1} [v_0, ..., \hat{v}_i, ..., \hat{v}_j, ...v_k] \\
&= \sum_{i=0}^{k} \sum_{j<i} (-1)^{i+j} [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_k] \\
&\quad + \sum_{i=0}^{k} \sum_{j<i} (-1)^{i+j-1} [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_k] \\
&= 0.
\end{aligned}$$

- **Chain complex:** The sequence $(C_k(\mathcal{K}), \partial_k)$:

$$0 \xrightarrow{i} C_m(\mathcal{K}) \xrightarrow{\partial_m} C_{m-1}(\mathcal{K}) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0$$

is called a chain complex associated with $\mathcal{K}$ and is denoted by $C(\mathcal{K})$. Here, i is the inclusion map.

- Example: We use Ex.4.1 as an example.

$$V = \begin{pmatrix} \sigma_{0,1} \\ \sigma_{0,2} \\ \sigma_{0,3} \\ \sigma_{0,4} \\ \sigma_{0,5} \end{pmatrix} = \begin{pmatrix} [1] \\ [2] \\ [3] \\ [4] \\ [5] \end{pmatrix}, \quad E = \begin{pmatrix} \sigma_{1,1} \\ \sigma_{1,2} \\ \sigma_{1,3} \\ \sigma_{1,4} \\ \sigma_{1,5} \\ \sigma_{1,6} \\ \sigma_{1,7} \\ \sigma_{1,8} \\ \sigma_{1,9} \end{pmatrix} = \begin{pmatrix} [12] \\ [13] \\ [14] \\ [23] \\ [24] \\ [25] \\ [34] \\ [35] \\ [45] \end{pmatrix},$$

$$F = \begin{pmatrix} \sigma_{2,1} \\ \sigma_{2,2} \\ \sigma_{2,3} \\ \sigma_{2,4} \\ \sigma_{2,5} \\ \sigma_{2,6} \\ \sigma_{2,7} \end{pmatrix} = \begin{pmatrix} [123] \\ [124] \\ [134] \\ [234] \\ [235] \\ [245] \\ [345] \end{pmatrix}, \quad T = \begin{pmatrix} \sigma_{3,1} \\ \sigma_{3,2} \end{pmatrix} = \begin{pmatrix} [1234] \\ [2345] \end{pmatrix}$$

The chains $C_k = \text{span}\{\sigma_{k,i} | i = 1, ..., n_k\}$. Thus, $C_0 = \mathbb{R}^5$, $C_1 = \mathbb{R}^9$, $C_2 = \mathbb{R}^7$ and $C_3 = \mathbb{R}^2$.

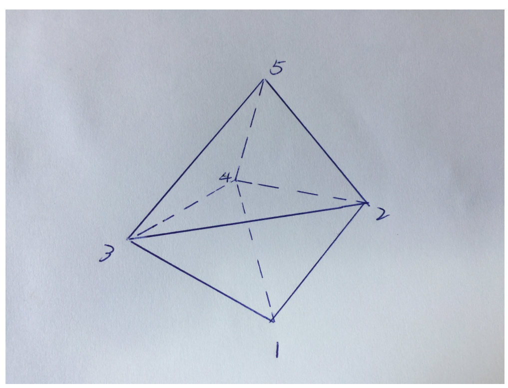


Figure 6.2: Copied from Chern and Schröder note

Homology Chain complex can be used to measure topological properties of the underlying simplicial complex $\mathcal{K}$.

- Recall the chain diagram

$$0 \xrightarrow{i} C_m(\mathcal{K}) \xrightarrow{\partial_m} C_{m-1}(\mathcal{K}) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0$$

From $\partial_k \circ \partial_{k+1} = 0$, we get

$$Im(\partial_{k+1}) \subset Ker(\partial_k).$$

We define

$$H_k(C(\mathcal{K})) := \frac{Ker(\partial_k)}{Im(\partial_{k+1})},$$

called the kth homology group of $C(\mathcal{K})$. Its dimension (number of generators) is called the kth *Betti number* of $\mathcal{K}$, and is denoted by b_k.

- We denote the dimension of $C_k(\mathcal{K})$ by I_k, and define the Euler characteristic $\chi(\mathcal{K})$ by

$$\chi(K) := \sum_{k=0}^{m} (-1)^k I_k. \tag{6.3}$$

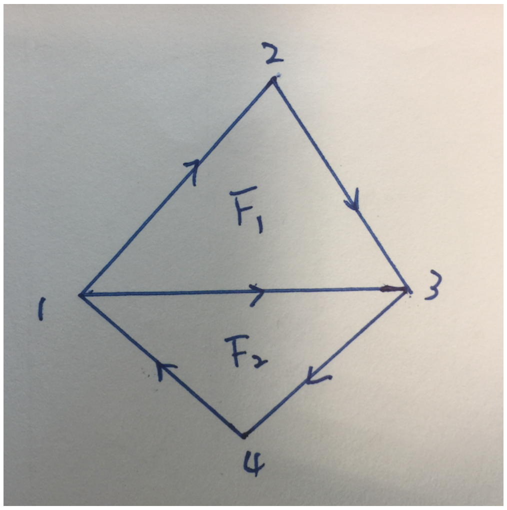


Figure 6.3: A disk-like simplicial complex.

- Example 1: Disk. Topologically, it has the following simplicial complex.

$$V = \begin{pmatrix} [1] \\ [2] \\ [3] \\ [4] \end{pmatrix} \quad E = \begin{pmatrix} [12] \\ [13] \\ [14] \\ [23] \\ [24] \end{pmatrix} \quad F = \begin{pmatrix} [123] \\ [314] \end{pmatrix}$$

  - Note that the dimensions of C_0 (i.e., V), C_1 (i.e., E), and C_2 (i.e., F) are 4, 5, and 2, respectively. The Euler characteristic is

$$\chi(\mathcal{K}) = I_0 - I_1 + I_2 = 4 - 5 + 2 = 1.$$

- In C_2, $Ker(\partial_2) = \{0\}$. Thus, $H_2 = \{0\}$ and $b_2 = 0$. In C_1, the following chains are closed loops

  $\ell_1 := [12] + [23] - [13]$
  $\ell_2 := [13] + [34] + [41]$

  There are only two independent loops in $C_1(\mathcal{K})$, that are ℓ_1 and ℓ_2. They are in $Ker(\partial_1)$. They are also in $Im(\partial_2)$ because their interiors are the 2-simplices [123] and [314]. Thus, $Ker(\partial_1) = Im(\partial_2)$ and $H^1(\mathcal{K}) = \{0\}$. We have $b_0 = 1$, $b_1 = 0$, $b_2 = 1$.

- In C_0, $Im(\partial_1) = \langle \{[2] - [1], [3] - [1], [4] - [1]\} \rangle$. Note that $\{[3] - [2], [4] - [3]\}$ are in $Im(\partial_1)$ but not independent. The $Ker(\partial_0) =< \{1, 2, 3, 4\} >$. Thus, $b_0 = \dim H_0 = \dim Ker(\partial_0) - \dim(Im\partial_1) = 4 - 3 = 1$. We get $b_0 - b_1 + b_2 = 1$.

- Example 2: Circle. $b_0 = 1$, $b_1 = 1$, $b_2 = 0$.
- Example 3: Sphere S^2. C_2 is nontrivial. But C_1 is trivial. $b_0 = 1$, $b_1 = 0$, $b_2 = 1$.
- Example 4: Torus. H_1 has two independent elements. $b_0 = 1$, $b_1 = 2$, $b_2 = 1$.

Theorem 6.1. *Homology is topological invariant: If X and Y are two triangulable topological spaces which are homeomorphic to each other. Let $\mathcal{K}_X$ and $\mathcal{K}_Y$ are the two simplicial complexes corresponding to X and Y. Then $H_k(C(\mathcal{K}_X)) = H_k(C(\mathcal{K}_Y))$.*

Since the homological is invariant in any triangulation, we can write $H_k(C(\mathcal{K}_X))$ by $H_k(X)$.

Theorem 6.2. *If X is connected, then* $\dim H_0(X) = 1$.

Proof. Any two points p_1 and p_2 are connected, which means that there exists edges $(p^{(1)}, p^{(2)})$, $(p^{(2)}, p^{(3)})$,$\cdots$, $(p^{(m-1)}, p^{(m)})$ with $p^{(1)} = p_1$ and $p^{(m)} = p_2$. Then $\partial \left((p^{(1)}, p^{(2)}) + \cdots (p^{(m-1)}, p^{(m)}) \right) = p_2 - p_1$. Thus, any two points in $\mathcal{K}$ are equivalent to each other w.r.t. $im(\partial)$. □

Theorem 6.3 (The Euler-Poincaré Theorem)**.** *Let $\mathcal{K}$ be an m-dimensional simplicial complex. Then $\chi(\mathcal{K})$ is related to the Betti number by*

$$\chi(\mathcal{K}) = \sum_{k=0}^{m} (-1)^k b_k(\mathcal{K}).$$

Proof.

1. Let I_k be the number of generators of C_k, then

$$I_k = \dim C_k = \dim(Ker\partial_k) + \dim(Im\partial_k).$$

2. Let b_k be the number of generators of the space H_k, where

$$H_k = Ker(\partial_k)/Im(\partial_{k+1}),$$

then

$$b_k = \dim(Ker\partial_k) - \dim(Im\partial_{k+1}).$$

3. The Euler characteristic is

$$\begin{aligned}\chi(\mathcal{K}) &= \sum_{k=0}^{m}(-1)^k I_k \\ &= \sum_{k=0}^{m}(-1)^k \left[\dim(Ker\partial_k) + \dim(Im\partial_k)\right] \\ &= \sum_{k=0}^{m}[(-1)^k \left[\dim(Ker\partial_k) - \dim(Im\partial_{k-1})\right] \\ &= \sum_{k=0}^{m}(-1)^k b_k.\end{aligned}$$

□

Reference: Nakahara, Geometry, Topology and Physics, 2003.

6.2.1 Co-chain complex and discrete differential forms

Co-chain and co-boundary operator

- Given a chain complex $C(\mathcal{K})$, its dual

$$C_k^*(\mathcal{K}) := \{\alpha : C_k(\mathcal{K}) \to \mathbb{R} \text{ linear}\}$$

  is called the dual chain. The dual boundary operator $\partial_k^* : C_{k-1}^* \to C_k^*$ is also called the co-boundary operator. The sequence $(C^*(\mathcal{K}), \partial^*)$

$$0 \xrightarrow{\partial_0^*} C_0^*(\mathcal{K}) \xrightarrow{\partial_1^*} C_1^*(\mathcal{K}) \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{m-1}^*} C_{m-1}^*(\mathcal{K}) \xrightarrow{\partial_m^*} C_m^*(\mathcal{K}) \xrightarrow{i^*} 0$$

  is called the co-chain complex of the chain complex $(C(\mathcal{K}), \partial)$.

- We can formally define the dual basis in C_k^* to be: for each $\sigma \in \mathcal{K}^{(k)}$, we formally define an element σ^* such that for any $\sigma' \in \mathcal{K}^{(k)}$, we have

$$\langle \sigma^*, \sigma' \rangle = \delta_{\sigma'}^{\sigma}.$$

- The matrix representation of the co-boundary operator

$$\partial_k^* : C_{k-1}^*(\mathcal{K}) \to C_k^*(\mathcal{K})$$

  w.r.t. the basis $\{\sigma^* | \sigma \in \mathcal{K}^{(k)}\}$ is exactly ∂_k^T, the transpose of ∂_k.

de Rham complex and Reduction The differential forms are natural linear functionals on chain complex.

- de Rham complex: The sequence $(\Omega^k(M), d_k)$:

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} \Omega^{m-1}(M) \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{d_m} 0$$

  is called a de Rham complex, where $d_{k+1}d_k = 0$.

- For any $\alpha \in \Omega^k(M)$, we define $\mathcal{R}(\alpha) \in C_k^*$ by

$$\mathcal{R} : \Omega^k(M) \to C_k^*(M), \quad \langle \mathcal{R}\alpha, c\rangle := \int_c \alpha := \sum_i c^i \int_{\sigma_{k,i}} \alpha,$$

  where $c = c^i \sigma_{k,i}$. $\mathcal{R}$ is called a reduction operator.

- By Stokes' theorem, for any $\alpha \in \Omega^{k-1}(M)$ and any $c \in C_k(\mathcal{K})$, we have

$$\int_c d\alpha = \int_{\partial c} \alpha.$$

- We have the commuting diagram

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) \\ \downarrow \mathcal{R} & & \downarrow \mathcal{R} \\ C_k^*(\mathcal{K}) & \xrightarrow{\partial_{k+1}^*} & C_{k+1}^*(\mathcal{K}) \end{array}$$

  This is because

$$\langle \mathcal{R}(d\alpha^k), c_{k+1}\rangle = \int_{c_{k+1}} d\alpha^k = \int_{\partial c_{k+1}} \alpha^k = \langle \mathcal{R}(\alpha^k), \partial c_{k+1}\rangle = \langle \partial^* \mathcal{R}(\alpha^k), c_{k+1}\rangle.$$

Whitney elements To bridge the co-chain and differential forms is a *representation of the co-chain in terms of differential form.* The simpliest example is the following Whitney basis.

For each $\sigma \in \mathcal{K}^{(k)}$, we will construct a corresponding differential forms ϕ^σ defined on M such that there is a 1-1 correspondence between $\sigma \in \mathcal{K}^{(k)}$ and ϕ^σ. We start from 0-form.

- 0-form: Suppose $\mathcal{K}^{(0)} = \{v_i | i = 1, ..., n_0\}$. For each $[i] := v_i \in \mathcal{K}^{(0)}$, we want to construct a function ϕ^i on M such that

$$\phi^i(v_j) = \delta^i_j$$

  and ϕ^j is linear on every simplices. More precisely, suppose $v \in \sigma_m \subset M$, where $\sigma_m = [w_0, w_1, ..., w_m]$ is an m-simplex containing v. Suppose $v = \sum_j x^j w_j$. Then

$$\phi^i(v) := \sum_j x^j \phi^i(w_j).$$

- 1-form: We look for the real-valued one form which is dual to $[i_0, i_1]$.

$$\phi^{[i_0,i_1]} := \phi^{i_0} d\phi^{i_1} - \phi^{i_1} d\phi^{i_0}.$$

Note that

$$\phi^{i_0} + \phi^{i_1} = 1 \text{ on } [i_0, i_1].$$

This leads to $d\phi^{i_0} = -d\phi^{i_1}$ on $[i_0, i_1]$. We have

$$\begin{aligned}\int_{[i_0,i_1]} \phi^{[i_0,i_1]} &= \int_{[i_0,i_1]} \phi^{i_0} d\phi^{i_1} - \phi^{i_1} d\phi^{i_0} \\ &= \int_{[i_0,i_1]} \phi^{i_0} d\phi^{i_1} + \phi^{i_1} d\phi^{i_1} \\ &= \int_{[i_0,i_1]} \left(\phi^{i_0} + \phi^{i_1}\right) d\phi^{i_1} \\ &= \int_{[i_0,i_1]} d\phi^{i_1} = 1.\end{aligned}$$

If we integrate $\phi^{[i_0,i_1]}$ over $[i_0, i_2]$ with $i_2 \neq i_1$. In this case, $\phi^{i_1} \equiv 0$ on $[i_0, i_2]$. This leads to $\phi^{[i_0,i_1]} = d\phi^{i_1} - \phi^{i_1} d\phi^{i_0} = 0$ on $[i_0, i_2]$. Thus,

$$\int_{[i_0,i_2]} \phi^{[i_0,i_1]} = 0.$$

In general, you can check that

$$\int_{[j_0,j_1]} \phi^{[i_0,i_1]} = \delta^{i_0 i_1}_{j_0 j_1},$$

where

$$\delta^{i_0,i_1}_{j_0,j_1} = \begin{cases} 1 & \text{if } (i_0, i_1) \leftrightarrow (j_0, j_1) \text{ is an even permutation} \\ -1 & \text{if } (i_0, i_1) \leftrightarrow (j_0, j_1) \text{ is an odd permutation} \\ 0 & \text{if } \{i_0, i_1\} \neq \{j_0, j_1\}. \end{cases}$$

- k-form: In general, let $\sigma = [i_0, ..., i_k]$ be a k-simplex, the corresponding dual k-form is

$$\phi^\sigma := k! \sum_{\ell=0}^{k} (-1)^\ell \phi^{i_\ell} d\phi^{i_0} \wedge \cdots \wedge \widehat{d\phi^{i_\ell}} \wedge \cdots \wedge d\phi^{i_k}.$$

First, we have

$$\sum_{\ell=0}^{k} \phi^{i_\ell} = 1 \text{ on } [i_0, ..., i_k].$$

This is because it is 1 at the vertices $i_0, ..., i_k$ and each function is a linear function between 0 and 1. The above equation leads to

$$d\phi^{i_0} = -\sum_{j=1}^{k} d\phi^{i_j} \text{ on } [i_0, ..., i_k].$$

We use this to replace $d\phi^{i_0}$ in the expression

$$\phi^\sigma := k! \sum_{\ell=0}^{k} (-1)^\ell \phi^{i_\ell} d\phi^{i_0} \wedge \cdots \wedge \widehat{d\phi^{i_\ell}} \wedge \cdots \wedge d\phi^{i_k}.$$

This gives

$$\phi^\sigma = k! \sum_{\ell=0}^{k} \phi^{i_\ell} d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_k} = k! d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_k}.$$

By taking the vector $X_\ell := v_{i_\ell} - v_{i_0}$, we see that

$$d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_k}(X_1, ..., X_k) = \frac{1}{k!},$$

the volume of the simplex $[i_0, ..., i_k]$. Thus,

$$\int_{[i_0,...,i_k]} \phi^{[i_0,...,i_k]} = 1.$$

In general, we have

$$\int_{[j_0,...,j_k]} \phi^{[i_0,...,i_k]} = \delta^{i_0...,i_k}_{j_0...,j_k}.$$

Discrete k-forms

- We consider the space spanned by the Whitney elements

$$\Omega^k_h(\mathcal{K}) := \text{Span } \{\phi^\sigma | \sigma \in \mathcal{K}^{(k)}\}$$

  The sequence $(\Omega^k_h(\mathcal{K}), d_k)$:

$$0 \xrightarrow{i} \Omega^0_h(\mathcal{K}) \xrightarrow{d_0} \Omega^1_h(\mathcal{K}) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} \Omega^{m-1}_h(\mathcal{K}) \xrightarrow{d_{m-1}} \Omega^m_h(\mathcal{K}) \xrightarrow{d_m} 0$$

  is called a Whitney complex.

- We note that

$$d\phi^{[i_0,...,i_k]} = d\left(k! \sum_{\ell=0}^{k} \phi^{i_\ell} d\phi^{i_1} \wedge \cdots \wedge d\phi^{i_k}\right) = (k+1)! d\phi^{i_0} \wedge \cdots \wedge d\phi^{i_k},$$

  Thus,

$$d_{k+1} d_k = 0.$$

- For $\alpha \in \Omega_h^k(\mathcal{K})$ and a chain $c = c^i \sigma_{k,i} \in C_k(\mathcal{K})$, we define

$$\langle \alpha, c \rangle := \int_c \alpha := \sum_i c^i \int_{\sigma_{k,i}} \alpha.$$

  There is a natural isomorphism

$$C_k^*(\mathcal{K}) \xrightarrow{i} \Omega_h^k(\mathcal{K}) \qquad \text{by } \sigma^* \mapsto \phi^\sigma$$

  By Stokes' theorem, for any $\alpha \in \Omega_h^{k-1}(\mathcal{K})$ and any $c \in C_k(\mathcal{K})$, we have

$$\langle d\alpha, c \rangle = \langle \alpha, \partial c \rangle = \langle \partial^* \alpha, c \rangle.$$

  Thus, we can identify d_k as ∂_k^*. The matrix representation of

$$d_{k-1} : \Omega_h^{k-1}(\mathcal{K}) \to \Omega_h^k(\mathcal{K})$$

  with respective to the Whitney basis is exactly ∂_k^*, which is ∂_k^T, an $n_k \times n_{k-1}$ matrix.

We can also choose other basis in $C^k(\mathcal{K})$. Say $\{\bar{\sigma}\}$, and there is an invertible map $\mathcal{F} : \{\sigma\} \to \{\bar{\sigma}\}$. The mapping from $\{\bar{\sigma}\}$ to a differential form by

$$\mathcal{I} : \bar{\sigma} \mapsto \bar{\phi}^{\bar{\sigma}}$$

is called a reconstruction operator. The composition:

$$\Omega^k(M) \xrightarrow{\mathcal{R}} C^k(\mathcal{K}) \xrightarrow{\mathcal{F}} C^k(\mathcal{K}) \xrightarrow{\mathcal{I}} \Omega_h^k(M)$$

is a natural projection $\pi_h : \Omega^k(M) \to \Omega_h^k(\mathcal{K})$. Here, h is the mesh size of the simplicial complex $\mathcal{K}$. An important formula is the following commuting diagram:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d_k} & \Omega^{k+1}(M) \\ \downarrow{\scriptstyle \pi_h} & & \downarrow{\scriptstyle \pi_h} \\ \Omega_h^k(\mathcal{K}) & \xrightarrow{\partial_{k+1}^*} & \Omega_h^{k+1}(\mathcal{K}) \end{array}$$

6.3 Hodge $\star$ operator

A discrete 0-form is defined on vertices. Its Hodge star is a discrete n-form defined on n-cells surrounding the corresponding vertices. These cells form a dual mesh of the original triangle mesh. The dual mesh of a Delaunay mesh is called a Voronoi mesh.

6.3.1 Dual mesh

- **Voronoi mesh:** Given a set of points $S = \{p_1, ..., p_N\} \subset \mathbb{R}^m$. A Voronoi cell R_k associated with p_k is the region

$$R_k = \{x \in \mathbb{R}^m | d(x, p_k) \le d(x, p_j) \text{ for all } j \ne k\}$$

  The Voronoi diagram is the tuple of cells $(R_k)_{p_k \in S}$. They can be generated from a set of seeds S, see the movie on wiki Voronoi Diagram.

- Let us use example to explain the dual mesh of a Delaunay mesh. Consider a simplicial complex $\mathcal{K}$ with dimension $m = 2$. On the plane, we have a vertex $[i_0]$ surrounded by six vertices $[i_1], ..., [i_6]$. The edges are $[i_0, i_j]$, $j = 1, ..., 6$. The faces are $[i_0, i_1, i_2]$, $[i_0, i_2, i_3], ..., [i_0, i_5, i_6]$ and $[i_0, i_6, i_1]$. For each face, we will define its dual, which will be a vertex. For instance, the dual of $\sigma_2 = [i_0, i_1, i_2]$ is a vertex. We define it to be the circumcenter of σ_2, and denote it by $c(\sigma_2)$. There are six such circumcenters corresponding the six faces. For a edge, say $[i_0, i_2]$, we define its dual to be an edge, which connecting $c([i_0, i_1, i_2])$ and $c([i_0, i_2, i_3])$. The orientation of this edge is chosen to be $[c([i_0, i_1, i_2]), c([i_0, i_2, i_3])]$ so that the orientation from v to w is consistent to the orientation of the orientations of the 2D plane. Here, $v = [i_0, i_2]$ and $w = [c([i_0, i_1, i_2]), c([i_0, i_2, i_3])]$. There are six oriented edges. Lastly, the dual of vertex $[i_0]$ is a honey comb surrounded by the dual edges $[[c([i_0, i_1, i_2]), c([i_0, i_2, i_3])], ..., [c([i_0, i_6, i_1]), c([i_0, i_1, i_2])]$. Its orientation is defined so that its is consistent to the plane orientation.

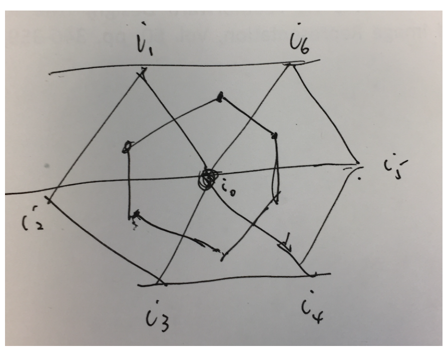


Figure 6.4: Dual Mesh

- Given a k-simplex $\sigma = [v_0, ..., v_k]$ in $\mathbb{R}^m$. The circumcenter is the unique point in the k-dimensional affine space spanned by σ that is equidistant from the $k+1$ nodes of σ. We denote it by $c(\sigma)$.

- The circumcenter $c[i_0, ..., i_k]$ can be found recursively from 1-simplices to k-simplex. For example, consider a 3-simplex $[i_0, i_1, i_2, i_3]$. We first find $c[i_0, i_1], ..., c[i_3, i_0]$, which are midpoints of the edges. Then we find $c[i_0, i_1, i_2]$, which is the intersection of the two straight lines emitted from $c[i_0, i_1]$ and $c[i_1, i_2]$ on the plane $[i_0, i_1, i_2]$ and are perpendicular to the simplices $[i_0, i_1]$ and $[i_1, i_2]$.

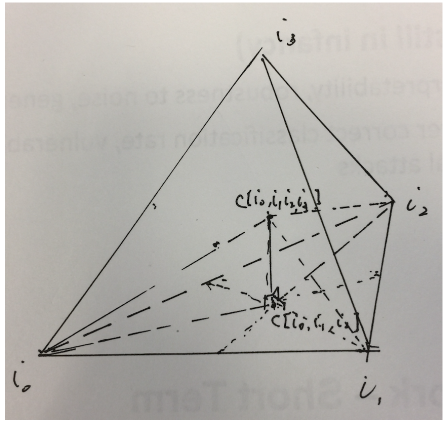


Figure 6.5: The vector $[c[i_0, i_1, i_2, i_3], c[i_0, i_1, i_2]] \perp [i_0, i_1, i_2]$. The circumcenters can be found recursively from $c[i_0, i_1], c[i_1, i_2], c[i_2, i_3]$, then $c[i_0, i_1, i_2]$ and then $c[i_0, i_1, i_2, i_3]$.

- The circumcentric subdivision of a simplicial complex is $[c(\sigma_0), ..., c(\sigma_k)]$, where $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_k$, or σ_i is a proper face of σ_j for $i < j$.

- The circumcentric dual of a k-simplex σ_k is defined to be

$$\widetilde{\sigma}_k := \sum_{\sigma_k \prec \cdots \prec \sigma_n} \varepsilon_{\sigma_k, ..., \sigma_n} [c(\sigma_k), ..., c(\sigma_n)].$$

  Here, the sum is over all circumcentric subdivision that containing σ_k. This dual $\widetilde{\sigma}_k$ is not a simplex in general. We call it a cell.

- The sign $\varepsilon_{\sigma_k, ..., \sigma_n}$ is chosen as below. First, for $\sigma_k = [v_0, v_1, ..., v_k]$, it determines an orientation of the k-vector $u_1 \wedge \cdots \wedge u_k$, where $u_i := v_i - v_0$. For the dual $[c(\sigma_k), ..., c(\sigma_n)]$, it also determines an orientation of the $(n-k)$-vector $w_{k+1} \wedge ... \wedge w_n$, where $w_j := c(\sigma_j) - c(\sigma_k)$. The sign $\varepsilon_{\sigma_k, ..., \sigma_n}$ is chosen so that the orientation of

$$u_1 \wedge \cdots \wedge u_k \wedge (\varepsilon_{\sigma_k, ..., \sigma_n} w_{k+1} \wedge \cdots \wedge w_n)$$

  is consistent to the volume form of the n-dimensional space.

- The boundary operator for $\widetilde{\sigma}_k$ is defined to be

$$\partial \widetilde{\sigma}_k := \sum_{\sigma_k \prec \cdots \prec \sigma_n} \varepsilon_{\sigma_k, ..., \sigma_n} \partial [c(\sigma_k), ..., c(\sigma_n)]$$

- The dual of a simplicial complex $C_k(\mathcal{K})$ is a cell complex. The k-cell $C_{n-k}(\widetilde{\mathcal{K}})$ is generated by
$$C_{n-k}(\widetilde{\mathcal{K}}) = \text{ Span } \{\widetilde{\sigma_k} | \sigma_k \in \mathcal{K}^{(k)}\}$$
The sequence $(C_k(\widetilde{K}), \partial)$ forms a cell complex.

Remark To construct a dual mesh of a triangle mesh, we don't have to choose circumcenters of the corresponding simplices. Any point inside a simplex works. Another common choice is the barycenter.

6.3.2 Discrete Hodge $\star$

- Given $\mathcal{K}$, we have chain complex $C_k(\mathcal{K})$. From $\mathcal{K}$, we can define dual mesh and dual complex $\tilde{\mathcal{K}}$, and the corresponding $(n-k)$-chain complex $C_{n-k}(\tilde{\mathcal{K}})$. For any $\sigma \in \mathcal{K}^{(k)}$, we define the orientation $\tilde{\sigma}$ so that
$$\sigma \wedge \widetilde{\sigma} = |\sigma||\widetilde{\sigma}|.$$

- For any $\sigma \in \mathcal{K}^{(k)}$, we can construct ϕ^σ which is a k-form. Thus, $\phi^\sigma \in C^k(\mathcal{K})$. For Whitney element ϕ^σ satisfy
$$\langle \phi^\sigma, \sigma' \rangle := \int_{\sigma'} \phi^\sigma = \delta^\sigma_{\sigma'}.$$
for any $\sigma, \sigma' \in \mathcal{K}^{(k)}$. In general, it satisfies
$$\langle \phi^\sigma, \sigma' \rangle := \int_{\sigma'} \phi^\sigma = a^\sigma_{\sigma'},$$
where the matrix $(a^\sigma_{\sigma'})$ is in general diagonal dominant and positive definite, unless the mesh is poor.

- We can define continuous Hodge star for ϕ^σ. For any pair $\sigma, \sigma' \in \mathcal{K}^{(k)}$, we want the matrix
$$b^\sigma_{\sigma'} := \int_{\widetilde{\sigma'}} (\star \phi^\sigma)$$
to be a positive-definite matrix. For Whitney element, we can compute this matrix directly, and find
$$\int_{\widetilde{\sigma'}} (\star \phi^\sigma) := \frac{|\widetilde{\sigma'}|}{|\sigma|} \delta^\sigma_{\sigma'}.$$
The corresponding matrix $(b^\sigma_{\sigma'})$ is a diagonal matrix.

- For Witney element, the star operator is then defined as the follows. It maps a k-form α to $\Omega_{n-k}(\widetilde{K})$ such that
$$\langle \star\alpha, \widetilde{\sigma} \rangle = \frac{|\widetilde{\sigma}|}{|\sigma|} \langle \alpha, \sigma \rangle \text{ for } \sigma \in C_k(\mathcal{K}).$$

6.3.3 A discrete Laplacian

Let us consider the Poisson equation $\triangle u = g$ on a 2-manifold M, with $g \in C^0(M)$. The Laplacian $\triangle = \star d \star d$. The poisson equation is

$$\star d \star du = g.$$

Or

$$d \star du = \star g.$$

In the discrete setting, let $\mathcal{K} = \{V, E, F\}$ and the above equation in discrete form is

$$\star_2 g = d_1^* \star_1 d_0 u = \partial_1 \frac{s|\tilde{e}|}{|e|} \partial_1^T u = Lu.$$

Here, $e = [i, j]$, $\tilde{e}$ is the orthogonal bisector of e connecting $c[i, j, k]$ and $c[i, j, \ell]$, and $s = 1$ (resp. -1) if the orientation of $(e, \tilde{e})$ is positive (resp. negative). One can show that

$$\frac{s|\tilde{e}|}{|e|} = \frac{1}{2} \left(\cot \theta_{ij}^k + \cot \theta_{ji}^\ell \right).$$

Here, the sizes of the matrices are: ∂_1^T is $|E| \times |V|$, $\star_1$ is $|V| \times |V|$, and ∂_1 is $|V| \times |E|$. The Laplacian L is $|V| \times |V|$.

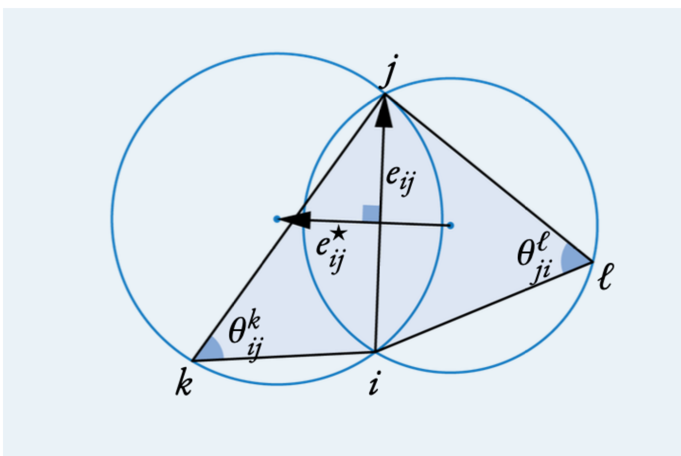


Figure 6.6: Copied from Albert Chern's note

6.4 Homology Generators and Harmonic Bases

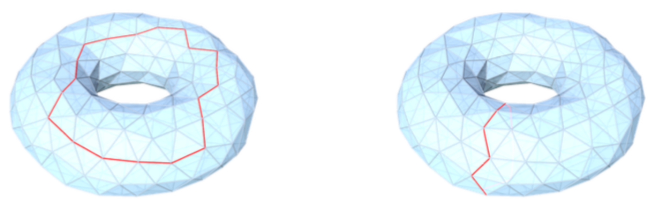


Figure 7.1: *Two examples of closed 1-chain that is not the boundary of any 2-chain.*

Figure 6.7: Copied from Chern and Schröder's lecture note

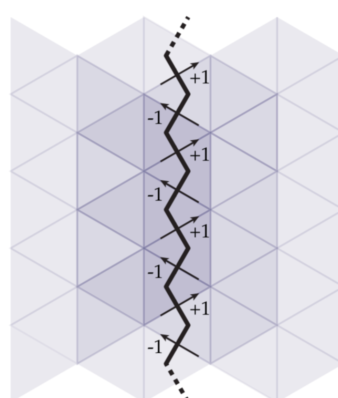


Figure 6.8: Copied from Crane's lecture note

Chapter 7

Connection, Parallel Transport, Curvatures

7.1 Motivation of covariant derivative and connection

- **Intuition for parallel transport** Let M be an m-manifold. A tangent vector field $X : M \to TM$ is a smooth map with $X_p \in T_p(M)$ for any $p \in M$. Let us denote the space of all tangent vector fields on M by $\mathfrak{X}(M)$.

  Suppose we want to study how a vector field Y changed with respect to another vector field X at a point p. Let $\boldsymbol{\gamma}(t)$ be the integral curve of X with $\boldsymbol{\gamma}(0) = p$. Let $q = \boldsymbol{\gamma}(\Delta t)$ for a small Δt. The relative change of Y w.r.t. X is the limit of

  $$\frac{Y(q) - Y(p)}{\Delta t}.$$

  But $Y(q) \in T_q(M)$ and $Y(p) \in T_pM$ are in different tangent space, which are not allowed to take subtraction unless they are both embedded in an ambient vector space. Therefore, we should "parallel transport" $Y(p)$ to $\tilde{Y}(q) \in T_q(M)$ in order to take subtraction. Thus, we should define what this "parallel transportation" means. Suppose (U, x) is a coordinate chart. A point p has coordinate x and a neighboring point q has coordinate $x + \Delta x$. The tangent vector Y can be expressed as

  $$Y = Y^\mu \frac{\partial}{\partial x^\mu}.$$

  Suppose the tangent vector $Y(x)$ is parallel transported to $\tilde{Y}(x + \Delta x)$ at $x + \Delta x$. We expect the tilde operator is linear and

  $$\tilde{Y}(x + \Delta x) - Y(x) \propto \Delta x$$

  This leads to

  $$\tilde{Y}^\mu(x + \Delta x) = Y^\mu(x) - \Gamma^\mu_{\nu\lambda}(x) Y^\lambda(x) \Delta x^\nu$$

for some smooth functions $\Gamma^\mu_{\nu\lambda}(x)$. With the concept of parallel transportation, the covariant derivative of Y with respective to $\frac{\partial}{\partial x^\nu}$ is then defined to be

$$\frac{Y^\mu(x+\Delta x)-\tilde{Y}^\mu(x+\Delta x)}{\Delta x^\nu}\frac{\partial}{\partial x^\mu} \to \left(\frac{\partial Y^\mu}{\partial x^\nu}+\Gamma^\mu_{\nu\lambda}Y^\lambda\right)\frac{\partial}{\partial x^\mu}$$

In general, $X = X^\nu \frac{\partial}{\partial x^\nu}$. We define the covariant derivative of Y w.r.t. X to be

$$\nabla_X Y := X^\nu \left(\frac{\partial Y^\mu}{\partial x^\nu}+\Gamma^\mu_{\nu\lambda}Y^\lambda\right)\frac{\partial}{\partial x^\mu}.$$

The operator $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ (by $(X,Y)\mapsto\nabla_X Y$) is called an affine connection. It has the following properties.

$$(a)\ \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$$
$$(b)\ \nabla_{fX+Y}Z = f\nabla_X Z + \nabla_Y Z$$
$$(c)\ \nabla_X fY = Xf + f\nabla_X Y.$$

To be more concise, we should take these properties as the definition of connection. And, in a coordinate chart (U,x), define $\Gamma^\mu_{\nu\lambda}$ to be

$$\nabla_{e_\nu}e_\lambda = \Gamma^\mu_{\nu\lambda}e_\mu$$

where

$$e_\mu = \frac{\partial}{\partial x^\mu}.$$

If $X = X^\nu e_\nu$ and $Y = Y^\mu e_\mu$, then

$$\begin{aligned}\nabla_X Y &= X^\nu \nabla_{e_\nu}(Y^\mu e_\mu)\\ &= X^\nu\left(\frac{\partial Y^\mu}{\partial x^\nu}e_\nu + Y^\lambda \nabla_{e_\nu}e_\lambda\right)\\ &= X^\nu\left(\frac{\partial Y^\mu}{\partial x^\nu}+Y^\lambda\Gamma^\mu_{\nu\lambda}\right)e_\mu.\end{aligned}$$

The coefficients $\Gamma^\mu_{\nu\lambda}$ are called the connection coefficients. The operator ∇ or the corresponding connection coefficients define how vectors are parallel transported locally. If we know $\Gamma^\mu_{\nu\lambda}$, we can parallel transport vector fields locally, the covariant derivatives are thus defined. Conversely, the definition of covariant derivative defines the concept of parallel transportation. This means that if $\boldsymbol{\gamma}$ is an integral curve of a vector field X, if Y is a vector field with $\nabla_X Y = 0$, then we say Y is parallel transported along $\boldsymbol{\gamma}$.

Remark Suppose M is a Riemannian manifold and $\{e_\mu\}$ is a frame. We have defined earlier that the connection $\omega_{\lambda\mu}$ to be the 1-form

$$\omega_{\lambda\mu}(e_\nu) = \langle\nabla_\nu e_\lambda, e_\mu\rangle = \Gamma^\mu_{\nu\lambda}.$$

- **Intuition for Gauge transformation** The parallel transport is not only applied to tangent vectors, it can also applied to cotangent vectors, tensor fields, in general sections of fibre bundles. Let us consider a $\mathbb{C}$-fiber bundle $\mathbb{C}M$. That is, given a manifold M, for each point $x \in M$, we associate it with a space $\mathbb{C}_x$, just like the tangent bundle TM where the associated fibre is the tangent space T_xM. For a point $s \in \mathbb{C}M$, it lies on $\mathbb{C}_xM$ for some x. We define the projection $\pi s = x$. A mapping $\Psi : M \to \mathbb{C}M$ with $\pi\Psi = id$ is called a section. On $\mathbb{C}_x$, we can choose an axis as the real axis, call it e_x, a unit vector. The corresponding imaginary axis is ie_x. We express Ψ as

$$\Psi(x) = \psi^r(x)e_x + \psi^i ie_x = \psi(x)e_x.$$

Let X be a tangent vector field on M. The covariant derivative of Ψ w.r.t. X is

$$\nabla_X\Psi = \nabla_X\psi e_x + \psi\nabla_X e_x$$

Here, $\nabla_X\psi := X\psi^r + iX\psi^i$. Since $\langle e_x, e_x\rangle = 1$, we have $\nabla_X e_x \perp e_x$. Therefore,

$$\nabla_X e_x = i\langle A_x, X\rangle e_x$$

for some 1-form A_x. Thus, we get

$$\nabla_X\Psi = [\nabla_X\psi + i\langle A_x, X\rangle\psi]e_x$$

A gauge transform is a transformation of the real unit vector on $\mathbb{C}_x$. That is $e_x \mapsto \tilde{e}_x$. Since both are unit vector, we can express $\tilde{e}_x = e^{-i\phi(x)}e_x$. With this transformation,

$$\Psi(x) = \psi(x)e_x = (\psi(x)e^{i\phi(x)})\tilde{e}_x := \tilde{\psi}(x)\tilde{e}_x.$$

The covariant derivative of $\tilde{e}_x$ is

$$\nabla_X\tilde{e}_x = \nabla_X(e^{-i\phi(x)}e_x) = -i\nabla_X\phi\tilde{e}_x + e^{-i\phi(x)}(i\langle A_x, X\rangle e_x) = i(\langle A_x, X\rangle - \nabla_X\phi(x))\tilde{e}_x.$$

The covariant derivative of Ψ under $\tilde{e}_x$ is

$$\begin{aligned}\nabla_X\Psi(x) &= \nabla_X(\tilde{\psi}(x)\tilde{e}_x)\\ &= \nabla_X\tilde{\psi}\tilde{e}_x + \tilde{\psi}\nabla_X\tilde{e}_x\\ &= \left(\nabla_X\tilde{\psi} + i(\langle A_x, X\rangle - \nabla_X\phi(x))\tilde{\psi}\right)\tilde{e}_x.\end{aligned}$$

Thus, gauge transformation $e_x \mapsto \tilde{e}_x := e^{-i\phi(x)}e_x$ corresponds to

$$\begin{aligned}\psi(x) &\mapsto \tilde{\psi}(x) := e^{i\phi(x)}\psi(x)\\ A_x &\mapsto \tilde{A}_x := A_x - d\phi(x).\end{aligned}$$

7.2 Affine connection and parallel transport

Let M be a differential manifold and $\mathfrak{X}(M)$ be the space of its tangent vector fields.

Affine connection for vector fields

Definition 7.1. An affine connection on $\mathfrak{X}(M)$ is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $(X, Y) \mapsto \nabla_X Y$ such that

$$
\begin{aligned}
&(a)\ \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \\
&(b)\ \nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z \\
&(c)\ \nabla_X fY = Xf + f\nabla_X Y.
\end{aligned}
$$

Property (c) means that ∇_X acts as a derivative and it satisfies Leibnitz rule. $\nabla_X Y$ is called the covariant derivative of Y in the direction X.
Remark We may also think ∇ maps X to ∇_X, which is an endomorphism on TM , that is, a smooth map $TM \to TM$ and linear on T_pM for all $p \in M$. That is

$$\nabla : \mathfrak{X}(M) \to End(TM).$$

Equivalently, ∇ is an $End(TM)$-valued 1-form which satisfies properties (a),(b),(c).

Coordinate representation of affine connection The affine connection has the following representation in a coordinate chart. Suppose (U, x) is a chart on M with tangent vector fields $e_i := \frac{\partial}{\partial x^i}$. Define

$$\nabla_{e_\nu} e_\lambda = \Gamma^\mu_{\nu\lambda} e_\mu$$

Suppose $X = X^i e_i$, $Y = Y^i e_i$. Then

$$
\begin{aligned}
\nabla_X Y &= X^\nu \nabla_{e_\nu} (Y^\mu e_\mu) \\
&= X^\nu \left(\frac{\partial Y^\mu}{\partial x^\nu} e_\nu + Y^\lambda \nabla_{e_\nu} e_\lambda \right) \\
&= X^\nu \left(\frac{\partial Y^\mu}{\partial x^\nu} + Y^\lambda \Gamma^\mu_{\nu\lambda} \right) e_\mu.
\end{aligned}
$$

Parallel Transport An affine connection on M naturally induces a covariant derivative of vector field along any curve $\boldsymbol{\gamma}$.

Proposition 7.1. *Let M be a manifold with an affine connection ∇. Let $\boldsymbol{\gamma}$ be a curve on M. Then for any vector field V defined $\boldsymbol{\gamma}$, it associates with a unique covariant derivative DV/dt which has the following properties*

(a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$

(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$

(c) If V is induced by a vector field $Y \in \mathfrak{X}(M)$, i.e. $V(t) = Y(\boldsymbol{\gamma}(t))$, then $\frac{DV}{dt} = \nabla_{\boldsymbol{\gamma}'} Y$.

Ref. Do Carmo, Riemannian Geometry.

Definition 7.2. Let M be a manifold with an affine connection ∇. Let $\boldsymbol{\gamma}$ be a curve on M and V be a vector field defined on $\boldsymbol{\gamma}$. We say V is parallel transported along $\boldsymbol{\gamma}$ if

$$\frac{DV}{dt} = 0.$$

In a local coordinate chart (U, x), we can express $\boldsymbol{\gamma}(t) = (x^1(t), ..., x^n(t))$ and $V(t) = v^1(t)e_1(\boldsymbol{\gamma}(t)) + \cdots + v^n(t)e_n(\boldsymbol{\gamma}(t))$, we have

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{ij} v^j \frac{dx^i}{dt} \Gamma^k_{ij} \right) e_k.$$

Affine connection for 1-forms If ω is a 1-form, we should define covariant derivative of ω so that the Leibnitz rule is valid. That is,

$$X\langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle.$$

The representation of this covariant derivative is the following. From

$$e_i \langle dx^j, e_k \rangle = \langle \nabla_{e_i} dx^j, e_k \rangle + \langle dx^j, \nabla_{e_i} e_k \rangle$$

we get

$$\nabla_{e_i} dx^j = -\Gamma^j_{ik} dx^k.$$

Suppose $\omega = \omega_i dx^i$ and $X = X^i e_i$. Then

$$\nabla_X \omega = X\omega_k dx^k - \Gamma^j_{ik} X^i \omega_j dx^k.$$

Affine connection for tensors Suppose T_1 and T_2 are two tensors on M and $X \in \mathfrak{X}(M)$. We define

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2).$$

Affine connection for sections on a complex line bundle $\mathbb{C}M$ Let $\mathbb{C}M$ be a complex line bundle and $\Psi : M \to \mathbb{C}M$ be a section. On $\mathbb{C}_x$ ($x \in M$) we choose a unit vector e_x as the real axis, called the gauge. We define ie_x to be the unit vector on $\mathbb{C}$ which rotates e_x by 90° counterclockwise. Then any section map Ψ can be express as

$$\Psi(x) = \psi^r e_x + \psi^i (ie_x) := (\psi^r + i\psi^i)e_x := \psi e_x.$$

Let $X \in \mathfrak{X}(M)$. Define the covariant derivative

$$\nabla_X e_x := i\langle A_x, X \rangle e_x$$

Here, A_x is a 1-form, called the connection. The covariant derivative of Ψ is

$$\nabla_X \Psi = \nabla_X (\psi e_x) := [(\nabla_X + i\langle A_x, X \rangle)\psi]e_x,$$

where

$$\nabla_X \psi := X\psi^r + iX\psi^i.$$

Affine connection on a surface Let M be a surface. Its tangent plane can be viewed as a complex plane and TM can be treated as a complex line bundle. On every T_xM, we associate a unit vector e_x. Suppose $\boldsymbol{\gamma}$ is a path on M. We view $e_{\boldsymbol{\gamma}(s)}$ being a unit vector moving along the curve $\boldsymbol{\gamma}$. The covariant derivative

$$\nabla_{\boldsymbol{\gamma}'} e_x = i\langle A_x, \boldsymbol{\gamma}'\rangle e_x$$

measures how e_x changes along $\boldsymbol{\gamma}$. The motion of e_x on T_xM is indeed a rotation. The rotation speed is $\langle A_{\boldsymbol{\gamma}(t)}, \boldsymbol{\gamma}'(t)\rangle$. If $\boldsymbol{\gamma}$ is a closed curve, the phase change of $e_{\boldsymbol{\gamma}}$ is

$$\int_0^t \langle A_{\boldsymbol{\gamma}(\tau)}, \boldsymbol{\gamma}'(\tau)\rangle \, d\tau.$$

Let $V(0)$ is a tangent vector. We parallel transport $V(0)$ along $\boldsymbol{\gamma}$. We can measure the angle between $V(t)$ and $\boldsymbol{\gamma}'(t)$, called it $\theta(t)$. Then $\theta'(t) = \langle A_{\boldsymbol{\gamma}(t)}, \boldsymbol{\gamma}'(t)\rangle$. This means that A is the angular velocity of V which parallel transport along $\boldsymbol{\gamma}$.

7.3 Riemannian connections and Levi-Civita connections

Definition 7.3. Let M be a Riemannian manifold with metric $\langle\cdot,\cdot\rangle$. An affine connection ∇ on M is said to be compatible with the metric $\langle\cdot,\cdot\rangle$ if for any path $\boldsymbol{\gamma}$ on M and any pair of parallel vectors $P_1(t)$ and $P_2(t)$ along $\boldsymbol{\gamma}$, we have $\langle P_1(t), P_2(t)\rangle$ is a constant.

Proposition 7.2. *∇ is compatible with the metric $\langle\cdot,\cdot\rangle$ if and only if for any vector fields V and W along any path $\boldsymbol{\gamma}$,*

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle.$$

Proof. ($\Leftarrow$)
If $\frac{DV}{dt} = 0$ and $\frac{DW}{dt} = 0$, then

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle = 0.$$

Hence $\langle V(t), W(t)\rangle$ is a constant.
($\Rightarrow$)
We choose an orthonormal basis $\{P_1(0), ..., P_n(0)\}$ at $\boldsymbol{\gamma}(0)$. We parallel transport them along $\boldsymbol{\gamma}$. Because ∇ is compatible to the metric, we get $\{P_1(t), ..., P_n(t)\}$ are orthonormal on $T_{\boldsymbol{\gamma}(t)}M$. We express $V = v^i P_i$ and $W = w^i P_i$ along $\boldsymbol{\gamma}$. It follows

$$\frac{DV}{dt} = \frac{dv^i}{dt}P_i + v^i\frac{DP_i}{dt} = \frac{dv^i}{dt}P_i,$$

$$\frac{DW}{dt} = \frac{dw^i}{dt}P_i$$

Hence

$$\langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle = \sum_i \left(\frac{dv^i}{dt}w^i + v^i\frac{dw^i}{dt}\right) P_i = \frac{d}{dt}\langle V, W\rangle.$$

□

Corollary 7.1. *A connection ∇ on a Riemannian manifold M is compatible if and only if*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad \text{for any } X, Y, Z \in \mathfrak{X}(M).$$

Definition 7.4. An affine connection ∇ is called symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

In coordinate representation, $\Gamma^k_{ij} := \langle \nabla_{e_i} e_j, dx^k\rangle$, the connection is symmetric if $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Theorem 7.1 (Levi-Civita)**.** *On a Riemannian manifold M, there exists a unique affine connection which is*

(a) compatible with the metric

(b) symmetric.

Proof. (See Do Carmo)
We claim that we can express $\nabla_Y X$ in terms of metric and Lie brackets:

$$\begin{aligned}\langle Z, \nabla_Y X\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\}\end{aligned}$$

which can be obtained from

$$\begin{aligned}X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle\end{aligned}$$

For detail, see Do Carmo. □

Example 1: S^2

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \le \theta \le \pi, 0 \le \phi < 2\pi.$$

- Tangents:

$$\begin{aligned}\frac{\partial}{\partial\theta} &= e_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \frac{\partial}{\partial\phi} &= e_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)\end{aligned}$$

- Connection

$$\nabla_{e_\theta} e_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$$
$$\nabla_{e_\phi} e_\theta = \nabla_{e_\theta} e_\phi = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)$$

$$\begin{aligned}
\Gamma^\theta_{\theta\theta} &= \Gamma^\phi_{\theta,\theta} = 0 \\
\Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \langle \nabla_{e_\theta} e_\phi, e_\phi\rangle e_\phi / \|e_\phi\|^2 = \cot\theta \\
\Gamma^\theta_{\theta\phi} &= \Gamma^\theta_{\phi\theta} = \langle \nabla_{e_\theta} e_\phi, e_\theta\rangle e_\theta / \|e_\theta\|^2 = 0 \\
\Gamma^\theta_{\phi\phi} &= \langle \nabla_{e_\phi} e_\phi, e_\theta\rangle e_\theta / \|e_\theta\|^2 = -\cos\theta\sin\theta \\
\Gamma^\phi_{\phi\phi} &= 0.
\end{aligned}$$

7.4 Torsion and Curvature

Torsion and curvature are tensors The operation $Z = \nabla_X Y$ which read two tangent vectors X, Y and output an tangent vector Z. It is not a type $(2,1)$ tensor. A tensor is multilinear in its argument. A type (r,s) tensor is a multilinear function on

$$TM \times \cdots \times TM \times T^*M \times \cdots \times T^*M$$

In order to make such covariant derivative to be a tensor form, we define torsion. For second order covariant derivatives, we define the Riemann curvature tensor.

- **Torsion** $T : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

  Here, $[X,Y] := XY - YX$.

- **Curvature** $K : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$K(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proposition 7.3. *T and R are tensors:*

$$T(fX, gY) = fgT(X,Y)$$
$$R(fX, gY)(hZ) = fghR(X,Y)Z.$$

Proof. 1.

$$\begin{aligned}
T(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY}(fX) - [fX, gY] \\
&= f\nabla_X(gY) - g\nabla_Y(fX) - fX(gY) + gY(fX) \\
&= f(Xg)Y + fg\nabla_X Y - g(Yf)X - fg\nabla_Y X - f(Xg)Y - fgXY + g(Yf)X + gfYX \\
&= fg(\nabla_X Y - \nabla_Y X - XY + YX) \\
&= fgT(X,Y).
\end{aligned}$$

2.

$$
\begin{aligned}
&R(fX,gY)(hZ) = \nabla_{fX}\nabla_{gY}(hZ) - \nabla_{gY}\nabla_{fX}(hZ) - \nabla_{fX(gY)-gY(fX)}(hZ)\\
&= f\nabla_X(g\nabla_Y(hZ)) - g\nabla_Y(f\nabla_X)(hZ) - \left(fXg\nabla_Y - fg\nabla_{XY} + gYf\nabla_X + gf\nabla_{YX}\right)(hZ)\\
&= f(Xg)\nabla_Y(hZ) + fg\nabla_X\nabla_Y(hZ) - g(Yf)\nabla_X(hZ) - gf\nabla_Y\nabla_X(hZ)\\
&\quad + \left(-fXg\nabla_Y + gYf\nabla_X - fg\nabla_{[X,Y]}\right)(hZ)\\
&= fg\left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)(hZ)\\
&= fg\left(\nabla_X((Yh)Z + h\nabla_Y Z) - \nabla_Y((Xh)Z + h\nabla_X Z - ([X,Y]h)Z - h\nabla_{[X,Y]}Z\right)\\
&= fg\left((XYh)Z + Yh\nabla_X Z + Xh\nabla_Y Z + h\nabla_X\nabla_Y Z - (YXh)Z - Xh\nabla_Y Z - Yh\nabla_X Z - h\nabla_Y\nabla_X Z\right.\\
&\quad \left. -(XYh - YXh)Z - h\nabla_{[X,Y]}Z\right)\\
&= fgh\left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)Z
\end{aligned}
$$

□

Representation of torsion and curvature in a coordinate chart Let (U,x) be a coordinate chart with $e_i := \partial/\partial x^i$.

Chapter 8

Integral Geometry

8.1 Variation formulation of curvature

8.1.1 Wedge product for vector-valued differential forms

In this section we will take approach of curvature through embedding. Let M be a surface and

$$f : M \to \mathbb{R}^3$$

be an embedding map. We can write $f = (x, y, z)$. It is convenient to introduce the following vector-valued differential forms and the corresponding wedge product.

Definition 8.1. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two $\mathbb{R}^3$-valued 1-forms, their wedge product is a $\mathbb{R}^3$-valued 2-form on M defined to be

$$\boxed{\boldsymbol{\alpha} \wedge \boldsymbol{\beta}(X, Y) := \boldsymbol{\alpha}(X) \times \boldsymbol{\beta}(Y) - \boldsymbol{\alpha}(Y) \times \boldsymbol{\beta}(X).}$$

Sometimes, we write

$$\begin{aligned} \boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= \begin{vmatrix} i & j & k \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}_\wedge \\ &:= (\alpha_2 \wedge \beta_3 - \alpha_3 \wedge \beta_2)i + (\alpha_3 \wedge \beta_1 - \alpha_1 \wedge \beta_3)j + (\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1)k. \end{aligned}$$

Properties

- Symmetry

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{\beta} \wedge \boldsymbol{\alpha}.$$

- $\boldsymbol{\alpha}$ is a vector-valued 1-form, then

$$d(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (d\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} - \boldsymbol{\alpha} \wedge (d\boldsymbol{\beta}).$$

- Suppose f, g are vector-valued 0-forms, then

$$d(f \times dg) = df \wedge dg$$

8.1.2 Variation formulation

In this section, we will see that surface curvatures appear in variation of volumes, areas, etc. To study such variation formulation, we use the embedding formulation of a surface. That is, $f : M \to \mathbb{R}^3$. We will assume that M is a closed surface. We want to express the surface area of $f(M)$ and the volume where $f(M)$ encloses in terms of f. Let also write $f = (x, y, z)$ to connect with the language in vector calculus.

- **Surface area form** σ. If M is parametrized by (u, v), then

$$\text{the normal is } N = f_u \times f_v / \|f_u \times f_v\|,$$

$$\text{the area form is } \sigma = \|f_u \times f_v\| du \wedge dv.$$

- **Area normal**

$$\boxed{N\sigma = \frac{1}{2} df \underset{\times}{\wedge} df = \frac{1}{2} d(f \times df).}$$

$$\begin{aligned} N\sigma = f_u \times f_v \, du \wedge dv &= \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du \wedge dv \\ &= (\partial(y,z)/\partial(u,v), \partial(z,x)/\partial(u,v), \partial(x,y)/\partial(u,v)) \, du \wedge dv \\ &= (dy \wedge dz, dz \wedge dx, dx \wedge dy) \\ &= \frac{1}{2} df \wedge df. \end{aligned}$$

The last line involves wedge product of a $\mathbb{R}^3$-valued 1-forms. Formally, this means that

$$df \underset{\times}{\wedge} df = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ dx & dy & dz \end{vmatrix} = 2(dy \wedge dz, dz \wedge dx, dx \wedge dy).$$

More precisely, for two $\mathbb{R}^3$-valued 1-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, their wedge product is defined to be

$$\boxed{\boldsymbol{\alpha} \wedge \boldsymbol{\beta}(X, Y) := \boldsymbol{\alpha}(X) \times \boldsymbol{\beta}(Y) - \boldsymbol{\alpha}(Y) \times \boldsymbol{\beta}(X).}$$

Note that

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{\beta} \wedge \boldsymbol{\alpha}.$$

From this definition, we have

$$df \wedge df(X, Y) = df(X) \times df(Y) - df(Y) \times df(X) = 2df(X) \times df(Y).$$

Note that

$$df \wedge df = d(f \times df) = d(ydz - zdy, zdx - xdz, xdy - ydx).$$

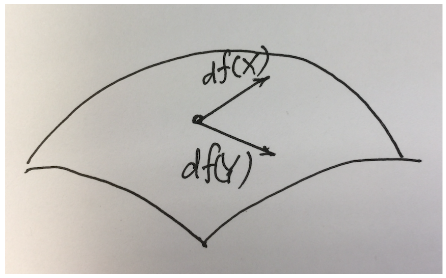


Figure 8.1: The vectors $df(X)$, $df(Y)$ are two tangent vectors. $df(X) \times df(Y)$ is in the normal direction. Its magnitude is the area element. Therefore $df \wedge df = 2N\sigma$.

Or in matrix form

$$df \wedge df = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ dx & dy & dz \end{vmatrix} = d \begin{vmatrix} i & j & k \\ x & y & z \\ dx & dy & dz \end{vmatrix}$$

- **Mean curvature Normal**

$$\boxed{\frac{1}{2} dN \wedge df = \frac{1}{2} d(N \times df) = HN\sigma.}$$

Suppose X_1 and X_2 are the two principal curvature directions of f. That is, $dN(X_1) = \kappa_1\, df(X_1)$ and $dN(X_2) = \kappa_2\, df(X_2)$. Then

$$\begin{aligned} dN \wedge df(X_1, X_2) &= dN(X_1) \times df(X_2) - dN(X_2) \times df(X_1) \\ &= \kappa_1 df(X_1) \times df(X_2) - \kappa_2 df(X_2) \times df(X_1) \\ &= (\kappa_1 + \kappa_2) df(X_1) \times df(X_2) \\ &= 2H df(X_1) \times df(X_2) \\ &= H df \wedge df(X_1, X_2) \end{aligned}$$

You can check that $dN \wedge df(X_2, X_1) = H df \wedge df(X_2, X_1)$ and $dN \wedge df(X_1, X_1) = dN \wedge df(X_2, X_2) = 0$. Since $\{X_1, X_2\}$ forms a basis of $T_p(M)$, we thus conclude

$$dN \wedge df = H df \wedge df = 2HN\sigma.$$

- **Gaussian curvature Normal**

$$\boxed{\frac{1}{2} dN \wedge dN = \frac{1}{2} d(N \times dN) = KN\sigma.}$$

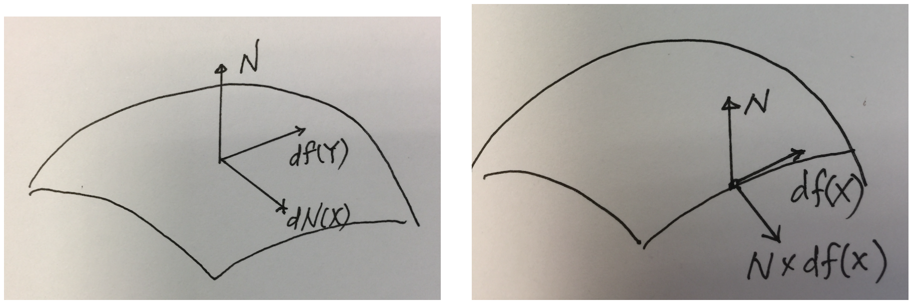


Figure 8.2: The vectors $dN(X)$, $df(Y)$ are two tangent vectors. $dN(X) \times df(Y)$ is in the normal direction. Its magnitude is the area element times the mean curvature. Therefore $dN \wedge df = 2HN\sigma$. $dN \wedge df = d(N \times df)$. $N \times df(X)$ is the tangential direction and pull to the normal to $df(X)$.

Let X_1 and X_2 be the two principal curvature directions with $dN(X_i) = \kappa_i df(X_i)$, $i = 1, 2$. Following the same step as before, we have

$$\begin{aligned} dN \wedge dN(X_1, X_2) &= 2dN(X_1) \times dN(X_2) \\ &= 2\kappa_1\kappa_2 df(X_1) \times df(X_2) \\ &= K df \wedge df(X_1, X_2). \end{aligned}$$

Following the same argument as previous step, we also get

$$dN \wedge dN = K df \wedge df = 2KN\sigma.$$

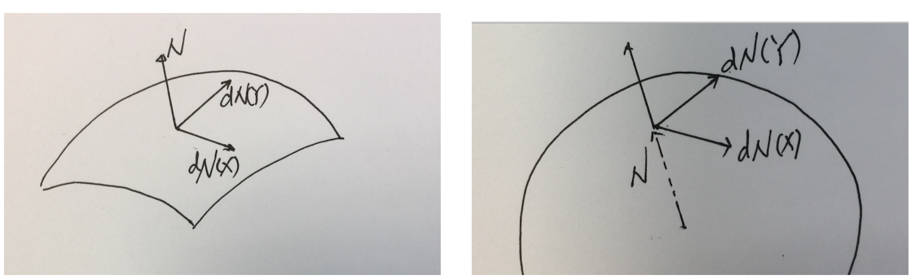


Figure 8.3: The vectors $dN(X)$, $dN(Y)$ are two tangent vectors. They can also be viewed as two tangent vectors on the unit sphere S^2 at N. $dN(X) \times dN(Y)$ is in the normal direction with magnitude K. Therefore $dN \wedge dN = 2KN\sigma$.

- **Area form** $\sigma = \frac{1}{2}\langle N, df \wedge df\rangle = det(N, df, df)$**.**
  Here,

$$\begin{aligned}\sigma(X,Y) &= \frac{1}{2}\langle N, df \wedge df(X,Y)\rangle \\ &= \frac{1}{2}\left[\langle N, df(X) \times df(Y)\rangle - \langle N, df(Y) \times df(X)\rangle\right] \\ &= det(N, df(X), df(Y)) := det(N, df, df)(X,Y).\end{aligned}$$

- **Volume form** $V = \frac{1}{6}\int_M \langle f, df \wedge df\rangle$**.**
  If $f(M)$ encloses a domain in $\mathbb{R}^3$, then its volume can be expressed as

$$\frac{1}{3}\int_M xdy \wedge dz + ydz \wedge dx + zdx \wedge dy = \frac{1}{6}\int_M \langle f, df \wedge df\rangle.$$

  This follows from Stokes' theorem, where

$$d(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = 3dx \wedge dy \wedge dz.$$

- **Gaussian curvature form**: $K\sigma = \frac{1}{2}\langle N, dN \wedge dN\rangle$. If $f(M)$ is parameterized by (u,v), then

$$\frac{1}{2}\langle N, dN \wedge dN\rangle(\partial_u, \partial_v) du \wedge dv = \langle N, N_u \times N_v\rangle du \wedge dv$$

  The last term is the area element on the unit sphere. Thus, we obtain

$$\int_U K\sigma = \text{Area}(N(U)).$$

Theorem 8.1. *Let $f : M \to \mathbb{R}^3$ be an embedding map. Let us denote variation of f by $\dot{f}$. We have the following variational formulae*

$$\begin{aligned}\dot{V}_M &= \int_M \langle \dot{f}, N\rangle\sigma, \\ \dot{A}_M &= \int_M \langle \dot{f}, N\rangle 2H\sigma, \\ \dot{H}_M &= \int_M \langle \dot{f}, N\rangle K\sigma, \\ \dot{K}_M &= 0.\end{aligned}$$

Remark The meaning of this theorem is:

- $\delta V = N\sigma$,
- $\delta A = 2NH\sigma$,

- $\delta H = KN\sigma$.

where δV is the variation of the volume V.

Proof. 1. From $V_M = \frac{1}{6}\int_M \langle f, df \wedge df\rangle$, we take variation in f to get

$$\begin{aligned}\dot{V}_M &= \frac{1}{6}\int_M \langle \dot{f}, df \wedge df\rangle + \frac{1}{3}\int_M \langle f, df \wedge d\dot{f}\rangle \\ &= \frac{1}{2}\int_M \langle \dot{f}, df \wedge df\rangle = \int_M \langle \dot{f}, N\rangle \sigma.\end{aligned}$$

In the second equality, we use

$$\begin{aligned}d\det(f, df, \dot{f}) &= d\langle f, df \times \dot{f}\rangle \\ &= \langle df\wedge, df \times \dot{f}\rangle - \langle f, df \wedge d\dot{f}\rangle \\ &= \langle df \wedge df, \dot{f}\rangle - \langle f, df \wedge d\dot{f}\rangle.\end{aligned}$$

and $\partial M = \emptyset$.

2. We use area formula

$$A_M = \frac{1}{2}\int_M \langle N, df \wedge df\rangle.$$

Take variation in f, we get

$$\begin{aligned}\dot{A}_M &= \frac{1}{2}\int_M \langle \dot{N}, df \wedge df\rangle + \int_M \langle N, df \wedge d\dot{f}\rangle \\ &= \int_M \langle N, df \wedge d\dot{f}\rangle \\ &= \int_M \langle \dot{f}, dN \wedge df\rangle - \int_M d\det(N, df, \dot{f}) \\ &= \int_M \langle \dot{f}, df \wedge df\rangle H - \int_{\partial M} \langle \dot{f}, N \times df\rangle \\ &= \int_M \langle \dot{f}, N\rangle 2H\sigma - \int_{\partial M} \langle \dot{f}, N \times df\rangle.\end{aligned}$$

In the second step, we use $df \wedge df = N\sigma$ and $\langle \dot{N}, N\rangle = 0$. In the third equality, we use

$$\begin{aligned}d\det(N, df, \dot{f}) &= d\langle N, df \times \dot{f}\rangle \\ &= \langle dN\wedge, df \times \dot{f}\rangle - \langle N, df \wedge d\dot{f}\rangle \\ &= \langle dN \wedge df, \dot{f}\rangle - \langle N, df \wedge d\dot{f}\rangle\end{aligned}$$

3. We use the total mean curvature formula

$$H_M = \int_M H\sigma = \frac{1}{2}\int_M \langle N, dN \wedge df\rangle.$$

Take variation in f, we get

$$
\begin{aligned}
\dot{H}_M &= \frac{1}{2}\int_M \Big(\langle \dot{N}, dN \wedge df\rangle + \langle N, d\dot{N} \wedge df\rangle + \langle N, dN \wedge d\dot{f}\rangle\Big) \\
&= \frac{1}{2}\int_M \Big(\langle N, d\dot{N} \wedge df\rangle + \langle N, dN \wedge d\dot{f}\rangle\Big) \\
&= \frac{1}{2}\int_M \langle \dot{f}, dN \wedge dN\rangle - \frac{1}{2}\int_{\partial M} \Big(\det(N, dN, \dot{f}) + \det(N, df, \dot{N})\Big) \\
&= \frac{1}{2}\int_M \langle \dot{f}, N\rangle K\sigma - \frac{1}{2}\int_{\partial M} \Big(\langle \dot{f}, N \times dN\rangle + \langle \dot{N}, N \times df\rangle\Big).
\end{aligned}
$$

The second equality uses that $det(\dot{N}, dN, df) = 0$ because all of the three entries lie on the 2D tangent plane. The third equality uses

$$
\begin{aligned}
d\det(N, dN, \dot{f}) &= \langle dN \wedge dN, \dot{f}\rangle - \langle N, dN \wedge d\dot{f}\rangle \\
d\det(N, df, \dot{N}) &= -\langle N, df \wedge d\dot{N}\rangle.
\end{aligned}
$$

4. The total Gaussian curvature is

$$
K_M = \int_M K\sigma = \frac{1}{2}\int_M \langle N, dN \wedge dN\rangle
$$

Its variation in f gives

$$
\begin{aligned}
\dot{K}_M &= \frac{1}{2}\int_M \langle \dot{N}, dN \wedge dN\rangle + \int_M \langle N, dN \wedge d\dot{N}\rangle \\
&= -\int_{\partial M} \det(N, dN, \dot{N}) \\
&= -\int_{\partial M} \langle \dot{N}, N \times dN\rangle.
\end{aligned}
$$

The second equality uses $dN \wedge dN(X,Y) = 2dN(X) \times dN(Y) \parallel N$ and $\dot{N} \perp N$.The third equality uses

$$
d\det(N, dN, \dot{N}) = -\langle N, dN \wedge d\dot{N}\rangle.
$$

□

Proposition 8.1 (Conservation laws of curvature normals)**.** *Let $f : M \to \mathbb{R}^3$ and $U \subset M$. We have*

$$
\begin{aligned}
\int_U N\sigma &= \frac{1}{2}\int_{\partial U} f \times df \\
\int_U HN\sigma &= \frac{1}{2}\int_{\partial U} N \times df \\
\int_U KN\sigma &= \frac{1}{2}\int_{\partial U} N \times dN.
\end{aligned}
$$

Proof. This follows from

$$\begin{aligned} df \wedge df &= d(f \times df) \\ dN \wedge df &= d(N \times df) \\ dN \wedge dN &= d(N \times dN). \end{aligned}$$

□

Remark Note that $N \times df$ is a $90°$ rotation of df. In two dimensions, this is exactly the orthogonal complement of df. That is, $N \times df = \star df$. Thus,

$$2HN\sigma = d(N \times df) = d \star df = \triangle f \sigma.$$

Or

$$\boxed{\triangle f = HN}$$

8.2 Discrete curvatures via Variational Approach

In tis subsection, we would like to define normal, mean curvature and Gaussian curvature for discrete surface. We take variation approach. This means that it will be in weak form. Recall for discrete curve $(\gamma_0, \gamma_1, ..., \gamma_{n-1})$, we have defined

- Tangent: $T_i = (\gamma_{i+1} - \gamma_i)/|\gamma_{i+1} - \gamma_i|$.
- Normal on $[\gamma_i, \gamma_{i+1}]$: $N_i := -iT_i$
- length weighted normal at γ_i: $\ell_{i-1}N_{i-1} + \ell_i N_i$, where $\ell_i := |\gamma_{i+1} - \gamma_i|$
- Curvature: $\kappa = \Delta\theta$, $\Delta\theta$ is the arc length distance between N_{i-1} and N_i on unit circle.

In this subsection, we will follow the same track. We will define several notions of normal: area weighted normal, mean curvature weighted normal. We will also define Gaussian curvature as the area of the polygon spanned by the normals on the unit sphere. They are defined through an integration over a patch surrounding a vertex. Such definition preserves conservation laws and make global geometric properties are valid in the discrete sense.

To begin with, we suppose we are given a discrete surface M with simplicial structure $\mathcal{K} = (V, E, F)$. The embedding map $f : V \to \mathbb{R}^3$. The vertices $V = \{p_1, ..., p_{n_0}\}$. The edges are $E = \{e_{ij}\}$ and the triangles are $F = \{t_{ijk}\}$. We would like to construct

- Area normal
- Mean curvature normal
- Gaussian curvature normal.

at a vertex p_i (or its embedded image f_i). Suppose p_i is surrounded by triangles t_{ijk}. We denote it by $p_i \prec t_{ijk}$. We will mainly integrate $df \wedge df$, $df \wedge dN$ and $dN \wedge dN$ over those triangles t_{ijk} with $p_i \prec t_{ijk}$.

Area normal Let us consider a triangle t_{ijk}. From the formula

$$\int_{t_{ijk}} df \wedge df = \int_{\partial t_{ijk}} f \times df = \int_{t_{ijk}} N\sigma,$$

The right-hand side is

$$\int_{t_{ijk}} N\sigma = N_{ijk} A_{ijk}.$$

For the left-hand side, let us integrate along an edge e_{ij}:

$$\int_{e_{ij}} f \times df = \frac{1}{2}(f_i + f_j) \times (f_j - f_i) = f_i \times f_j$$

We sum over all faces with common vertex p_i to get

$$\boxed{\frac{\partial V}{\partial f_i} = \sum_{p_i \prec t_{ijk}} N_{ijk} A_{ijk} = \sum_{p_i \prec t_{ijk}} f_j \times f_k}$$

This formula gives the gradient of volume, which is the *area normal*, in terms of the positions of f_j.

Mean curvature normal The mean curvature normal involves variation of area:

$$\frac{\delta A_U}{\delta f_i} = \int_U 2HN\sigma = \int_U dN \wedge df = \int_{\partial U} N \times df.$$

Let us take $U = t_{ijk}$ and denote $f_j - f_i$ by u_{ij}. We take variation of A_{ijk} with respect to f_i. We get

$$\frac{\partial A_{ijk}}{\partial f_i} = \frac{1}{2} N_{ijk} \times u_{jk} = \frac{1}{2}\left(\frac{u_{ji} \times u_{ki}}{2A_{ijk}}\right) \times u_{jk}.$$

Use the product formula: $A \times (B \times C) = \langle A, C\rangle B - \langle A, B\rangle C$, we get

$$\begin{aligned}
\frac{\partial A_{ijk}}{\partial f_i} &= \frac{1}{2}\frac{\langle u_{jk}, u_{ji}\rangle}{2A_{ijk}} u_{ki} - \frac{1}{2}\frac{\langle u_{jk}, u_{ki}\rangle}{2A_{ijk}} u_{ji} \\
&= \frac{1}{2}\frac{|u_{jk}|u_{ji}| \cos\alpha}{|u_{jk}||u_{ji}| \sin\alpha} u_{ki} + \frac{1}{2}\frac{|u_{ki}||u_{kj}| \cos\beta}{|u_{ki}||u_{kj}| \sin\beta} u_{ji} \\
&= \frac{1}{2}\cot\alpha\, u_{ki} + \frac{1}{2}\cot\beta\, u_{ji}.
\end{aligned}$$

The variation of area with respect to f_i is

$$\frac{\partial A}{\partial f_i} = \sum_{p_i \prec t_{ijk}} \frac{\partial A_{ijk}}{\partial f_i} = \frac{1}{2}\sum_{p_i \prec t_{ijk}} (\cot\alpha_{ij} + \cot\beta_{ij})(f_i - f_j). \tag{8.1}$$

We summarize the formula from variation of area:

$$\boxed{\frac{\partial A}{\partial f_i} = 2(HN)_i = \frac{1}{2}\sum_{p_i \prec t_{ijk}} N_{ijk} u_{jk} = (\triangle f)_i = \frac{1}{2}\sum_{p_i \prec t_{ijk}} (\cot\alpha_{ij} + \cot\beta_{ij})(f_i - f_j).} \tag{8.2}$$

This gives the mean curvature normal in terms of the position function f_i.

Chapter 9

PDEs on Manifolds

We will discuss the standard Poisson equation, heat equation on manifolds.

Physical Units Physical laws are conservation of mass, momentum, energy, etc. Physical quantities have units. We discuss some of them.

- Density: $\rho \in \Omega^3(M, \mathbb{R}, kg)$. Its dimension is $[\rho] = [kg/m^3]$.
- Temperature: $u \in \Omega^0(M, \mathbb{R}K)$. Its dimension is $[u] = [K]$.
- Velocity: $v \in \Omega^1(M, \mathbb{R}m^2/s)$. Its dimension is $[v] = [m/s]$.
- Mass flux: $J \in \Omega^2(M, \mathbb{R}kg/s)$. Its dimension is $[J] = [kg/m^2 s]$.

The Hodge star $\star$ maps physical quantities with different units:

$$\star_k : \Omega^k(M, \mathbb{R}\text{unit}_1) \to \Omega^{n-k}(M, \mathbb{R}\text{unit}_2)$$

which encode a material property that relates unit_1/m^k with unit_2/m^{n-k}. For instance, the Fourier law

$$Q = - \star dT$$

relates temperature gradient with the heat flux. $[Q] = J/sm^2$, where $[dT] = [K/m]$.

9.1 Heat Equation

9.1.1 Continuous version

- The physical quantities are
  - Temperature: $u \in \Omega^0(M, \mathbb{R}K)$
  - Heat Flux: $Q \in \Omega^{n-1}(M, \mathbb{R}J/s)$.

- The Fourier law relates du and Q by

$$Q = -\kappa \star_1 d_0 u.$$

  The physical parameter κ is called the heat conductivity. The temperature and energy density is connected by physical property of the material:

$$C \star_0 u_t = E \in \Omega^n(M, \mathbb{R}J/s)$$

  Here, C is called the heat capacitance.

- The physical law is the conservation of energy, which reads

$$\boxed{C \star_0 u_t = -dQ = d\kappa \star_1 d_0 u.}$$

- Let us make life simpler by taking κ and C being constants and $\kappa/C = 1$. The equation now reads

$$u_t = \star_0^{-1} d_{n-1} \star_1 d_0 u := \triangle u.$$

  If there is a heat source $\beta(t) \in \Omega^n(M, \mathbb{R}J/s)$, then the equation is

$$\star_0 u_t = -dQ + \beta,$$

  or

$$u_t = \triangle u + \star_0^{-1}\beta.$$

- We plot the commuting diagram:

$$\begin{array}{ccc} \Omega^0(M) & \xrightarrow{d_0} & \Omega^1(M) \\ \uparrow \star_0^{-1} & & \downarrow \star_1 \\ \Omega^n(M) & \xleftarrow{d_{n-1}} & \Omega^{n-1}(M) \end{array}$$

9.1.2 Discrete Heat equation – Discrete Exterior Calculus Approach

Let M be a discrete n-manifold with simplicial complex $\mathcal{K}$. Let $\mathcal{K}^*$ be its Voronoi dual complex.

- Physical quantities:
  - Temperature u is defined on vertices. That is, $u \in C^0(M, \mathbb{R}K)$, where C^k is the k-cochain.
  - The temperature difference $d_0 u \in C^1(M, \mathbb{R}K)$ sits on edges.
  - Heat flux Q is defined on the dual $(n-1)$-cell. That is, $Q \in C^{n-1}(M^*, \mathbb{R}J/s)$.

- The Fourier law is given by the Hodge $\star$ operator:

$$\star_1 = \operatorname{diag}\left(\frac{|e^*|}{|e|}\right)$$

where $e \in \mathcal{K}^1$ and e^* is its dual face in $\mathcal{K}^{*,n-1}$. This is an $n_1 \times n_1$ matrix. The connection between temperature and energy is

$$\star_0 u = E.$$

where

$$\star_0 = \operatorname{diag}(|C_i|), \quad C_i \text{ is the dual cell centered at vertex } i.$$

- The conservation of energy reads

$$\star_0 u_t = -d_{n-1} \star_1 d_0 u.$$

$$\begin{array}{ccc} \Omega^0(M) & \xrightarrow{d_0} & \Omega^1(M) \\ \uparrow \star_0^{-1} & & \downarrow \star_1 \\ \Omega^n(M^*) & \xleftarrow{d_{n-1}} & \Omega^{n-1}(M^*) \end{array}$$

We have seen that $d_0(\mathcal{K}) = \partial_1^T(\mathcal{K})$, an $n_1 \times n_0$ matrix. Note that, there are 1-1 correspondences: $\mathcal{K}^0 \leftrightarrow \mathcal{K}^{*n}$, $\mathcal{K}^1 \leftrightarrow \mathcal{K}^{*(n-1)}$. For $d_{n-1}(\mathcal{K}^*)$ on the dual cell, it is $\partial_n^T(\mathcal{K}^*)$, an $n_0 \times n_1$ matrix. But this is exactly $\partial_1(\mathcal{K})$. We summary these relation

$$d_0(\mathcal{K}) = \partial_1^T(\mathcal{K}), \quad d_{n-1}(\mathcal{K}^*) = \partial_n^T(\mathcal{K}^*) = \partial_1(\mathcal{K}).$$

$$d_{n-1}(\mathcal{K}^*) = d_0^T(\mathcal{K}).$$

Thus, we rewrite the above equation as

$$\boxed{\star_0 u_t = -d_0^T \star_1 d_0 u, \quad M u_t = Lu.}$$

For a 2-manifold M, let us compute $\star_1$ now. Suppose $e_{ij} = [ij] \in \mathcal{K}^1$ is an edge. e_{ij}^* is its dual. Let t_{ijk} and $t_{ij\ell}$ be the two triangles with e_{ij} as their common edge. Let α_{ij}^k and α_{ij}^ℓ be the angles of the triangle at k and ℓ. Then

$$\frac{|e_{ij}^*|}{|e_{ij}|} = \frac{1}{2}\left(\cot \alpha_{ij}^k + \cot \alpha_{ij}^\ell\right).$$

We get that

$$L = -d_0^T \star_1 d_0,$$

$$(Lu)_i = \frac{1}{2} \sum_{p_i \prec e_{ij}} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i).$$

$$M = \operatorname{diag}|C_i|, \quad C_i \text{ is the Voronoi cell at vertex } i.$$

9.1.3 Discrete Heat equation – Finite Element Approach

- Finite element basis In finite element method, we first choose finite element basis ϕ^i. Here, we choose the Whitney element ϕ^i, which is a piecewise linear function on $(M, \mathcal{K})$ with
$$\phi^i([j]) = \delta_{ij}.$$
The approximate space is $\Omega_h^0(M)$, spanned by $\{\phi^i | [i] \in \mathcal{K}^0\}$. Here, h is the maximum of $|e_{ij}|$ with $[ij] \in \mathcal{K}^1$. A function $u \in \Omega^0(M)$ is approximated by
$$\hat{u} = \sum_{[i] \in \mathcal{K}^0} u_i \phi^i.$$

- Project the equation to $\Omega_0(M)$. The heat equation in weak form is
$$\int_M \langle \dot{u}, v \rangle \mu = -\int_M \langle du, dv \rangle \mu$$
for all $v \in C^1(M)$. Here, $\dot{u} := \partial u / \partial t$. We now take $u = \hat{u}$ and $v = \phi^i$ to get
$$\langle \phi^i, \sum_j \dot{u}_j \phi^j \rangle = -\langle d\phi^i, \sum_j d\phi^j u_j \rangle.$$
This leads to
$$\sum_j \langle \phi^i, \phi^j \rangle \dot{u}_j = -\sum_j \langle d\phi^i, d\phi^j \rangle u_j.$$
In matrix form
$$M\dot{u} = Lu.$$
where
$$M = \star_0^{\text{FEM}} = (\langle \phi^i, \phi^j \rangle)_{n_0 \times n_0},$$
$$L = -d_0^T \star_1^{\text{FEM}} d_0 = (\langle d\phi^i, d\phi^j \rangle)_{n_0 \times n_0}.$$
M and L are called the mass matrix and the stiff matrix, respectively.

For surface, one can show that
$$(Lu)_i = \frac{1}{2} \sum_{p_i \prec e_{ij}} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i),$$
where α_{ij} and β_{ij} are the angles of opposite vertices across from e_{ij} in the two adjacent triangles.

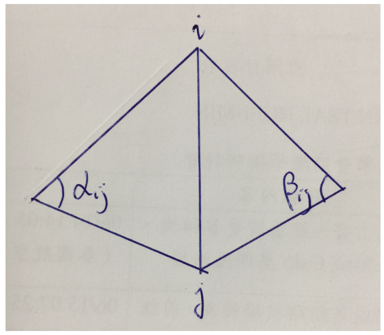


Figure 9.1: α_{ij} and β_{ij} are the angles of opposite vertices across from e_{ij} in the two adjacent triangles.

Time Discretization Let us discretize time axis by $n\Delta t$. Δt is called a temporal mesh size. We denote $u(n\Delta t)$ by u^n. The equation

$$M\dot{u} = Lu$$

is a system of ODE. There are standard time discretization method. The simplest two are the forward Euler method and the backward Euler method, which are

- Forward Euler Method:
$$M\left(\frac{u^{n+1} - u^n}{\Delta t}\right) = Lu^n.$$
- Backward Euler Method:
$$M\left(\frac{u^{n+1} - u^n}{\Delta t}\right) = Lu^{n+1}.$$

For forward Euler method, we have

$$u^{n+1} = \left(I + \Delta t M^{-1} L\right) u^n.$$

For backward Euler method,

$$u^{n+1} = (M - \Delta t L)^{-1} M u^n.$$

In numerical PDE, the above scheme is called *stable* if $\|u^n\|$ remains bounded for all $n \geq 0$ (uniformly w.r.t. n). This is necessary for convergence. Because for a fixed $t = n\Delta t$, we expect $u^n \to u(t)$ as $\Delta t \to 0$, or equivalently, $n \to \infty$. The convergence of u^n leads to boundedness of $\|u^n\|$.

For forward Euler method, the stability condition is that the Δt should satisfy

$$\frac{\Delta t}{h^2} \leq \frac{1}{2}.$$

where $h = \max\{|e| | e \in \mathcal{K}^1\}$. For backward Euler method, it is unconditional stable. One way to prove such a result is to use maximum principle. This means that we can show that u^{n+1} is convex combination of u^n. That is, for any $i \in V$, we can show that

$$u_i^{n+1} = \sum_{j \in V} a_{ij} u_j^n,$$

$$a_j \geq 0, \quad \sum_j a_{ij} = 1$$

Such a result can lead $|u^n|_\infty$ remain boundedness. This is a stability result in L^∞ sense. I shall not give details of stability results.

9.2 Mean Curvature Flow

Let $f : M \to \mathbb{R}^3$ be an imbedding map. If f also depends on time, $f(t)$ can describe surface motion. A particular example is the mean curvature flow, where the surface moves in its mean curvature normal direction at speed equals to the mean curvature of the surface. We have seen that

$$\triangle f = 2HN.$$

Thus, the mean curvature flow satisfies

$$\boxed{\partial_t f = \frac{1}{2} \triangle f.}$$

The mean curvature can smooth the surface because it is a diffusion process. Usually, it will shrink to a point. During the shrinking process, it may develop singularities.

Chapter 10

Surface Parametrization

10.1 Conformal structure

Following the notation of previous chapter on surface, let $f : M(\subset \mathbb{R}^2) \to \mathbb{R}^3$ be an imbedding map. In this chapter, we want to parametrize the immersed surface $f(M)$ such that the map $h : M \to \mathbb{C}$ preserves angles. First, we can define a conformal structure J on M by: for any $p \in M$,

$$J : T_pM \to T_pM, \quad df_p(JX) = N_p \times df_p(X) \text{ for all } X \in T_pM.$$

Such J has the following property:

$$J^2 = -id.$$

where id is the identity map. Such J is called a complex structure. It just a $90°$ degree rotation. A Riemann surface is a surface with a complex structure J.

The complex plan $\mathbb{C}$ also has such a complex structure. That is, $i^2 = -1$. We look for $h : M \to \mathbb{C}$ such that

$$\boxed{dh(JX) = i\, dh(X) \text{ for all } X \in TM.}$$

This means that it preserves the complex structure. Such function h is called a *holomorphic* function. When M is endowed with a metric, then this also means that: if $X \perp Y$, then $dh(X) \perp dh(Y)$. That is, dh preserves right angles. In this case, in fact, dh preserves all angles. This is the following proposition.

Proposition 10.1. *Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear and preserves right angles. The latter means that if $\langle X, Y\rangle = 0$, then $\langle AX, AY\rangle = 0$. Then A is an angle preserving map:*

$$\frac{\langle AX, AY\rangle}{|AX||AY|} = \frac{\langle X, Y\rangle}{|X||Y|}.$$

Proof. Suppose $A = (a_1, ..., a_n)$, a_j are the column vectors of A. Then $a_j = Ae_j$. From $\langle e_i, e_j\rangle = 0$, we get $\langle a_i, a_j\rangle = 0$ for $i \neq j$. From $(e_i + e_j) \perp (e_i - e_j)$, we get $\langle A(e_i + e_j), A(e_i - e_j)\rangle = 0$. This leans to $|Ae_i| = |Ae_j|$ for all $i \neq j$.Thus, $A = sR$, where R is a rotation matrix (i.e. $R^T R = id$) and $s \neq 0$. This leads to the angle preserving property. $\square$

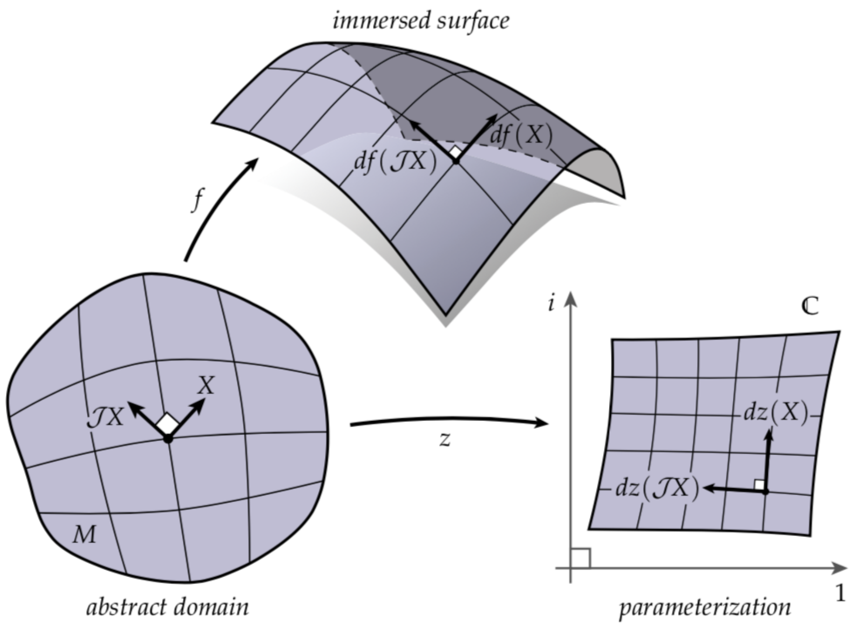


Figure 10.1: Copied from Crane's note. The map $z : M \to \mathbb{C}$ is holomorphic. The holomorphic function z is the function h in this note.

Remarks

1. Conformal structure is a structure weaker than the inner product structure. The former has only the concept of 90° turn, or the concept of orthogonality. The latter has the concepts of both angle and length. In inner product spaces, a map which preserves the conformal structure also an angle preserving map. But it may not preserve length. A length-preserving linear map also angle preserving. Such a map is called an isometry.

2. Let $h : M(\subset \mathbb{R}^2) \to \mathbb{C}$ be holomorphic. Let us write $h(x, y) = (u(x, y), v(x, y))^T$. The differential
$$dh = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$
is conformal means that dh has the form
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$
This leads to
$$u_x = v_y \quad u_y = -v_x,$$
called the Cauchy-Riemann equations. If we write $h = u + iv$, then the above Cauchy-Riemann equations have a simpler form
$$\partial_y h = i\,\partial_x h,$$

or

$$dh(J\partial_x) = i\, dh(\partial_x),$$

where $J\partial_x = \partial_y$.

3. We can express the Cauchy-Riemann equation in terms of differential form and Hodge $\star$ operator. Recall $\star dx = dy$. We have

$$\star dh = \star(h_x dx + h_y dy) = h_x dy - h_y dx.$$

On the other hand,

$$-dh \circ J(\partial_x) = -dh(\partial_y) = -h_y, \quad -dh \circ J(\partial_y) = -dh(-\partial_x) = h_x.$$

Thus,

$$\star dh = -dh \circ J.$$

The Cauchy-Riemann equation now reads

$$\boxed{\star dh + i\, dh = 0.}$$

4. To find such a holomorphic function, we seek for minimum of the conformal energy

$$\boxed{E_C[h] := \frac{1}{4}\| \star dh + idh\|^2}$$

subject to some suitable constraints. Here, the norm $\| \cdot \|$ for a complex 1-form is defined in the next section. If there exists a nontrivial minimum with zero minimal value, then this minimum satisfies the Cauchy-Riemann equation. There are basically two approaches to find such a holomorphic function.

   - Fixed boundary approach: The boundary $h(\partial M)$ is fixed during minimization process. When $M <$ is simply connected, a popular one is $h(\partial M) = S^1$. That is, we minimize

$$\min\{E_C[h] \mid h(\partial M) = S^1\}$$

     and show that the minimal value is 0.
   - Free boundary approach: That is, we let $h(\partial M)$ be free during minimization process. Since the kernel of δE_C is too big, we need to impose some constraints to exclude trivial solutions. One trivial solution is the constant $\mathbf{1}$. We impose

$$\langle\langle h, \mathbf{1} \rangle\rangle = 0.$$

     It means that $h(M)$ is centered at origin on the complex plane. Another one is a normalization: $\|h\|^2 = 1$. This is due to the following reason. If we rescale h by ch, its image on $\mathbb{C}$ is rescaled by c and $E_C[ch] = c^2 E_C[h]$. When c is shrunk to 0, E_C is the minimal value. We get trivial solution. Thus, we choose our minimizer to be

$$\boxed{\min_h E_C[h] \quad \text{subject to } \|h\|^2 = 1, \quad \langle\langle h, \mathbf{1} \rangle\rangle = 0.}$$

     and show that the minimal value is 0.

We shall discuss this in the next section.

10.2 Variational Approach

10.2.1 Complex-valued differential forms

We introduce the following definitions.

1. For $u, v \in \mathbb{C}$, define $\langle u, v\rangle := \bar{u}v$.
2. For $\alpha, \beta \in \Omega^1(M, \mathbb{C})$, define

$$\alpha \wedge \beta(X, Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

   for $X, Y \in TM$.
3. For $\alpha \in \Omega^1(M, \mathbb{C})$, define

$$\star\alpha := -\alpha \circ J \quad \text{and} \quad \langle \alpha, \beta\rangle\sigma := \bar{\alpha} \wedge \star\beta.$$

   Here, σ is the area form of M.

We have the following properties.

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
- $\langle iu, v\rangle = -\langle u, iv\rangle, \quad \langle iu, iv\rangle = \langle u, v\rangle$.
- $\overline{i\alpha} = -i\bar{\alpha}$.
- $\alpha \wedge \beta = -\beta \wedge \alpha$
- $\overline{\alpha \wedge \beta} = \bar{\alpha} \wedge \bar{\beta}$
- Let $h = u + iv$. Then $du \wedge dv = \frac{i}{2} dh \wedge d\bar{h} = -\frac{i}{2} d\bar{h} \wedge dh$

  Proof. $dh \wedge d\bar{h} = d(u + iv) \wedge d(u - iv) = -2i\, du \wedge dv$. □

- $\star\star = -1, \quad \star i = i\star, \quad \overline{\star\alpha} = \star\bar{\alpha}$.
- $\langle \star\alpha, \beta\rangle = -\langle \alpha, \star\beta\rangle, \quad \langle \star\alpha, \star\beta\rangle = \langle \alpha, \beta\rangle$
- $\langle \alpha, \beta\rangle = \overline{\langle \beta, \alpha\rangle}$

Proof. We show $\bar{\alpha} \wedge \star\beta = \overline{\bar{\beta} \wedge \star\alpha}$. Let $X \in T_pM$ and $Y = JX$. Then $\{X, Y\}$ is a basis of T_pM. We have

$$\begin{aligned}
\bar{\alpha} \wedge \star\beta(X, Y) &= \bar{\alpha}(X)(-\beta(JY)) - \bar{\alpha}(Y)(-\beta(JX)) \\
&= \bar{\alpha}(X)\beta(X) + \bar{\alpha}(Y)\beta(Y)
\end{aligned}$$

$$\begin{aligned}
\bar{\beta} \wedge \star\alpha(X, Y) &= \bar{\beta}(X)(-\alpha(JY)) - \bar{\beta}(Y)(-\alpha(JX)) \\
&= \bar{\beta}(X)\alpha(X) + \bar{\beta}(Y)\alpha(Y)
\end{aligned}$$

Thus, we get

$$\bar{\alpha} \wedge \star\beta = \overline{\bar{\beta} \wedge \star\alpha}.$$

which leads to $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$. □

Energy functional Define

$$\langle\langle \alpha, \beta \rangle\rangle := \int_M \langle \alpha, \beta \rangle \sigma := \int_M \bar{\alpha} \wedge \star\beta, \quad \|\alpha\|^2 := \langle\langle \alpha, \alpha \rangle\rangle.$$

Here, recall that σ is the area element on M induced by the imbedding map f.

10.2.2 Computing conformal energy

Continuous conformal energy

Proposition 10.2. *Let $h : M \to \mathbb{C}$, define*

$$E_C[h] := \frac{1}{4}\| \star dh + i\, dh\|^2.$$

Then

$$\boxed{E_C[h] = E_D[h] - A[h],}$$

$$\boxed{E_D[h] := \frac{1}{2}\|dh\|^2, \quad A[h] := -\frac{i}{2}\int_M d\bar{h} \wedge dh.}$$

Proof.

$$\begin{aligned}
\langle \star dh + i\, dh, \star dh + i\, dh \rangle &= \langle \star dh, \star dh \rangle + \langle i\, dh, i\, dh \rangle + \langle \star dh, i\, dh \rangle + \langle i\, dh, \star dh \rangle \\
&= 2\langle dh, dh \rangle + \langle i\, dh, \star dh \rangle + \overline{\langle i\, dh, \star dh \rangle}
\end{aligned}$$

Note that

$$\langle i\, dh, \star dh \rangle \sigma = (-i\, d\bar{h}) \wedge \star(\star dh) = i\, d\bar{h} \wedge dh,$$

we get

$$\int_M \langle \star dh + i\, dh, \star dh + i\, dh \rangle \sigma = 2\langle\langle dh, dh \rangle\rangle + 2i \int_M d\bar{h} \wedge dh.$$

□

1. Since
$$A[h] = -\frac{i}{2}\int_M d\bar{h}\wedge dh = \int_M du\wedge dv$$
we get that $A[h]$ is the area of $h(M)$ in $\mathbb{C}$.

2. From $d\bar{h}\wedge dh = d(\bar{h}\,dh)$, we get
$$\boxed{A[h] = -\frac{i}{2}\int_M d(\bar{h}\,dh) = -\frac{i}{2}\int_{\partial M}\bar{h}\,dh.}$$
$A[h]$ only depends on $h(\partial M)$.

3. Dirichlet energy:
$$\boxed{E_D[h] := \frac{1}{2}\langle\langle dh, dh\rangle\rangle = \frac{1}{2}\int_M d\bar{h}\wedge\star dh.}$$

Discrete Conformal Energy

1. By taking finite element approach, choosing ϕ^i be the Whitney element with vertex $i\in V$, we get
$$E_D[h] = \frac{1}{2}\sum_{i,j\in V}\bar{h}_i L_{ij} h_j := \frac{1}{2}\bar{h}^T L h,$$
$$\begin{aligned} L_{ij} &= \frac{1}{2}(\cot\alpha_{ij}+\cot\beta_{ij}) \qquad \text{when } e_{ij}\in E \\ L_{ii} &= -\frac{1}{2}\sum_{p_i\prec e_{ij}}(\cot\alpha_{ij}+\cot\beta_{ij}). \end{aligned}$$

2. In the discrete setting, let E_∂ be the edges of ∂M, then
$$\begin{aligned} A[h] &= -\frac{i}{2}\sum_{e_{ij}\in E_\partial}\int_{e_{ij}}\bar{h}\,dh \\ &= -\frac{i}{2}\sum_{e_{ij}\in E_\partial}\frac{\bar{h}_i+\bar{h}_j}{2}(h_j-h_i) \\ &= -\frac{i}{4}\sum_{e_{ij}\in E_\partial}\bar{h}_i h_j - \bar{h}_j h_i. \end{aligned}$$

3. Discrete conformal energy
$$E_C[h] = \frac{1}{2}\bar{h}^T C h, \quad C = L - 2A,$$
where
$$L = (L_{ij})_{|V|\times|V|}, \quad \bar{h}^T A h := -\frac{i}{4}\sum_{e_{ij}\in E_\partial}\bar{h}_i h_j - \bar{h}_j h_i.$$

10.2.3 Fixed boundary approach

1. Suppose M is simply connected. In this fixed boundary approach, we require $h(\partial M) = S^1$. Since $A[h]$ is the area of $h(M)$, which is fixed now, we get

$$\min E_C[h] := \min(E_D[h] + A[h]) = \min E_D[h].$$

2. Note that

$$\min\{E_D[h] \| h| = 1 \text{ on } \partial M\} = \min_{g:\partial M \to S^1} \min_h \{E_D[h] | h = g \text{ on } \partial M\}$$

The last minimization problem is the standard Dirichlet problem:

$$\min E_D[h] := \frac{1}{2} \int_M \langle dh, dh \rangle \sigma, \quad h = g \text{ on } \partial M$$

Its variation is

$$\begin{aligned} \langle \delta E_D[h], \dot{h} \rangle &:= \frac{d}{d\varepsilon}|_{\varepsilon=0} E_D[h + \varepsilon \dot{h}] \\ &= \frac{1}{2} \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_M \langle d(h + \varepsilon \dot{h}), d(h + \varepsilon \dot{h}) \rangle \sigma \\ &= -\int_M \langle \triangle h, \dot{h} \rangle \sigma. \end{aligned}$$

Here, $\dot{h} : M \to \mathbb{C}$ with $\dot{h} = 0$ on ∂M. Thus, the corresponding Euler-Lgrange equation is

$$\triangle h = 0.$$

The standard elliptic theory shows that such minimum exists uniquely and satisfies the Laplace equation

$$\triangle h = 0 \text{ in } M, \quad h = g \text{ on } \partial M.$$

Such function h is called a harmonic function with Dirichlet boundary data g. We denote it by h_g. The minimization problem now becomes

$$\min\{E_D[h_g] | g : \partial M \to S^1\}$$

3. In the discrete setting, we look for

$$\min_{\substack{h : V \to \mathbb{C} \\ |h| = 1}} E_C(h) = \frac{1}{2} \bar{h}^T C h, \quad C = L - 2A.$$

In the continuous setting, the condition $|h| = 1$ gives fixed $area(h(\partial M))$. In the discrete setting, however, $|h| = 1$ may not be enough to give a fixed area of the polygon $h(\partial M)$.

Nevertheless, we impose such fixed area condition. Namely, we look for h such that the variation of the area $\bar{h}Ah$ is fixed. This gives the condition

$$h \in Ker(A).$$

The minimization problem now reads

$$\min\{\frac{1}{2}\bar{h}^T Lh | h \in Ker(A)\}.$$

Note that the operator A only depends on h on ∂M. Let us denote $h|_{\partial M}$ by g. The above problem becomes

$$\min\{\frac{1}{2}\bar{h}^T Lh | h = g \text{ on } \partial M\}$$

Thus, h satisfies

$$Lh = 0 \text{ in } M, \quad h = g \text{ on } \partial M.$$

But this is equivalent tp

$$h \in Ker(L) \cap Ker(A).$$

4. The existence of h follows from Riemann mapping theorem: A conformal mapping between a simply connected domain and the unit disk always exists. Such solution is unique up to Möbius transforms with dimension 3. A Möbius transform $\tau : \mathbb{C} \to \mathbb{C}$ of the form

$$\tau(z) = \frac{(z - z_0)(z_1 - z_0)}{(z - z_2)(z_1 - z_2)},$$

which is conformal and maps (z_0, z_1, z_2) to $(0, 1, \infty)$. Given (z_0, z_1, z_2) and (w_0, w_1, w_2), we can construct a Möbius transform $\tau_2^{-1}\tau_1$ to map (z_0, z_1, z_2) to (w_0, w_1, w_2). If z_i and w_i are all on S^1, there are three degrees of freedoms of such transforms. In order to have a unique solution for our conformal map, we should fix three boundary point images, or one interior point image and one boundary point image.

5. We can use gradient descent method, or an iteration between g and h to minimize E_C to find a minimum.
Ref.

- M.-H. Yueh, W.-W. Lin, C.-T. Wu, and S.-T. Yau, An Efficient Energy Minimization for Conformal Parameterizations, J. Sci. Comput. 73(1): 203-227, 2017.
- Gu and Yau, Computational Conformal Geometry, 2008.

10.2.4 Free boundary approach

1. In this approach, we let $h(\partial M)$ be free and solve the following constrained minimization problem:

$$\min_h E_C[h] \quad \text{subject to } \|h\|^2 = 1, \quad \langle\langle h, \mathbf{1} \rangle\rangle = 0.$$

The discrete version is

$$\min_h \frac{1}{2}\bar{h}^T Ch, \quad \text{subject to } \bar{h}^T Mh = 1, \quad \bar{h}^T M\mathbf{1} = 0,$$

where

$$C = L - 2A, \quad M_{ij} = \langle\langle \phi^i, \phi^j \rangle\rangle$$

is the mass matrix.

2. The minimization of $\bar{h}^T Ch$ subject to the first constraint leads to an eigenvalue problem

$$Ch = \lambda Mh.$$

This together with

$$\bar{h}^T Mh = 1, \quad \bar{h}^T M\mathbf{1} = 0$$

is our generalized eigenvalue problem.

3. All eigenvalues of C w.r.t. M are non-negative because $C \geq 0$. We look for an eigenvector corresponding to the zero eigenvalue and also perpendicular to $\mathbf{1}$. This solution set is $Ker(C) \cap \mathbf{1}^\perp$. We apply the inverse power method to find such an eigenvector. To apply the inverse power method, we replace

$$\tilde{C} \leftarrow C + \lambda_0 M, \quad \lambda_0 > 0 \text{ (some small constant)}$$

in order to take $\tilde{C}^{-1}$ in the inverse power method. Now, $\tilde{C} > 0$. We look for

$$\min_h \frac{1}{2}\bar{h}^T \tilde{C}h, \quad \text{subject to } \bar{h}^T Mh = 1, \quad \bar{h}^T M\mathbf{1} = 0.$$

The eigenvector corresponding to λ_0 is the one satisfying $Ch = 0$. The algorithm is the inverse power method-2 in the next subsection.

10.2.5 Inverse Power method for solving generalized eigenvalue problem

A standard numerical method to find a largest eigenvalue and the corresponding eigenvector is the power method. A variant of the power method, called the inverse power method, is to find a smallest eigenvalue and the corresponding eigenvector. Let us make a short description on power method. Then go to the inverse power method with constraints. Below, B is symmetric positive definite matrix.

Power method-1 To find $Ax = \lambda x$, $\|x\| = 1$ with largest $|\lambda|$.
Here, Residual$(A, x) := Ax - (\bar{x}^T Ax)x$.

Algorithm Power Method 1

Input: A, x^0(initial guess)
Output: $x, \lambda = \bar{x}^T Ax, x$, the largest eigenvalue in magnitude.

```
while Residual(A, x^{n-1}) > ε do
    y ← A x^{n-1}
    x^n ← y/|y|
end while
return x^n, λ
```

Algorithm Power Method 2

Input: A, x^0(initial guess)
Output: $x, \lambda = \bar{x}^T Ax$, the largest eigenvalue in magnitude.

```
while Residual(A, x^{n-1}) > ε do
    y ← A x^{n-1}
    x^n ← y/sqrt(ȳ^T B y)
end while
return x^n, λ
```

Power method-2 To find $Ax = \lambda Bx$, $\bar{x}^T Bx = 1$ with largest $|\lambda|$. Here, we define $\text{Residual}(A, x) := Ax - (\bar{x}^T Ax)Bx$.

Inverse Power Method-1 The inverse power method is to find the smallest eigenvalue (in magnitude) of A. This is equivalent to find the largest eigenvalue of A^{-1}. The algorithm reads

Algorithm Inverse Power Method – 1

Input: A, x^0(initial guess)
Output: $x, \lambda = \bar{x}^T Ax$, the smallest eigenvalue in magnitude.

```
while Residual(A, x^{n-1}) > ε do
    y ← A^{-1} x^{n-1}
    x^n ← y/|y|
end while
return x^n, λ
```

Inverse Power method for generalized eigenvalue problem wit constraints Let A, B are Hermitian matrices and $B > 0$. We assume the smallest eigenvalue of A is λ_0. We want to solve $Ax = \lambda_0 Bx$, $\bar{x}^T Bx = 1$ and $\bar{x}^T B\mathbf{1} = 0$.

Algorithm Inverse Power Method – 2

Input: A, x^0(initial guess)

Output: $x, \lambda = \bar{x}^T A x$, the largest eigenvalue in magnitude.

```
while Residual(A, x^{n-1}) > ε do
    y ← A^{-1} x^{n-1}
    y ← y − ȳ^T B1/(1^T B1)
    x^n ← y/sqrt(ȳ^T By)
end while
return x^n, λ
```

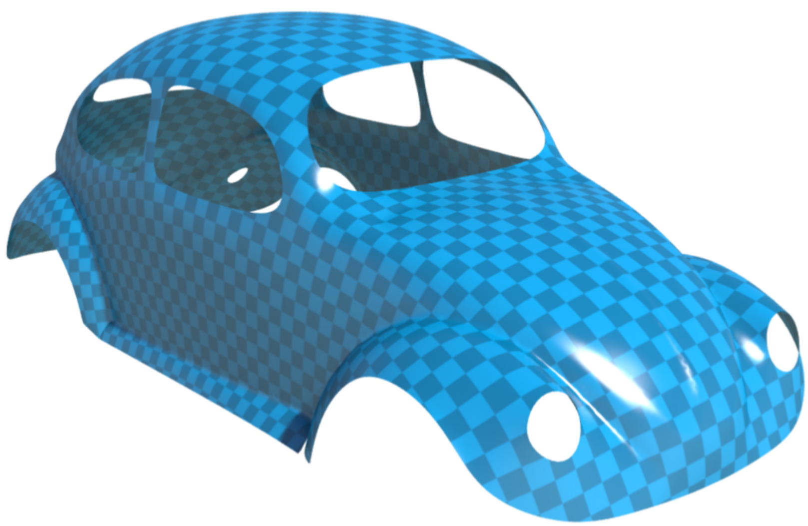


Figure 6.2: *Result of using the first non-zero eigenvector to parameterize the VW bug model. Note that this model is path connected but not simply connected.*

Figure 10.2: Copied from Chern and Schröder's lecture note

Chapter 11

Vector Field Design

Definition of vector fields Vector fields on manifolds are those tangent vector valued functions on manifolds. Directional fields are those vector fields that we care only on its direction, not magnitude. N-vector fields are referred to that at each point $p \in M$, we associate it with N vectors.

Applications of vector field design include

- Mesh generation
- Parametrization
- Deformation
- Shape analysis
- Texture mapping and synthesis
- Architectural geometry
- $\cdots$

Framework of design

- Representation of vector fields
- Set up an objective function and constraints, formulate the problem as an optimization problem with constraints.

objectives include:

- Parallelity
- Orthogonality

- Coons interpolation
- $\cdots$

Constraints are

- Alignment
- Symmetry
- Differential constraints
- Topological constraints
- $\vdots$

A good review article is

- Amir Vaxman, Marcel Campen, Olga Diamanti, Daniele Panozzo, David Bommes, Klaus Hildebrandt and Mirela Ben-Chen, Directional Field Synthesis, Design, and Processing, EUROGRAPHICS 2016.