

### DOUBLE INTEGRALS OVER RECTANGLES

### TRANSPARENCIES AVAILABLE

#48 (Figures 4 and 5), #49 (Figures 7 and 8), #50 (Figure 11), #51 (Figures 12 and 13)

### SUGGESTED TIME AND EMPHASIS

 $\frac{1}{2}$ -1 class **Essential Material** 

### **POINTS TO STRESS**

- I. The definition and properties of the double integral.
- **2.** The analogy between single and double integration.
- **3.** Volume interpretations of double integrals.

### **QUIZ QUESTIONS**

• Text Question: Compute  $\sum_{i=1}^{2} \sum_{j=1}^{3} 2^{i} 3^{j}$ .

**Answer:** 234

• **Drill Question:** If we partition [a, b] into m subintervals of equal length and [c, d] into n subintervals of equal length, what is the value of  $\Delta A$  for any subrectangle  $R_{ij}$ ?

Answer: 
$$\left(\frac{b-a}{m}\right)\left(\frac{d-c}{n}\right)$$

### MATERIALS FOR LECTURE

- Review single variable integration, including Riemann sums. Then show how double integration extends the concepts of single variable integration.
- Use a geometric argument to directly compute  $\iint_R (3+4x) dA$  over  $R = [0, 1] \times [0, 1]$ .
- Discuss what happens when f(x, y) takes negative values over the region of integration. Start with an odd function such as  $x^3 + y^5$  being integrated over  $R = [-1, 1] \times [-1, 1]$ . Then discuss what happens when a function that is not odd has negative values, for example, z = 2 - 3x on  $R = [0, 1] \times [0, 1]$ . Illustrate numerically. (This can also be done using the group work "An Odd Function".)

### WORKSHOP/DISCUSSION

• Do a problem involving numerical estimation, such as estimating  $\iint (x^2 + 2y^2) dA$  over  $0 \le x \le 2$ ,  $0 \le y \le 2$ , using both the lower left corners and midpoints as sample points. Also approximate  $f_{ave}$  over R.

- The following example can be used to help solidify the idea of approximating an area. Consider a square pyramid with vertices at (1, 1, 0), (1, -1, 0), (-1, 1, 0), (-1, -1, 0) and (0, 0, 1). Derive an equation for the surface of the pyramid, using functions such as  $z = 1 \max(|x|, |y|)$ , and then the equations of the planes containing each of the five faces (z = 0, y + z = 1, and so on). Approximate the volume using the Midpoint Rule for the following equal subdivisions. Note that *m* is the number of equal subdivisions in the *x*-direction and *n* is the number of equal subdivisions in the *y*-direction.
  - **I.** m = n = 2 (Approximation is 2)
  - **2.** m = n = 3 (Approximation is  $\frac{44}{27}$ )
  - **3.** m = n = 4 (Approximation is  $\frac{3}{2}$ )
  - **4.** m = n = 5 (Approximation is  $\frac{36}{25}$ )

Compare your answers with the actual volume  $\frac{4}{3}$  computed using the formula  $V = \frac{1}{3}Bh$ .

• Consider  $\iint \sqrt{4 - y^2} \, dA$ , with  $R = [0, 3] \times [-2, 2]$ . Use a geometric argument to compute the actual volume after approximating as above with m = n = 3 and m = n = 4. Show that the average value is  $\frac{6\pi}{12} = \frac{\pi}{2}$ , and that the point  $\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) \approx (0, 1.24)$  in *R* satisfies  $f\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) = \frac{\pi}{2}$ .

### **GROUP WORK I: Back to the Park**

This group work is similar to Example 4 and uses the Midpoint Rule.

Answer: Roughly 81 meters

#### **GROUP WORK 2: An Odd Function**

The goal of the exercise is to estimate  $\iint_{[-2,2] \times [-2,2]} (x^3 + y^5) dA$  numerically. There are three different problem sheets, each one suggesting a different strategy to obtain sample points for the estimation. Groups seated near each other should get different problem sheets, without this fact necessarily being announced. After all the students are finished, have them compare their results. The exact value of the integral is zero, by symmetry.

### **Answers:**

The answers for arbitrarily chosen points will vary.

### Version 1:

Four regions:  $(2^2) (f(-2, 0) + f(-2, -2) + f(0, 0) + f(0, -2)) = -320$ Nine regions:  $(\frac{4}{3})^2 \left[ f\left(-2, \frac{2}{3}\right) + f\left(-2, -\frac{2}{3}\right) + f\left(-2, -2\right) + f\left(-\frac{2}{3}, \frac{2}{3}\right) + f\left(-\frac{2}{3}, -\frac{2}{3}\right) + f\left(-\frac{2}{3}, -2\right) + f\left(\frac{2}{3}, \frac{2}{3}\right) + f\left(\frac{2}{3}, -2\right) \right] = -\frac{640}{3} \approx -213.33$ Sixteen regions:  $(1^2) \left[ f(-2, 1) + f(-2, 0) + f(-2, -1) + f(-2, -2) + f(-1, 1) + f(-1, 0) + f(-1, -1) + f(-1, -2) + f(0, 1) + f(0, 0) + f(0, -1) + f(0, -2) + f(1, 1) + f(1, 0) + f(1, -1) + f(1, -2) \right] = -160$ 

### Version 2:

Four regions:  $(2^2) (f (2, 0) + f (2, 2) + f (0, 0) + f (0, 2)) = 320$ Nine regions:  $(\frac{4}{3})^2 \left[ f \left( 2, -\frac{2}{3} \right) + f \left( 2, \frac{2}{3} \right) + f \left( 2, 2 \right) + f \left( \frac{2}{3}, -\frac{2}{3} \right) + f \left( \frac{2}{3}, \frac{2}{3} \right) + f \left( \frac{2}{3}, 2 \right) + f \left( -\frac{2}{3}, 2 \right) \right] = \frac{640}{3} \approx 213.33$ Sixteen regions:  $(1^2) \left[ f (2, -1) + f (2, 0) + f (2, 1) + f (2, 2) + f (1, -1) + f (1, 0) + f (1, 1) + f (1, 2) + f (0, -1) + f (0, 0) + f (0, 1) + f (0, 2) + f (-1, -1) + f (-1, 0) + f (-1, 1) + f (-1, 2) \right] = 160$ 

### Version 3:

Four regions: 
$$(2^2) (f(1, 1) + f(1, -1) + f(-1, 1) + f(-1, -1)) = 0$$
  
Nine regions:  $(\frac{4}{3})^2 \left[ f(\frac{4}{3}, -\frac{4}{3}) + f(\frac{4}{3}, 0) + f(\frac{4}{3}, \frac{4}{3}) + f(0, -\frac{4}{3}) + f(0, 0) + f(0, \frac{4}{3}) + f(-\frac{4}{3}, -\frac{4}{3}) + f(-\frac{4}{3}, 0) + f(-\frac{4}{3}, \frac{4}{3}) \right] = 0$ 

Sixteen regions: 0 (by symmetry)

### **GROUP WORK 3: Justifying Properties of Double Integrals**

Put the students into groups. Have them read the section carefully, and then have some groups try to justify Equation 7, some Equation 8, and some Equation 9 for nonnegative functions f and g. They don't have to do a formal proof, but they should be able to justify these equations convincingly, either using sums or geometrical reasoning.

There is no handout for this activity.

### **GROUP WORK 4: Several Ways to Compute Double Integrals**

The students may need help finding good points to use for Problem 1. The Midpoint Rule works best.

### **Answers:**

Let f(x, y) = 2 - x - y. **I.**  $\left(\frac{1}{2}\right)^2 \left(f\left(\frac{1}{4}, \frac{1}{4}\right) + f\left(\frac{3}{4}, \frac{1}{4}\right) + f\left(\frac{1}{4}, \frac{3}{4}\right) + f\left(\frac{3}{4}, \frac{3}{4}\right)\right) = 1$  **2.**  $A(x) = \frac{3}{2} - x$  **3.**  $\int_0^2 \left(\frac{3}{2} - x\right) dx = 1$ 

### **HOMEWORK PROBLEMS**

**Core Exercises:** 1, 5, 6, 7, 9, 13, 14

Sample Assignment: 1, 2, 4, 5, 6, 7, 9, 10, 12, 13, 14, 17

Exercise	D	Α	Ν	G
1		×		
2		×		
4		×		
5		×	×	
6			×	
7	×			

Exercise	D	Α	Ν	G
9		×		×
10		×		×
12		×		
13		×		
14				×
17		×		

## GROUP WORK 1, SECTION 16.1 Back to the Park



The following is a map with curves of the same elevation of a region in Orangerock National Park:

Estimate (numerically) the average elevation over this region using the Midpoint Rule.

## GROUP WORK 2, SECTION 16.1 An Odd Function (Version 1)

In this exercise, we are going to try to approximate the double integral  $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5) dA$ . We start by partitioning the region  $[-2, 2] \times [-2, 2]$  into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the lower left corner of every small region, like so:



Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

## GROUP WORK 2, SECTION 16.1 An Odd Function (Version 2)

In this exercise, we are going to try to approximate the double integral  $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5) dA$ . We start by partitioning the region  $[-2, 2] \times [-2, 2]$  into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the upper right corner of every small region, like so:



Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

## GROUP WORK 2, SECTION 16.1 An Odd Function (Version 3)

In this exercise, we are going to try to approximate the double integral  $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5) dA$ . We start by partitioning the region  $[-2, 2] \times [-2, 2]$  into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the midpoint of every small region, like so:



Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

Approximation for four regions:

Approximation for nine regions:

Approximation for sixteen regions:

## GROUP WORK 4, SECTION 16.1 Several Ways to Compute Double Integrals

Consider the double integral  $\iint (2 - x - y) dA$ , where  $R = [0, 1] \times [0, 1]$ .

I. Estimate the value of the double integral using two equal subdivisions in each direction.

**2.** Fix x such that  $0 \le x \le 1$ . What is the area A(x) of the slice shown below?



**3.** Find the exact volume of the solid with cross-sectional area A(x) using single variable calculus.

16.2 ITERATED INTEGRALS

### SUGGESTED TIME AND EMPHASIS

 $\frac{1}{2}$ -1 class Essential material

### POINTS TO STRESS

- **I.** The meaning of  $\int_a^b \int_c^d f(x, y) \, dy \, dx$  for a positive function f(x, y) over a rectangle  $[a, b] \times [c, d]$ .
- 2. The statement of Fubini's Theorem and how it makes computations easier.
- 3. The geometric meaning of Fubini's Theorem: slicing the area in two different ways.

### **QUIZ QUESTIONS**

• **Text Question:** Consider Figures 1 and 2 in the text. Why is  $\int_a^b A(x) dx = \int_c^d A(y) dy$ ?



Answer: The two integrals express the volume of the same solid.

• Drill Question: Compute  $\int_0^3 \int_3^4 x^2 y \, dy \, dx$ . Answer:  $\frac{63}{2}$ 

### MATERIALS FOR LECTURE

- Revisit the example  $\iint_{[-2,2] \times [-2,2]} (x^3 + y^5) dA$  using integrated integrals, to illustrate the power of the technique of iteration.
- Use an alternate approach to give an intuitive idea of why Fubini's Theorem is true. Using equal intervals of length  $\Delta x$  and  $\Delta y$  in each direction and choosing the lower left corner in each rectangle, we can write the double sum in Definition 16.1.5 as the iterated sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta x \ \Delta y = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta x\right) \Delta y$$

which in the limit gives an iterated integral. Go through some examples such as  $\iint_{[-1,3] \times [-1,3]} xy \, dA$  and  $\iint_{[0,1] \times [0,1]} (2 - x - y) \, dA$  to demonstrate this approach.

• Remind the students how, in single-variable calculus, volumes were found by adding up cross-sectional areas. Take the half-cylinder  $f(x, y) = \sqrt{1 - y^2}$ ,  $0 \le x \le 2$ , and find its volume, first by the "old" method, then by expressing it as a double integral. Show how the two techniques are, in essence, the same.

### WORKSHOP/DISCUSSION

- Have the students work several examples, such as  $\iint_{[0,1] \times [0,1]} y \sqrt{1-x^2} \sin(2\pi y^2) dA$ , which can be computed as  $\int_0^1 f(x) dx \int_0^1 g(y) dy$ .
- Find the volume of the solids described by  $\iint_{\left[-\sqrt{2},\sqrt{2}\right]\times\left[-2,3\right]}(2-x^2) dA$  and  $\iint_{\left[-1,1\right]\times\left[-1,1\right]}(1+x^2+y^2) dA$ .

### **GROUP WORK I: Regional Differences**

If the students get stuck on this one, give them the hint that Problem 1(b) can be done by finding the double integral over the square, and then using the symmetry of the function to compute the area over R. The regions in the remaining problems can be broken into rectangles.

**Answers:** 1.(a)  $\frac{3}{4}$  (b)  $\frac{1}{2}$  **2.**(a) -8 (b) -5 (c)  $-\frac{9}{2}$ 

### **GROUP WORK 2: Practice with Double Integrals**

It is a good idea to give the students some practice with straightforward computations of the type included in Problem 1. It is advised to put the students in pairs or have them work individually, as opposed to putting them in larger groups. The students are not to actually compute the integral in Problem 2; rather, they should recognize that each slice integrates to zero. Think about what happens when integrating with respect to x first.

### Answers:

**1.** (a)  $\frac{24}{5}\sqrt{6} - \frac{10}{3}\sqrt{5} - \frac{6}{5}\sqrt{3} + \frac{8}{15}\sqrt{2}$  (b)  $-\frac{3}{2}\ln 3 + \frac{1}{2} + 2\ln 2$  (c)  $e^2 - 1 - e$  **2.** True

### **GROUP WORK 3:** The Shape of the Solid

In Problem 3, the students could first change the order of integration to more easily recognize the "pup tent" shape of the resulting solid.

### **Answers:**

**I.** Irregular tetrahedron, volume  $\frac{3}{2}$ 

**2.** Half a cylinder, volume  $12\pi$  **3.** "Pup tent", volume 5

### HOMEWORK PROBLEMS

Core Exercises: 1, 3, 6, 17, 23, 26, 32, 35

Sample Assignment: 1, 2, 3, 6, 9, 13, 15, 17, 20, 23, 26, 28, 32, 35

Exercise	D	Α	Ν	G
1		×		
2		×		
3		×		
6		×		
9		×		
13		×		
15		×		

Exercise	D	Α	Ν	G
20		×		
23				×
26		×		
28		×		
32		×		×
35		×		

## GROUP WORK 1, SECTION 16.2 Regional Differences

**I.** Calculate the double integral  $\iint_R (x + y) dA$  for the following regions *R*:



**2.** Calculate the double integral  $\iint_R (xy - y^3) dA$  for the following regions *R*.



## GROUP WORK 2, SECTION 16.2 Practice with Double Integrals

Compute the following double integrals:

I. (a) 
$$\iint_{[1,2] \times [0,1]} x \sqrt{1+y+x^2} \, dA$$

(b) 
$$\iint_{[0,1] \times [1,2]} \frac{x}{x+y} \, dx \, dy$$

(c) 
$$\iint_{[0,1]\times[1,2]} y e^{xy} dx dy$$

**2.** Is the statement  $\iint_{[0,1] \times [0,1]} \cos(2\pi (y^2 + x)) dA = 0$  true or false?

## GROUP WORK 3, SECTION 16.2 The Shape of the Solid

For each of the following integrals, describe the shape of the solid whose volume is given by the integral, then compute the volume.

 $I. \iint_{[0,1] \times [0,1]} (3 - 2x - y) \, dA$ 

**2.** 
$$\int_{-3}^{3} \int_{-2}^{2} \sqrt{4 - y^2} \, dy \, dx$$

**3.** 
$$\int_{-1}^{1} \int_{-2}^{3} (1 - |x|) dy dx$$

16.3 DOUBLE INTEGRALS OVER GENERAL REGIONS

### SUGGESTED TIME AND EMPHASIS

1 class Essential Material

### **POINTS TO STRESS**

- **I.** The geometric interpretation of  $\int_a^b \int_{f(y)}^{g(y)} dx \, dy$  and  $\int_c^d \int_{h(x)}^{k(x)} dy \, dx$ .
- 2. Setting up the limits of double integrals, given a region over which to integrate.
- **3.** Changing the order of integration.

### **QUIZ QUESTIONS**

• **Text Question:** Sketch a region that is type II and not type I, and then sketch one that is both type II and type I.

**Answer:** 



Type I; not type II

Both type I and type II

• Drill Question: Is it true that  $\int_0^1 \int_x^1 f(x, y) dy dx = \int_0^1 \int_y^1 f(x, y) dx dy$ ? Answer: Yes, because the limits of integration describe the same region.

### MATERIALS FOR LECTURE

- Clarify why we bother with the fuss of defining  $F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$  instead of "just integrating over *D*".
- Show that the order of integration matters when computing  $\iint e^{x^2} dA$ , where D is the region shown below.



- Point out that  $\iint_D 1 \, dA$  gives the area of *D*, and that  $\int_a^b \int_{f(x)}^{g(x)} 1 \, dy \, dx$  gives the usual formula for the area between curves.
- Evaluate  $\iint_D h \, dA$ , where *D* is a circle of radius *r* and *h* is constant. Show how this gives the general formula for the volume of a cylinder. Ask the students to evaluate  $\iint_D h \, dA$ , where *D* is the parallelogram with vertices (1, 1), (2, 3), (5, 1), and (6, 3). Have them interpret their answer in terms of volume.

### WORKSHOP/DISCUSSION

• Let *D* be the region shown below. Set up  $\iint_D f(x, y) dA$  both as a type I integral  $\left[\iint_D f(x, y) dx dy\right]$ and a type II integral  $\left[\iint_D f(x, y) dy dx\right]$ .



- Evaluate  $\iint_D yx^2 dA$ , where D is the unit circle, both as a type I integral and as a type II integral.
- Show how to change order of integration in  $\int_0^1 \int_{x^4}^{x^{1/3}} f(x, y) dy dx$ .
- Change order of integration for  $\int_0^1 \int_{(2/\pi) \arcsin y}^{y^{1/3}} f(x, y) dx dy$ .

### GROUP WORK I: Type I or Type II?

Before handing out this activity, remind the students of the definitions of type I and type II regions given in the text.

#### **Answers:**

I. Both 2. Both 3. Type II 4. Neither 5. Type II 6. Type I 7. Both 8. Type II 9. Neither 10. Both

### **GROUP WORK 2: Fun with Double Integrals**

In this section, it is probably best to have the students get experience in setting up many double integrals, rather than have them spend a lot of time solving only one. During the homework, of course, both skills should be practiced.

### **Answers:**

- **I.** (a)  $\int_0^{3/4} \int_{x/4}^{x-x^2} dy \, dx$ . This can also be done as two integrals in  $dx \, dy$ .
  - (b)  $\int_0^1 \int_{x^2}^{\sqrt[4]{x}} dy \, dx$  or  $\int_0^1 \int_{y^4}^{\sqrt{y}} dx \, dy$
  - (c)  $\int_0^{1/2} \int_{y^2}^{\sqrt[4]{y}} dx \, dy$ . This can also be done as two integrals in  $dy \, dx$ .
  - (d)  $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} dx \, dy$ . This can also be done as two integrals in  $dy \, dx$ .
- 2. A hemisphere

### **GROUP WORK 3: Bounding on a Disk**

You may or may not tell your students that there is no such thing as Nentebular science. The surface  $\frac{1}{x^2 + y^2 + 1}$  lends itself naturally to this type of analysis, as seen by its graph.



Before handing out this activity, review the method of finding the area of an annulus. Close the activity by pointing out that if we continued the process of dividing *D* into smaller and smaller rings, the upper and lower bounds would approach each other, converging on the actual area:

$$\int_0^{2\pi} \int_0^5 \frac{1}{r^2 + 1} r \, dr \, d\theta = \pi \ln 2 + \pi \ln 13 \approx 3.2581\pi$$

#### **Answers:**

- 1. Answers will vary here. The upper bound should be no worse than  $25\pi \approx 78.5$  (setting the function equal to the constant 1, its maximum) and the lower bound should be no worse than  $\frac{25\pi}{26} \approx 3.02$  (setting the function equal to the constant  $\frac{1}{26}$ , its minimum).
- **2.** Upper:  $4\pi (1) + (25\pi 4\pi) \left(\frac{1}{5}\right) = \frac{41}{5}\pi \approx 25.8$ . Lower:  $4\pi \left(\frac{1}{5}\right) + (25\pi 4\pi) \left(\frac{1}{26}\right) = \frac{209}{130}\pi \approx 5.05$ .
- **3.** Upper:  $\pi (1) + (9\pi \pi) \left(\frac{1}{2}\right) + (25\pi 9\pi) \left(\frac{1}{10}\right) = \frac{33}{5}\pi \approx 20.7.$ Lower:  $\pi \left(\frac{1}{2}\right) + (9\pi - \pi) \left(\frac{1}{10}\right) + (25\pi - 9\pi) \left(\frac{1}{26}\right) = \frac{249}{130}\pi \approx 6.02.$

### **HOMEWORK PROBLEMS**

**Core Exercises:** 1, 5, 10, 13, 22, 32, 43, 45, 51

Sample Assignment: 1, 5, 8, 10, 13, 17, 20, 22, 25, 29, 32, 33, 40, 43, 45, 51, 53, 58, 59

Exercise	D	Α	Ν	G
1		×		
5		×		
8		×		
10		×		
13		×		
17		×		
20		×		
22		×		
25		×		
29				×

Exercise	D	Α	Ν	G
32		×		
33				×
40				×
43				×
45		×		
51		×		×
53		×		
58	×			×
59		×		

## GROUP WORK I, SECTION 16.3 Type I or Type II?

Classify each of the following regions as type I, type II, both, or neither.



### **GROUP WORK 2, SECTION 16.3**

### Fun with Double Integration

**I.** Write double integrals that represent the following areas.





**2.** What solid region of  $\mathbb{R}^3$  do you think is represented by  $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dy \, dx$ ?

## GROUP WORK 3, SECTION 16.3 Bounding on a Disk

One integral that is very important in Nentebular science is the Ossidoot integral:

$$\iint_D \frac{1}{x^2 + y^2 + 1} \, dA$$

For these sorts of integrals the domain *D* is usually a disk. In this exercise, we are going to find upper and lower bounds for this integral, where *D* is the disk  $0 \le x^2 + y^2 \le 25$ .

- I. It is possible to get some crude upper and lower bounds for this integral over D without any significant calculations? Find upper and lower bounds for this integral (perhaps crude ones) and explain how you know for sure that they are true bounds.
- **2.** One can get a better estimate by splitting up the domain as shown in the graph below, and bounding the integral over the inside disk and then over the outside ring. Using this method, what are the best bounds you can come up with?



3. Now refine your bounds by looking at the domain D as the union of three domains as shown.



16.4 DOUBLE INTEGRALS IN POLAR COORDINATES

### SUGGESTED TIME AND EMPHASIS

1 class Essential material. If polar coordinates have not yet been covered, the students should read Section 11.3.

### **POINTS TO STRESS**

- **I.** The definition of a polar rectangle: what it looks like, and its differential area  $r dr d\theta$
- 2. The idea that some integrals are simpler to compute in polar coordinates
- 3. Integration over general polar regions

### **QUIZ QUESTIONS**

• Text Question: Is the area of a polar rectangle  $R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$  equal to  $(b-a)(\beta - \alpha)$ ?

Answer: No

• Drill Question: Convert  $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$  to polar coordinates.

**Answer:** 
$$\int_0^{2\pi} \int_0^1 r^3 dr d\theta$$

### MATERIALS FOR LECTURE

- Start by reminding the students of polar coordinates, and ask them what they think the polar area formula will be: what will replace dx dy?
- Draw a large picture of a polar rectangle, and emphasize that it is the region between the gridlines  $r = r_0$ ,  $r = r_0 + \Delta r$ ,  $\theta = \theta_0$ , and  $\theta = \theta_0 + \Delta \theta$ .



The following method can be used to show that  $Area(\Delta R) \approx r \Delta r \Delta \theta$  if  $\Delta r$  and  $\Delta \theta$  are small:

I. Using the formula for the area of a circular sector, we obtain

Area 
$$(\Delta R) = \frac{1}{2} (r + \Delta r)^2 \Delta \theta - \frac{1}{2} (r)^2 \Delta \theta$$
  
=  $r \Delta r \Delta \theta + \frac{1}{2} (\Delta r)^2 \Delta \theta$ 

**2.** We now take the limit as  $\Delta r$ ,  $\Delta \theta \to 0$ , of  $\frac{\text{Area}(\Delta R)}{r \Delta r \Delta \theta}$  and show that this limit is equal to 1.

$$\lim_{\Delta r \to 0, \ \Delta \theta \to 0} \frac{\operatorname{Area} \left(\Delta R\right)}{r \ \Delta r \ \Delta \theta} = \lim_{\Delta r \to 0, \ \Delta \theta \to 0} \frac{r \ \Delta r \ \Delta \theta + \frac{1}{2} \ (\Delta r)^2 \ \Delta \theta}{r \ \Delta r \ \Delta \theta}$$
$$= \lim_{\Delta r \to 0, \ \Delta \theta \to 0} 1 + \frac{\Delta r}{2r} = 1$$

**3.** The result follows, since when  $\Delta r$  and  $\Delta \theta$  are small,

$$\frac{\operatorname{Area}\left(\Delta R\right)}{r\ \Delta r\ \Delta \theta}\approx 1$$

or

Area 
$$(\Delta R) \approx r \Delta r \Delta \theta$$

• Show how to set up the two general types of polar regions. (The second type occurs less frequently, and may be omitted.) Then indicate to the students that polar coordinates are most useful when one has an obvious center of symmetry for the region *R* in the *xy*-plane.



• Point out that polar areas can be found by setting up (double) polar integrals of the function  $f(r \cos \theta, r \sin \theta) = 1$  and compare this method to using the formula  $A = \int_a^b \frac{1}{2}r^2 d\theta$ . Calculate

- 1. The area between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 2$  in the second quadrant.
- 2. The area inside the spiral  $r = \theta$  where  $0 \le \theta \le \pi$ .

3. 
$$A = \{(x, y) \mid -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2} \}.$$

4. The area inside the first loop of the curve  $r = 2 \sin \theta \cos \theta$ .

### WORKSHOP/DISCUSSION

Illustrate that difficult problems can sometimes be simplified using geometry and symmetry by finding the area inside the circles x<sup>2</sup> + y<sup>2</sup> = 1 and (x - 1)<sup>2</sup> + y<sup>2</sup> = 1 (or r = 2 cos θ). First find the point of intersection (1, π/3) (in polar coordinates). Next, break the double integral for the portion in the first quadrant into two pieces as shown in the figure. The lighter region is a sector of angle π/3 of the unit disk (which contributes π/6 to the area), the darker region is a simple polar integral, and the total area between the circles is double the area above the x-axis.



#### CHAPTER 16 MULTIPLE INTEGRALS

- Present some different uses of polar integrals:
  - **I.** Compute  $\iint_R (x^2 + y^2)^2 dA$ , where *R* is the region enclosed by  $x^2 + y^2 = 4$  between the lines y = xand y = -x.



**2.** Compute the volume between the cone  $z^2 = x^2 + y^2$  and the paraboloid  $z = 4 - x^2 - y^2$ .

### **GROUP WORK I: The Polar Area Formula**

Perhaps start by having students guess the final answer to Problem 1. Notice that the students will have to be able to antidifferentiate  $\cos^2 \theta$  and  $\sin^2 \theta$  to complete this activity.

#### **Answers:**

**1.** 
$$2\int_0^{\pi} \frac{1}{2} (1 + \cos\theta)^2 d\theta = \frac{3\pi}{2}$$
 **2.**  $\int_0^{\pi} \left[ \frac{1}{2} (1 + \sin\theta)^2 - \frac{1}{2} (1)^2 \right] d\theta = \frac{\pi}{4} + 2$ 

### **GROUP WORK 2: Fun with Polar Area**

Many students will write  $\int_0^{3\pi} \int_0^{\theta} r \, dr \, d\theta$ . Try to get them to see for themselves that they must now subtract the area of the region that is counted twice in this expression.

### **Answers:**

**2.**  $\int_0^{\pi} \int_0^2 e^{-r^2} r \, dr \, d\theta = \frac{\pi}{2} \left( 1 - e^{-4} \right)$ 1.  $\frac{9}{2}\pi^3 - \frac{19}{6}\pi^3 = \frac{4}{3}\pi^3$ 

### **GROUP WORK 3: Fun with Polar Volume**

If the students get stuck on Problem 2, point out that the intersection can be shown to be  $x^2 + y^2 = \frac{3}{4}$ ,  $z = \frac{1}{2}$ , and hence the integral is over the region  $x^2 + y^2 \le \frac{3}{4}$ . If time is an issue, the students can be instructed to set up the integrals without solving them.

### **Answers:**

$$1. \int_{0}^{2\pi} \int_{0}^{1} \left(\sqrt{1-r^{2}}+1\right) r \, dr \, d\theta = \frac{5\pi}{3}$$
$$2. \int_{0}^{2\pi} \int_{0}^{3/4} \left[\sqrt{1-r^{2}}-\left(1-\sqrt{1-r^{2}}\right)\right] r \, dr \, d\theta = \frac{\pi}{48} \left(37-7\sqrt{7}\right)$$

### HOMEWORK PROBLEMS

**Core Exercises:** 2, 5, 11, 15, 22, 30, 33, 35

Sample Assignment: 2, 5, 8, 11, 15, 17, 20, 22, 25, 30, 31, 33, 35

Exercise	D	Α	Ν	G
2	×	×		
5		×		×
8		×		
11		×		
15		×		
17		×		
20		×		
22		×		
25		×		
30		×		
31		×		
33	×	×		
35		×		

## GROUP WORK 1, SECTION 16.4 The Polar Area Formula

The following formula is used for finding the area of a polar region described by the polar curve  $r = f(\theta)$ ,  $a \le \theta \le b$ :

$$A = \int_{a}^{b} \frac{1}{2} \left[ f(\boldsymbol{\theta}) \right]^{2} d\boldsymbol{\theta}$$

**I.** Use this formula to compute the area inside the cardioid  $r = 1 + \cos \theta$ .



**2.** Use the formula to find the area inside the cardioid  $r = 1 + \sin \theta$  and outside the unit circle r = 1.



### **GROUP WORK 2, SECTION 16.4**

Fun with Polar Area

**I.** Find the area of the region inside the curve  $r = \theta$ ,  $0 \le \theta \le 3\pi$ .



**2.** Rewrite  $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx$  as a polar integral and evaluate it.

### **GROUP WORK 3, SECTION 16.4**

### Fun with Polar Volume

I. Find the volume of the region bounded above by the upper hemisphere of the sphere  $x^2 + y^2 + (z - 1)^2 = 1$ and bounded below by the *xy*-plane.



**2.** Find the volume of the region bounded above by the sphere  $x^2 + y^2 + z^2 = 1$  and below by the sphere  $x^2 + y^2 + (z - 1)^2 = 1$ .

16.5 APPLICATIONS OF DOUBLE INTEGRALS

### SUGGESTED TIME AND EMPHASIS

1 class Optional material

### POINTS TO STRESS

I recommend stressing one of the following topics:

- I. Density, mass, and centers of mass (for an engineering- or physics-oriented course)
- 2. Probability and expected values (for a course oriented toward biology or the social sciences)

### **QUIZ QUESTIONS**

• **Text Question:** What is a logical reason that the total area under a joint density curve should be equal to 1?

Answer: The total area is the probability that *some* outcome will occur.

Drill Question: If a lamina has a uniform density and an axis of symmetry, what information do we then have about the location of the center of mass?
 Answer: It is on the axis of symmetry.

### MATERIALS FOR LECTURE

- Describe the general ideas behind continuous density functions, computations of mass, and centers of mass.
- Do one interesting mass problem. A good exercise is the mass over the unit disk if  $\rho(x, y) = |x| + |y|$ . This reduces to  $4 \int_0^1 \int_0^{\sqrt{1-x^2}} (x + y) \, dy \, dx$  which, surprisingly, equals  $\frac{8}{3}$ .
- Describe the general idea of the joint density function of two variables. Similarly, describe the concept of expected value. If time permits, show that  $f(x, y) = \frac{1}{2\pi} \frac{1}{(1 + x^2 + y^2)^{3/2}}$  describes a joint density function.

### WORKSHOP/DISCUSSION

• Define the centroid  $(\overline{x}, \overline{y})$  of a plane region R as the center of gravity, obtained by using a density of 1 for the entire region. So  $\overline{x} = \frac{1}{A(R)} \iint x \, dx \, dy$  and  $\overline{y} = \frac{1}{A(R)} \iint y \, dx \, dy$ . Show the students how to find the centroid for two or three figures like the following:

Show the students that if x = 0 is an axis of symmetry for a region *R*, then  $\overline{x} = 0$ , and more generally, that  $(\overline{x}, \overline{y})$  is on the axis of symmetry. Point out that if there are two axes of symmetry, then the centroid  $(\overline{x}, \overline{y})$  is at their intersection.

• Consider the triangular region *R* shown below, and assume that the density of an object with shape *R* is proportional to the square of the distance to the origin. Set up and evaluate the mass integral for such an object, and then compute the center of mass  $(\overline{x}, \overline{y})$ .



### **GROUP WORK I: Fun with Centroids**

Have the students find the centroids for some of the eight regions described earlier in Workshop/Discussion. Perhaps give each group of students two different ones to work on, and have them present their answers at the board.

Answers: I.  $\left(0, \frac{8}{3\pi}\right)$  2.  $\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$  3.  $\left(\frac{4}{5}, 0\right)$  4.  $\left(0, \frac{4}{5}\right)$  5.  $\left(0, \frac{2}{3}\right)$  6.  $\left(\frac{2}{3}a, \frac{2}{3}b\right)$  7.  $\left(\frac{1}{3}a, \frac{1}{3}b\right)$  8. (0, 0)

### **GROUP WORK 2: Verifying the Bivariate Normal Distribution**

Go over, in detail, Exercise 36 from Section 12.4, which involves computing the integral  $\int_{-\infty}^{\infty} e^{-x^2} dx$  using its double integral counterpart. Since these functions are related to the normal distribution and the bivariate normal distribution, the students can then actually show that  $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x/\mu)^2/(2\sigma)} = 1$ , as it should be.

Show that this is also true for  $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-a)/\mu)^2/(2\sigma)}$  by noting that replacing x by x-a just corresponds

to a horizontal shift of the integrand. There is no handout for this activity.

### **GROUP WORK 3: A Slick Model**

This activity requires a CAS and is based on the results of Group Work 2.

### **Answers:**

- **I.**  $K = \frac{1}{2\pi}$
- **2.** Both are 0. This is because the oil spread is radial. The expected values correspond to the center of mass, which is at the origin.
- **3.** Radius  $\approx 3.0348$

### **HOMEWORK PROBLEMS**

Core Exercises: 1, 4, 7, 15, 19, 24, 27, 30

Sample Assignment: 1, 4, 5, 7, 13, 15, 16, 19, 21, 24, 27, 30, 33

Exercise	D	Α	Ν	G
1		×		
4		×		
5		×		
7		×		
13		×		
15		×		
16		×		
19		×		
21		×		
24		×		
27		×		
30	×	×		
33	×	×		

GROUP WORK I, SECTION 16.5

## Fun with Centroids

Find the centroids of the following figures.

















### GROUP WORK 3, SECTION 16.5 A Slick Model

An oil tanker has leaked its entire cargo of oil into the middle of the Pacific Ocean, far from any island or continent. The oil has spread out in all directions in a thin layer on the surface of the ocean. The slick can be modeled by the two-dimensional density function  $K \exp\left(-2\frac{x^2 + y^2}{w^2}\right)$ , where w is a fixed constant and the origin of the xy-plane represents the location of the tanker. Assuming that none of the oil evaporates, the density function must account for all of the oil and hence can be interpreted as a probability distribution.

**I.** Suppose w = 2. Find the value of K which ensures that  $K \exp\left(-2\frac{x^2 + y^2}{w^2}\right)$  is a probability distribution.

**2.** Find the expected values  $\mu_x$  and  $\mu_y$  of x and y in the probability distribution from Problem 1. Interpret your answer geometrically.

3. Find the radius of the circle centered at the origin which contains exactly 99% of the oil in the slick.

I 6.6 TRIPLE INTEGRALS

### SUGGESTED TIME AND EMPHASIS

1 class Essential material

### POINTS TO STRESS

- **I.** The basic definition of a triple integral.
- 2. The various types of volume domain, and how to set up the volume integral based on each of them.
- **3.** Changing the order of integration in triple integrals.

### **QUIZ QUESTIONS**

- **Text Question:** Give an example of *one* region that is type I, type II, and type III. **Answer:** A tetrahedron and a sphere are both correct answers. Other answers are possible.
- Drill Question: In the expression ∫∫∫<sub>E</sub> f (x, y, z) dV, what does the dV mean?
   Answer: Either dx dy dz or a description of the geometric meaning of dV (the infinitesimal volume element) should be counted as correct.

### MATERIALS FOR LECTURE

• One way to introduce volume integrals is by revisiting the concept of area, pointing out that area integrals can be viewed as double integrals (for example  $\int_0^{10} f(x) dx = \int_0^{10} \int_0^{f(x)} dy dx$ ) and then showing how some volume integrals work by an analogous process. Set up a typical volume integral of a solid *S* using double integrals and similarly transform it into a triple integral:

$$V = \iint_R f(x, y) \, dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x, y) \, dy \, dx = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_0^{f(x, y)} dz \, dy \, dx$$

Then "move" the bottom surface of S up to z = g(x, y), so S has volume  $V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g(x,y)}^{f(x,y)} dz \, dy \, dx$ .



If we now have a function k(x, y, z) defined on S, then the triple integral of k over S is

$$\iiint_{S} k(x, y, z) \, dV = \int_{a}^{b} \int_{h_{1}(x)}^{h_{2}(x)} \int_{g(x, y)}^{f(x, y)} k(x, y, z) \, dz \, dy \, dx$$

• Show the students why the region shown is type 1, type 2, and type 3, and describe it in all three ways.



• Show how to identify the region of integration *E* shown for the volume integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} f(x, y, z) \, dz \, dy \, dx,$$

and then rewrite the integral as an equivalent iterated integral of the form

 $\iiint_E f(x, y, z) \, dx \, dz \, dy.$ 



• Do a sample computation, such as the triple integral of  $f(x, y, z) = z + xy^2$  over the volume V bounded by the surface sketched at right, in the first octant.



• Set up a triple integral for the volume V of the piece of the sphere of radius 1 in the first octant, with different orders of integration.

### CHAPTER 16 MULTIPLE INTEGRALS

• Sketch the solid whose volume is given by the triple integral

$$\int_{-\sqrt{(\sqrt{5}+1)/2}}^{\sqrt{(\sqrt{5}+1)/2}} \int_{y^2}^{\sqrt{1-y^2}} \int_0^{3-x-y} dz \, dy \, dx.$$
 Such a

sketch is shown at right.

• Compute  $\iiint_E (x^2 + y^2)^{1/2} dV$ , where *E* is the solid pictured at right, by first integrating with respect to *z* and then using polar coordinates in place of dx dy.



### **GROUP WORK I: The Square-Root Solid**

### Answers:



2. 
$$\int_{0}^{1} \int_{-\sqrt{1-z^{2}}}^{\sqrt{1-z^{2}}} \int_{-\sqrt{y^{2}-z^{2}}}^{\sqrt{y^{2}-z^{2}}} dx \, dy \, dz$$
  
3. 
$$\int_{-1}^{1} \int_{-\sqrt{1-y^{2}}}^{\sqrt{1-y^{2}}} \int_{\sqrt{x^{2}+y^{2}}}^{1} dz \, dx \, dy$$
  
4. 
$$\int_{-1}^{1} \int_{-\sqrt{1-x^{2}}}^{\sqrt{1-x^{2}}} \int_{\sqrt{x^{2}+y^{2}}}^{1} dz \, dy \, dx = \int_{0}^{2\pi} \int_{0}^{1} r^{2} \, dr \, d\theta = \frac{\pi}{3}$$
  
The students don't yet know how to convert the integral into cylindrical coordinates, but they can do polar integrals, and they can also use the formula for the volume of a cone.

### **GROUP WORK 2: Setting Up Volume Integrals**

### **Answers:**

1. Symmetry gives 
$$4\left(\int_{0}^{1}\int_{1}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-y^{2}}}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-y^{2}}}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx\right)$$
 or  
 $8\left(\int_{0}^{1}\int_{1}^{\sqrt{4-x^{2}}}\int_{0}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}}}\int_{0}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx\right)$ .  
2.  $\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{4-x^{2}}}^{\sqrt{4-x^{2}}}\int_{1}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx$   
3. Symmetry gives  $4\left(\int_{0}^{1}\int_{\sqrt{1-x^{2}}}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-y^{2}}}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-y^{2}}}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx\right)$  or  
 $8\left(\int_{0}^{1}\int_{\sqrt{1-x^{2}}}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}-y^{2}}}dz\,dy\,dx\right)$ .

#### SECTION 16.6 TRIPLE INTEGRALS

**4.** Symmetry gives 
$$4\left(\int_{0}^{1}\int_{1}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-z^{2}}}^{\sqrt{4-x^{2}-z^{2}}}dy\,dz\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}}}\int_{-\sqrt{4-x^{2}-z^{2}}}^{\sqrt{4-x^{2}-z^{2}}}dy\,dz\,dx\right)$$
 or  $8\left(\int_{0}^{1}\int_{1}^{\sqrt{4-x^{2}}}\int_{0}^{\sqrt{4-x^{2}-z^{2}}}dy\,dz\,dx + \int_{1}^{2}\int_{0}^{\sqrt{4-x^{2}}}\int_{0}^{\sqrt{4-x^{2}-z^{2}}}dy\,dz\,dx\right)$ 

### **GROUP WORK 3: An Unusual Volume**

This is a challenging group work for more advanced students. The idea is to show that just because a solid looks simple, the computation of its volume may be difficult. The line generated by  $P_1$  and  $P_2$  has equation  $z = -\frac{3}{4}x + 4$ , and hence this equation, interpreted in three dimensions, is also the equation of the plane *S*. The integral  $V(E) = \int_{-2}^{2} \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} \left(-\frac{3}{4}x + 4\right) dy dx$  requires polar coordinates to solve by hand (since the bounding circle has equation  $r = 4\cos\theta$ ,  $0 \le \theta \le \pi$ ) and also requires the students to remember how to integrate  $\cos^2 \theta$  and  $\cos^4 \theta$ . The volume is  $16\pi$ .

Note that the problem can be simplified by moving the solid so that the *z*-axis runs through the center of *D*. Point out that a simple geometric solution can be obtained by replacing *S* by the horizontal plane  $z = \frac{5}{2}$ , thus giving a standard cylinder.

### HOMEWORK PROBLEMS

Core Exercises: 5, 11, 13, 19, 24, 25, 27, 34, 39

Sample Assignment: 2, 5, 8, 10, 11, 13, 17, 19, 21, 23, 24, 25, 26, 27, 30, 34, 39, 42, 46, 50, 51

Exercise	D	Α	Ν	G
2		×		
5		×		
8		×		
10		×		
11		×		
13		×		
17		×		
19		×		
21		×		
23		×		
24		×		

Exercise	D	A	N	G	
25		×			
26		×			ſ
27				×	ſ
30		×			ſ
34		×		×	ſ
39		×			ſ
42		×			ſ
46	×	×			ſ
50		×			ſ
51		×			ſ

## GROUP WORK 1, SECTION 16.6 The Square-Root Solid

Consider the volume integral over the solid *S* given by  $V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz \, dy \, dx$ . I. Identify the solid *S* by drawing a picture.

**2.** Rewrite the volume integral as  $V = \iiint_S dx \, dy \, dz$ .

**3.** Rewrite the volume integral as  $V = \iiint_S dz \, dx \, dy$ .

4. Starting with the original iterated integral, compute the volume by any means at your disposal.

## GROUP WORK 2, SECTION 16.6 Setting Up Volume Integrals

Set up volume integrals for the following solids:

**I.** *S* is the solid formed by hollowing out a square cylinder of side length 1 centered along the *z*-axis through the solid sphere  $x^2 + y^2 + z^2 = 4$ .

**2.** S is the part of the solid sphere  $x^2 + y^2 + z^2 = 4$  lying above the plane z = 1.

**3.** *S* is the solid formed by drilling a cylindrical hole of radius 1 centered along the *z*-axis through the solid sphere  $x^2 + y^2 + z^2 = 4$ .

**4.** *S* is the solid formed by hollowing out a square cylinder of side length 1 centered along the *y*-axis through the solid sphere  $x^2 + y^2 + z^2 = 4$ .

## GROUP WORK 3, SECTION 16.6 An Unusual Volume

Consider the solid *E* shown below.



**I.** Find the equation of the plane S, parallel to the y-axis, which forms the top cap of E.

**2.** Set up a volume integral for *E* of the form  $V(E) = \int_{-2}^{2} \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} \left(-\frac{3}{4}+4\right) dx dy.$ 

**3.** Compute *V*(*E*) by any means at your disposal. *Hint:* Try polar coordinates.

## DISCOVERY PROJECT Volumes of Hyperspheres

Problems 1 and 2 review computations that the students may already know. Problem 4 is optional for this project, but it is highly recommended. To extend this project, students can be asked to find a book or article that discusses hyperspheres, and add some geometric discussion of these objects to their reports.

16.7 TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

### TRANSPARENCY AVAILABLE

#52 (Figure 7)

### SUGGESTED TIME AND EMPHASIS

 $\frac{1}{2}$ -1 class Essential material

### **POINTS TO STRESS**

- **I.** The cylindrical coordinate system as an extension of polar coordinates in  $\mathbb{R}^2$ .
- **2.** The basic shapes of cylindrical solids.
- **3.** The idea that the cylindrical coordinate system can be used to simplify equations and volume integrals of certain three-dimensional surfaces and solids.

### **QUIZ QUESTIONS**

• **Text Question:** Does the region of Example 3 have an axis of symmetry? If so, what does it say about the choice of using cylindrical coordinates?

**Answer:** The solid is symmetric about the *z*-axis, which implies that cylindrical coordinates should be considered.

• **Drill Question:** Describe in your own words the surface given by the equation  $r = \theta$ ,  $0 \le \theta \le 6\pi$  in cylindrical coordinates.

**Answer:** It looks something like a rolled-up piece of paper.

### MATERIALS FOR LECTURE

- Compute the intersection of the surfaces  $z = x^2 + y^2$  and z = x, first in rectangular coordinates, then in cylindrical coordinates. (Here is a case where the rectangular coordinates are the easiest to visualize, even though there is an  $x^2 + y^2$  term.)
- Point out that while  $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$  does not exist,  $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$  does exist. To see this, use cylindrical coordinates to get  $\frac{xyz}{x^2+y^2+z^2} = \frac{r^2z\sin\theta\cos\theta}{r^2+z^2}$ , and compare to  $\frac{r^2z}{r^2+z^2}$ , which approaches 0, as seen in Section 14.2.
- Convert a typical cylindrical volume integral of a solid *S* computed using double integrals into a triple integral:

$$V = \iint_{R} f(r, \theta) r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{h_{1}(\theta)}^{h_{2}(\theta)} f(r, \theta) r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{h_{1}(\theta)}^{h_{2}(\theta)} \int_{0}^{f(r, \theta)} r \, dz \, dr \, d\theta$$

#### SECTION 16.7 TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Move the bottom surface of *S* up to  $z = g(r, \theta)$ , as pictured at right. The volume now becomes  $\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} r \, dz \, dr \, d\theta$ . The basic volume element is given in Figure 3 of the text. Conclude with the situation where we have  $h(r, \theta, z)$  defined on *S*. Then the triple integral of *h* on *S* is  $\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} h(r, \theta, z) r \, dz \, dr \, d\theta$ .



#### WORKSHOP/DISCUSSION

- Describe in terms of cylindrical coordinates the surface of rotation formed by rotating z = 1/x about the *z*-axis, noting that there is an axis of symmetry. Point out that the equations come from simply replacing *x* by *r*.
- Develop a straightforward example such as the region depicted below:



The Capped Cone

Set this volume up as a triple integral in cylindrical coordinates, and then find the volume. (The computation of this volume integral is not that hard, and can be assigned to the students.) Conclude by setting up the volume integral of  $h(r, \theta, z) = rz$  over this region.

### **GROUP WORK: A Partially Eaten Sphere**

Notice that you are removing "ice cream cones" both above and below the xy-plane.

#### Answers:

**1.** 
$$36\pi - 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-r\sqrt{2}}^{\sqrt{6}} r \, dz \, dr \, d\theta = \pi \left( 36 - 10\sqrt{6} \right)$$
  
**2.**  $36\pi - 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-r\sqrt{2}}^{\sqrt{6}} zr^2 \sin \theta \, dz \, dr \, d\theta = 36\pi$ 

### HOMEWORK PROBLEMS

**Core Exercises:** 2, 3, 6, 9, 12, 15, 22, 27

Sample Assignment: 2, 3, 6, 8, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29

Exercise	D	Α	Ν	G
2				×
3	×	×		
6	×			
8	×			
9	×	×		
12				×
14				×
15		×		×
17		×		
20		×		
22		×		
24		×		×
25		×		
27		×		
29	×	×		

## GROUP WORK, SECTION 16.7 A Partially Eaten Sphere

I. Compute the volume of the solid *S* formed by starting with the sphere  $x^2 + y^2 + z^2 = 9$ , and removing the solid bounded below by the cone  $z^2 = 2(x^2 + y^2)$ .

**2.** Set up the triple integral  $\iiint_S yz \, dV$  in the same coordinate system you used for Problem 1.

## DISCOVERY PROJECT The Intersection of Three Cylinders

This discovery project extends the problem of finding the volume of two intersecting cylinders given in Exercise 66 in Section 6.2. This is a very thought-provoking project for students with good geometric intuition. It is worthwhile for students to work on it even if they don't wind up with the correct answer. If the students are mechanically inclined, perhaps ask them to build a model of the relevant solid. A good solution is given in the *Complete Solutions Manual*.

Transparency 53 can be used with this project.

16.8 TRIPLE INTEGRALS IN SPHERICAL COORDINATES

### TRANSPARENCY AVAILABLE

#52 (Figure 7)

### SUGGESTED TIME AND EMPHASIS

1–2 classes Essential material

### **POINTS TO STRESS**

- **I.** The geometry of the spherical coordinate system.
- 2. The basic shapes of solids in spherical coordinates.
- **3.** The idea that the spherical coordinate system can be used to simplify equations and volume integrals of certain three-dimensional surfaces and solids.

### **QUIZ QUESTIONS**

- Text Question: What surface is given by the equation  $\rho = 3$ ? Answer: A sphere
- Drill Question: What is the solid described by the integral  $\int_{\pi/2}^{\pi} \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} \rho^{2} \sin \varphi \, d\rho \, d\theta \, d\varphi$ ? Answer: The bottom half of a sphere with radius  $\sqrt{3}$

### MATERIALS FOR LECTURE

- One way to demonstrate spherical coordinates is to sit in a pivoting office chair, holding a yardstick or, better yet, an extendable pointer. Now  $\rho$  can be demonstrated in the obvious way,  $\theta$  by spinning in the chair, and  $\varphi$  by raising and lowering the yardstick. The instructor can do this, or students can come up and be asked to "touch" certain points with the ruler.
- Describe the coordinates of all points 3 units from the origin in each of the three coordinate systems. Repeat for the coordinates of all points on a circular cylinder of radius 2 with central axis the z-axis, and the coordinates of all points on the line through the origin with direction vector  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ . Conclude that cylindrical coordinates are the most useful in problems that involve symmetry about an axis, and that spherical coordinates are most useful where there is symmetry about a point.
- Identify the somewhat mysterious surface  $\rho \cos \phi = \rho^2 \sin^2 \phi \cos 2\theta$  given in spherical coordinates by using the formula  $\cos 2\theta = \cos^2 \theta \sin^2 \theta$  and changing to rectangular coordinates.

• Draw a basic spherical rectangular solid S and compute that its volume is approximately  $\rho^2 \sin \phi \, \Delta \rho \, \Delta \phi \, \Delta \theta$  if  $\Delta \rho$ ,  $\Delta \phi$ , and  $\Delta \theta$  are small. Calculate the volume of the solid pictured below to be



As usual, if there is a function  $l(\rho, \phi, \theta)$  on *S*, then the triple integral is  $\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(\phi,\theta)}^{f(\phi,\theta)} l(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$ 

• Give the students some integrands and ask them which coordinate system would be most convenient for integrating that integrand.

### WORKSHOP/DISCUSSION

• Discuss conversions from cylindrical and spherical to rectangular coordinates and back. For example, the surface  $\sin \phi = \frac{1}{\sqrt{2}}$  becomes the cone  $z = \sqrt{x^2 + y^2}$ , the surface  $r = z \cos \theta$  becomes  $xz = x^2 + y^2$ , and the surface  $z = r^2 (1 + 2 \sin^2 \theta)$  becomes the elliptic paraboloid  $z = x^2 + 3y^2$ .



#### SECTION 16.8 TRIPLE INTEGRALS IN SPHERICAL COORDINATES

• Indicate to the students why using spherical coordinates is a good choice for calculating the volume of *E*, the solid bounded above by the sphere  $x^2 + y^2 + z^2 = 4$  and below by the paraboloid  $4z = 4 - x^2 - y^2$ . Ask them if they think cylindrical coodinates would work just as well. Then compute *V*(*E*) using either method.



- Give a geometric description of the solid *S* whose volume is given in spherical coordinates by  $V = \int_0^{\pi} \int_{\pi/4}^{\pi/3} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ and then show the students how to write the volume of *S* as a triple integral in cylindrical coordinates.
- Revisit the "capped cone" example from Section 15.7 using spherical coordinates. Set the volume up as a triple integral in spherical coordinates, and then find the volume.



- Give the students some three-dimensional regions and ask them which coordinate system would be most convenient for computing the volume of that region. Examples:
  - 1. { $(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \le 4 \text{ and } -2 \le z \le 3$ }

2. 
$$\{(x, y, z) \in \mathbb{R}^3 \mid z^2 + y^2 \le 4 \text{ and } |x| \le 1\}$$

3.  $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 \ge x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \le 1\}$ 

4. 
$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{1}{4} \le x^2 + y^2 + z^2 \le 9, x \ge 0, y \ge 0, \text{ and } z \ge 0 \right\}$$

### **GROUP WORK I: Describe Me!**

This activity has problems that range from straightforward to conceptually tricky. When students have completed it, they will have a good intuitive sense of the three coordinate systems. They also tend to find this one enjoyable, particularly if they know ahead of time that the solutions will be distributed at the end. Answers follow the student handout.

### **GROUP WORK 2: Setting Up a Triple Integral**

If a group finishes Problems 1–4 quickly, have them choose one of their integrals and compute it, and explain why they made the choice they did. Note that for Problem 5, the solid is a truncated piece of the cone z = -r + R in cylindrical coordinates.

#### CHAPTER 16 MULTIPLE INTEGRALS

Answers:

1. 
$$\int_{-R}^{R} \int_{-\sqrt{R^{2}-x^{2}}}^{\sqrt{R^{2}-x^{2}}} \int_{-\sqrt{R^{2}-x^{2}-y^{2}}}^{\sqrt{R^{2}-x^{2}-y^{2}}} (x^{2} + y^{2}) dz dy dx = \frac{8}{15}\pi R^{5}$$
  
2. 
$$\int_{0}^{R} \int_{0}^{2\pi} \int_{-\sqrt{R^{2}-r^{2}}}^{\sqrt{R^{2}-r^{2}}} r^{3} dz d\theta dr = \frac{8}{15}\pi R^{5}$$
  
3. 
$$\int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{R} \rho^{4} \sin^{3} \varphi d\rho d\theta d\varphi = \frac{8}{15}\pi R^{5}$$
  
4. 
$$\int_{-1}^{1} \int_{-x}^{x} \int_{-\sqrt{x^{2}-y^{2}}}^{\sqrt{x^{2}-y^{2}}} dz dy dx = \frac{2}{3}\pi$$
  
5. 
$$\int_{0}^{2\pi} \int_{0}^{R/2} \int_{0}^{-z+R} zr^{3} dr dz d\theta = \frac{19}{1280}\pi R^{6}$$

### **GROUP WORK 3: Many Paths**

This activity should help the students to discover the utility of spherical coordinates. Start by defining "great circle" for the students. If a group finishes early, have the students replace (0, 0, -1) with (1, 0, 0) and parametrize the path of shortest length between these two points.

### Answers:

**1.**  $(\sin \pi t, 0, \cos \pi t), 0 \le t \le 1$  **2.**  $\left(\frac{1}{\sqrt{2}}\sin \pi t, \frac{1}{\sqrt{2}}\sin \pi t, \cos \pi t\right), 0 \le t \le 1$ **3.**  $\left(\sqrt{1-t^2}\sin \pi t, \sqrt{1-t^2}\cos \pi t, -t\right), -1 \le t \le 1$ 

### LABORATORY PROJECT: Equation of a Goblet

This project is fairly open-ended, giving the students a chance to stretch as far as they can. In groups, the students try to get the best equations describing a wine glass, or goblet. As a hint, explain how cylindrical coordinates are best suited for the task. There will be an equation for the base, one for the stem, and then one for the glass itself. If a CAS is available, the students can then graph their equations and pick a winning glass.



### **HOMEWORK PROBLEMS**

**Core Exercises:** 1, 4, 10, 17, 21, 28, 41

Sample Assignment: 1, 4, 5, 10, 13, 16, 17, 21, 24, 28, 30, 35, 40, 41, 44

Exercise	D	Α	Ν	G
1				×
4	×	×		
5	×			
10	×	×		
13				×
16	×	×		
17	×	×		
21		×		

Exercise	D	Α	Ν	G
24		×		
28		×		
30		×		
35		×		
40		×		
41				×
44		×		

## GROUP WORK I, SECTION 16.8 Describe Me!

Sketch, or describe in words, the following surfaces whose equations are given in cylindrical coordinates:

**1.**  $r = \theta$ **2.** z = r

**3.**  $z = \theta$ 

Sketch, or describe in words, the following surfaces whose equations are given in spherical coordinates:

**4.**  $\theta = \frac{\pi}{4}$  **5.**  $\phi = \frac{\pi}{4}$ **6.**  $\rho = \phi$ 

GROUP WORK I, SECTION 16.8

## Describe Me! (Solutions)





3.  $z = \theta$ 

4

z 0-

\_4

4





**5.**  $\phi = \frac{\pi}{4}$ 



 $\begin{array}{c}
0 \\
x
\end{array}$ 



y

-0



## GROUP WORK 2, SECTION 16.8 Setting Up a Triple Integral

**I.** Set up the triple integral  $\iiint_S (x^2 + y^2) dV$  in rectangular coordinates, where *S* is a solid sphere of radius *R* centered at the origin.

2. Set up the same triple integral in cylindrical coordinates.

- **3.** Set up the same triple integral in spherical coordinates.
- **4.** Set up an integral to compute the volume of the solid inside the cone  $x^2 = y^2 + z^2$ ,  $|x| \le 1$ , in a coordinate system of your choice.
- **5.** Compute  $\iiint_E z (x^2 + y^2) dV$ , where *E* is the solid shown below.



## GROUP WORK 3, SECTION 16.8 Many Paths

Three people live on the unit sphere, and are going to walk from the North Pole (0, 0, 1) to the South Pole (0, 0, -1). The first person walks along the arc of a great circle that lies in one of the coordinate planes. The second walks along an arc of a great circle that does *not* lie in one of the coordinate planes, and the third walks along a curve that spirals once around the sphere. Find parametric equations that describe possible paths for each person.

## APPLIED PROJECT Roller Derby

This project has a highly dramatic outcome: a real race can be run whose results are predicted by mathematics. An in-class demonstration of the "roller derby" can be done before this project is assigned.

If this project is used, it should be assigned in its entirety, in order to predict the outcome of the race.

16.9 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

### SUGGESTED TIME AND EMPHASIS

 $1-1\frac{1}{2}$  classes Optional material (essential if Chapter 17 is to be covered)

### **POINTS TO STRESS**

- **I.** Reason for change of variables: to reduce a complicated multiple integration problem to a simpler integral or an integral over a simpler region in the new variables
- **2.** What happens to area over a change in variables: The role of the Jacobian  $\frac{\partial (x, y)}{\partial (u, v)}$
- 3. Various methods to construct a change of variables

### **QUIZ QUESTIONS**

- Text Question: When we convert a double integral from rectangular coordinates to spherical coordinates, where does the ρ<sup>2</sup> sin φ term come from?
   Answer: It is the magnitude of the Jacobian.
- Drill Question: The *unit square* is the square with side length 1 and lower left corner at the origin. What is the area of the image *R* (in the *xy*-plane) of the unit square *S* (in the *uv*-plane) under the transformation x = u + 2v, y = -6u v? Answer: 11

### MATERIALS FOR LECTURE

- One good way to begin this section is to discuss *u*-substitution from a geometric point of view. For example,  $\int (\sin^2 x) \cos x \, dx$  is a somewhat complicated integral in *x*-space, but using the change of coordinate  $u = \sin x$  reduces it to the simpler integral  $\int u^2 du$  in *u*-space. If the students are concurrently taking physics or chemistry, discuss how the semi-logarithmic paper that they use is an example of this type of coordinate transformation.
- Note that it is very important that we take the absolute value of the Jacobian determinant. For example, point out that the Jacobian determinant for spherical coordinates is always negative (see Example 4). Another example of a negative Jacobian is the transformation x = u + 2v, y = 3u + v, which takes  $\langle 1, 0 \rangle$  to  $\langle 1, 3 \rangle$  and  $\langle 0, 1 \rangle$  to  $\langle 2, 1 \rangle$ .

• Consider the linear transformation x = u - v, y = u + v. This takes the unit square *S* in the *uv*-plane into a square with area 2.



Note that the Jacobian of this transformation is  $\frac{\partial(x, y)}{\partial(u, v)} = 2$ . In general, we have

$$A(R) = \iint_{R} 1 \, dA = \iint_{S} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| A(S)$$

So for linear transformations, the Jacobian is the determinant of the matrix of coefficients, and the absolute value of this determinant describes how area in uv-space is magnified in xy-space under the transformation T.

• Pose the problem of changing the rectangle  $[0, 1] \times [0, 2\pi]$  in the *uv*-plane into a disk in the *xy*-plane by a change of variable  $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$ . Show that  $x = u \cos v$ ,  $y = u \sin v$  will work.



Then  $u^2 = x^2 + y^2$  and  $\tan v = y/x$ , so u can be viewed as the distance to the origin and v is the angle with the positive x-axis. This implies that the u, v transformation is really just the polar-coordinate transformation. Note that for this transformation the Jacobian is equal to u, so we get  $u \, du \, dv$  as in polar coordinates. The grid lines  $u = c \ge 0$  go to circles, and v = c go to rays. Therefore uv-rectangles go to polar rectangles in xy-space. Perhaps note that trying to look at this transformation in reverse leads to problems at the origin. See if the students can determine what happens to the line u = 0. Also note that there are other transformations that work, such as  $x = u \sin v$ ,  $y = u \cos v$ .

• Care should be taken near singularities in the coordinate system. The following is a good historical example:

Consider the curve  $y = x^{1/3}$ . This curve has a singularity in its derivative at x = 0. However, if we make the change of coordinates u = x,  $v = y^3$ , then our curve becomes the line u = v. So was the original singularity really a singularity at all, or just a problem with the original coordinate system?

This question came up in physics at the beginning of this century. Schwarzchild solved a problem in general relativity involving the nature of space-time near a star. However, it turned out that the coordinates that he used to solve the problem had two singularities: one at the center of the star, and one at a nonzero

distance away from the center, which distance depended on the mass of the star. For some time, people wondered if this second singularity was a genuine singularity or, in fact, a problem with the coordinate system. It was shown later (by Kruskal) that the second singularity was not real, but merely an artifact of Schwarzchild's coordinate system.

#### WORKSHOP/DISCUSSION

• Pose the problem of changing a rectangle into the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} \le 1$ . Posit that the answer might again be of the form  $x = c_1 u \cos v$ ,  $y = c_2 u \sin v$  and solve  $\frac{x^2}{c_1^2} + \frac{y^2}{c_2^2} = u^2$ . So chose  $c_1 = 2$ ,  $c_2 = 3$  and then the rectangle  $[0, 1] \times [0, 2\pi]$  maps into the specified ellipse. This time,  $\frac{2}{3} \tan v = \frac{y}{x}$  so  $\tan v = \frac{3y}{2x}$ . The grid line u = c > 0 goes to the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} = c$  and v = k goes to the ray  $y = \frac{2}{3}x \tan k$ . This means that (u, v) gives an elliptical coordinate system. The Jacobian is  $\begin{vmatrix} 2\cos v & 3\sin v \\ -2u\sin v & 3u\cos v \end{vmatrix} = 6u$ , leading to  $6u \, du \, dv$ . Using this new coordinate system to compute  $\iint_R x^2 \, dA$  where R is the region bounded by the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ , we have  $\iint_R x^2 \, dA = \frac{1}{2} \int_R x^2 \, dA$ 

$$\int_0^{2\pi} \int_0^1 (2u\cos v)^2 |6u| \, du \, dv = \int_0^{2\pi} 6\cos^2 v \, dv = 6\pi.$$

• Describe how to find a transformation which maps the *uv*-plane as follows:



Show how it is sufficient to check what happens to (0, 1) and (1, 0).

• Consider the change of variables  $x = u^2 - v^2$ , y = 2uv described in Example 1 of the text. Show that the grid lines u = a give the parabolas  $x = a^2 - \frac{y^2}{4a^2}$ , and the grid lines v = b give the parabolas  $x = \frac{y^2}{4b^2} - b^2$ .

### **GROUP WORK I: Many Changes of Variables**

#### **Answers:**

1. We change variables:  $u = \frac{1}{2}x - y$ , v = 3x - y gives us the rectangle in the *uv*-plane bordered by  $u = -\frac{5}{2}$ , u = 0, v = 0, and v = 10. The inverse transformation is  $x = -\frac{2}{5}u + \frac{2}{5}v$ ,  $y = -\frac{6}{5}u + \frac{1}{5}v$ , giving an area element  $dA = \frac{2}{5} dv du$ . The integral becomes  $\int_{-5/2}^{0} \int_{0}^{10} \left(-\frac{2}{5}u + \frac{2}{5}v\right) \left(-\frac{6}{5}u + \frac{1}{5}v\right)^{2} dv du = \frac{6875}{12}$ . 2. x = 5u + 5v, y = -2u + 2v. The Jacobian is 20.

### **GROUP WORK 2: Transformed Parabolas**

### **Answers:**

**I.** By symmetry, we get  $2 \int_0^1 \int_0^1 (u^2 - v^2)^2 4 (u^2 + v^2) du dv = \frac{128}{105}$ .



The rectangle maps to the region bounded by the straight line from (-4, 0) to (1, 0), then up the parabola  $x = 1 - \frac{1}{4}y^2$  until the point (-3, 4), then down the parabola  $x = \frac{1}{16}y^2 - 4$ .

### **HOMEWORK PROBLEMS**

**Core Exercises:** 1, 3, 8, 13, 15, 17, 19

Sample Assignment: 1, 3, 4, 8, 12, 13, 15, 17, 19, 20, 21, 23

Exercise	D	Α	Ν	G
1		×		
3		×		
4		×		
8		×		×
12		×		
13		×		
15		×		
17	×	×		
19		×		
20		×		
21		×		
23		×		

### **GROUP WORK I, SECTION 16.9**

### Many Changes of Variables

**I.** We wish to find  $\iint_R xy^2 dA$ , where *R* is given below.



Hint: Use a change of variables to create an equivalent integral over a rectangular region in the uv-plane.

**2.** Find a mapping T which maps the triangle bounded by (0, 0), (0, 1), and (1, 0) to the triangle bounded by (0, 0), (5, 2), and (5, -2). What is the Jacobian of T? What is the area of R?

# GROUP WORK 2, SECTION 16.9

### **Transformed Parabolas**

Consider the change of variables  $x = u^2 - v^2$ , y = 2uv described in Example 1 of the text.

**I.** Compute  $\iint_R x^2 dA$ , where *R* is the region shown below.

*Hint:* How is  $\iint_R x^2 dA$  related to  $\iint_{R_1} x^2 dA$ , where  $R_1$  is the portion of R above the *x*-axis?



**2.** Let *S* be the rectangle  $\{(u, v) \mid 0 \le u \le 1, 0 \le v \le 2\}$ . What is the image *T* in the *xy*-plane of *S* under this change of variables?

**3.** What is the area of T?

### I6 SAMPLE EXAM

Problems marked with an asterisk (\*) are particularly challenging and should be given careful consideration.

- **I.** Consider the function  $f(x, y) = x^y$  on the rectangle  $[1, 2] \times [1, 2]$ .
  - (a) Approximate the value of the integral  $\int_{1}^{2} \int_{1}^{2} f(x, y) dy dx$  by dividing the region into four squares and using the function value at the lower left-hand corner of each square as an approximation for the function value over that square.
  - (b) Does the approximation give an overestimate or an underestimate of the value of the integral? How do you know?
- **2.** Given that  $\int_0^{\pi/2} \frac{dx}{1 + \sin^2 x} = \frac{\pi}{2\sqrt{2}}$ ,
  - (a) evaluate the double integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{\left(1 + \sin^2 x\right) \left(1 + \sin^2 y\right)} \, dx \, dy$$

(b) evaluate the triple integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz \, dx \, dy$$

**3.** Consider the rugged region below:



- (a) Divide the region into smaller regions, all of which are Type I.
- (b) Divide the region into smaller regions, all of which are Type II.
- **4.** Rewrite the integral

$$\int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta$$

in rectangular coordinates.

### CHAPTER 16 SAMPLE EXAM

**5.** Evaluate  $\iint_D \cos(x^2 + y^2) dA$ , where  $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4\}$  is a washer with inner radius 1 and outer radius 2.



**6.** Consider the ellipse  $x^2 + 2y^2 = 1$ .



- (a) Rewrite this equation in polar coordinates.
- (b) Write an integral in polar coordinates that gives the area of this ellipse. *Note:* Your answer will not look simple.
- 7. Consider the rectangular prism *R* pictured below:



Compute  $\iiint_R 10 \, dV$  and  $\iiint_R x \, dV$ .

**8.** (a) Compute

$$\int_0^1 \int_{-1}^1 \int_0^{xy} 1 \, dz \, dx \, dy$$

and give a geometric interpretation of your answer.

(b) Compute

$$\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1 \, dz \, dx \, dy$$

and give a geometric interpretation of your answer.

9. A light on the *z*-axis, pointed at the origin, shines on the sphere  $\rho = 1$  such that  $\frac{1}{4}$  of the total surface area is lit. What is the angle  $\phi$ ?



**10.** Consider the region R enclosed by y = x, y = -x + 2,  $y = -\sqrt{1 - (x - 1)^2}$ :



Set up the following integrals as one or more iterated integrals, but do not actually compute them:

- (a)  $\iint_R (x+y) \, dy \, dx$
- (b)  $\iint_R (x+y) dx dy$

**II.** Consider the region *R* enclosed by y = x + 1, y = -x + 1, and the *x*-axis.



- (a) Set up the integral  $\iint_R xy \, dx \, dy$  in polar coordinates.
- (b) Compute the integral  $\iint_R xy \, dx \, dy$  using any method you know.
- **12.** Consider the double integral



where R is given by the region between the two semicircles pictured below:



- (a) Compute the shaded area.
- (b) Show that the function  $\frac{1}{9 (x^2 + y^2)^{3/2}}$  is constant on each of the two bounding semicircles.
- (c) Give a lower bound and an upper bound for the double integral using the above information.
- **I3.** Observe the following Pac-Man:



- (a) Describe him in polar coordinates.
- (b) Evaluate  $\iint_{Pac-Man} x \, dA$  and  $\iint_{Pac-Man} y \, dA$ .

**14.** Consider the triple integral

$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz \, dx \, dy$$

representing a solid S. Let R be the projection of S onto the plane z = 0.

- (a) Draw the region *R*.
- (b) Rewrite this integral as  $\iiint dz \, dy \, dx$ .
- **15.** Consider the transformation T: x = 2u + v, y = u + 2v.
  - (a) Describe the image S under T of the unit square  $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$  in the uv-plane using a change of coordinates.
  - (b) Evaluate  $\iint_{S} (3x + 2y) dA$
- 16. What is the volume of the following region, described in spherical coordinates:  $1 \le \rho \le 9, 0 \le \theta \le \frac{\pi}{2}$ ,  $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}$ ?
- **17.** Consider the transformation  $x = v \cos 2\pi u$ ,  $y = v \sin 2\pi u$ .
  - (a) Describe the image S under T of the unit square  $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ .
  - (b) Find the area of S.
- **18.** Consider the function f(x, y) = ax + by, where a and b are constants. Find the average value of f over the region  $R = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}.$

1. 
$$f(x, y) = x^{y}$$
  
(a)  
 $\int_{1}^{2} \int_{1}^{2} f(x, y) \, dy \, dx \approx f(1, 1) \cdot \frac{1}{2} \cdot \frac{1}{2} + f\left(\frac{3}{2}, 1\right) \cdot \frac{1}{2} \cdot \frac{1}{2} + f\left(\frac{3}{2}, \frac{3}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{2} + f\left(\frac{3}{2}, \frac{3}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac$ 

(b) This estimate is an underestimate since the function is increasing in the x- and y-directions as x and y go from 1 to 2.

2. (a) 
$$\int_{0}^{\pi/2} \int_{0}^{\pi/2} \frac{1}{(1+\sin^{2}x)(1+\sin^{2}y)} dx dy = \left(\int_{0}^{\pi/2} \frac{dx}{1+\sin^{2}x}\right) \left(\int_{0}^{\pi/2} \frac{dy}{1+\sin^{2}y}\right) = \left(\frac{\pi}{2\sqrt{3}}\right)^{2} = \frac{\pi^{2}}{12}$$
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### CHAPTER 16 SAMPLE EXAM SOLUTIONS

(b) 
$$\int_{0}^{\pi/2} \int_{0}^{\pi/2} \int_{1/(1+\sin^{2} x)}^{1/(1+\sin^{2} y)} dz \, dx \, dy = \int_{0}^{\pi/2} \int_{0}^{\pi/2} \left(\frac{1}{1+\sin^{2} y} - \frac{1}{1+\sin^{2} x}\right) dx \, dy$$
$$= \int_{0}^{\pi/2} \int_{0}^{\pi/2} \left(\frac{1}{1+\sin^{2} y}\right) dx \, dy - \int_{0}^{\pi/2} \int_{0}^{\pi/2} \left(\frac{1}{1+\sin^{2} x}\right) dx \, dy$$
$$= \frac{\pi^{2}}{4\sqrt{3}} - \frac{\pi^{2}}{4\sqrt{3}} = 0$$



4. 
$$\int_{0}^{2\pi} \int_{0}^{1} r^{2} dr d\theta = \int_{-1}^{1} \int_{-\sqrt{1-x^{2}}}^{\sqrt{1-x^{2}}} \sqrt{x^{2} + y^{2}} dy dx$$

5. 
$$\int_{0}^{2\pi} \int_{1}^{2} \cos(r^{2}) r \, dr \, d\theta = \pi (\sin 4 - \sin 1)$$
  
6.  $x^{2} + 2y^{2} = 1$   
(a)  $r^{2} (\cos^{2} \theta + 2 \sin^{2} \theta) = r^{2} (1 + \sin^{2} \theta) = 1, r \ge 0$   
(b)  $\int_{0}^{2\pi} \int_{0}^{1/\sqrt{1 + \sin^{2} \theta}} r \, dr \, d\theta$ 

**7.** Since the parallelepiped has volume 60, we have  $\iiint_R 10 \, dV = 600$ .

$$\iiint_R x \, dV = 12 \int_0^5 x \, dx = 12 \left(\frac{25}{2}\right) = 150$$

**8.** (a)  $\int_0^1 \int_{-1}^1 \int_0^{xy} 1 \, dz \, dx \, dy = \int_0^1 \int_{-1}^1 xy \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^2 y \right]_{-1}^1 dy = 0$ . The region between z = 0 and z = xy in the first quadrant is above the *xy*-plane, while a symmetric region is below the *xy*-plane in the second quadrant.



#### CHAPTER 16 MULTIPLE INTEGRALS

- (b)  $\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1 \, dz \, dx \, dy = 2 \int_0^1 \int_0^1 \int_0^{xy} dz \, dx \, dy = 2 \int_0^1 \left[ \frac{1}{2} x^2 y \right]_0^1 dy = 2 \int_0^1 \frac{1}{2} y \, dy = \frac{1}{2} y^2 \Big]_0^1 = \frac{1}{2}.$ This is the total volume between z = 0 and z = xy. Because we take the absolute value, the volumes do not cancel.
- **9.** Since the surface area is  $4\pi$ , we need to find  $\phi$  so that the area lit is  $\pi$ .

$$\pi = \int_0^{\phi} \int_0^{2\pi} \sin \phi \, d\theta \, d\phi = 2\pi \int_0^{\phi} \sin \phi \, d\phi = 2\pi \left( -\cos \phi + \cos 0 \right), \text{ so } \frac{1}{2} = 1 - \cos \phi \implies \cos \phi = \frac{1}{2}$$
$$\Rightarrow \phi = \frac{\pi}{3}.$$

- **10.** (a)  $\int_0^1 \int_{-\sqrt{1-(x-1)^2}}^x (x+y) \, dy \, dx + \int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{2-x} (x+y) \, dy \, dx$ 
  - (b)  $\int_{-1}^{0} \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} (x+y) \, dx \, dy + \int_{0}^{1} \int_{y}^{2y} (x+y) \, dx \, dy.$

Note that the circular part of the curve is  $y = -\sqrt{1 - (x - 1)^2}$  or  $x = 1 \pm \sqrt{1 - y^2}$ . (a)  $\int \frac{\pi/2}{2} \int \frac{1}{(\sin\theta + \cos\theta)} x^3 \sin\theta \cos\theta \, dx \, d\theta + \int \frac{\pi}{2} \int \frac{1}{(\sin\theta - \cos\theta)} x^3 \sin\theta \cos\theta \, dx \, d\theta$ 

**11.** (a) 
$$\int_0^{\pi/2} \int_0^{\pi/(\sin\theta + \cos\theta)} r^3 \sin\theta \cos\theta \, dr \, d\theta + \int_{\pi/2}^{\pi} \int_0^{\pi/(\sin\theta - \cos\theta)} r^3 \sin\theta \cos\theta \, dr \, d\theta$$
  
(b) 0

12. 
$$\iint_{R} \frac{1}{9 - (x^2 + y^2)^{3/2}} \, dA$$
  
(a)  $\frac{1}{2} (4\pi - \pi) = \frac{3\pi}{2}$ 

**I4.** (a)

- (b) Since the semicircles satisfy  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ , we have on  $x^2 + y^2 = 1$ ,  $\frac{1}{9(x^2 + y^2)^{3/2}} = \frac{1}{8}$  and on  $x^2 + y^2 = 4$ ,  $\frac{1}{9(x^2 + y^2)^{3/2}} = 1$ .
- (c) A lower bound is the minimum value times the area, that is,  $\frac{1}{8} \cdot \frac{3\pi}{2} = \frac{3\pi}{16}$ . An upper bound is the maximum value times the area, that is,  $1 \cdot \frac{3\pi}{2} = \frac{3\pi}{2}$ .
- **13.** (a)  $\{(r, \theta) \mid 0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{7\pi}{4}\}$ 
  - (b)  $\iint_{\text{Pac-Man}} x \, dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2 \cos\theta \, d\theta \, dr = \int_0^1 \left[ r^2 \sin\theta \right]_{\pi/4}^{7\pi/4} dr = -\sqrt{2} \int_0^1 r^2 \, dr = -\frac{\sqrt{2}}{3} \iint_{\text{Pac-Man}} y \, dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2 \sin\theta \, d\theta \, dr = \int_0^1 \left[ -r^2 \cos\theta \right]_{\pi/4}^{7\pi/4} dr = 0$



(b) 
$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz \, dx \, dy = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} \int_0^{xy} dz \, dy \, dx$$
  
**15.**  $x = 2u + v, y = u + 2v$ 



T maps the unit square in the uv-plane to the unit circle in the xy-plane.

(b) The area of S is  $\pi$ .

**18.** 
$$f_{\text{ave}} = \frac{\int_{-1}^{1} \int_{-1}^{1} (ax + by) \, dy \, dx}{\int_{-1}^{1} \int_{-1}^{1} 1 \, dy \, dx} = \frac{\int_{-1}^{1} 2ax \, dx}{4} = 0$$