# 12 INFINITE SEQUENCES AND SERIES

## 12.1 SEQUENCES

### SUGGESTED TIME AND EMPHASIS

1 class Essential material

### POINTS TO STRESS

**1.** The basic definition of a sequence; the difference between the sequences $\{a_n\}$ and the functional value $f(n)$.

**2.** The meanings of the terms "convergence" and "the limit of a sequence".

**3.** The notion of recursive sequences (including the use of induction and the Monotonic Sequence Theorem to establish convergence).

### QUIZ QUESTIONS

- **Text Question:** Could there be a sequence $\{a_n\} = \{f(n)\}$ such that $\lim_{x\to\infty} f(x)$ exists, but $\lim_{n\to\infty} a_n$ does not? Could $\lim_{n\to\infty} a_n$ exist, but not $\lim_{x\to\infty} f(x)$?

  **Answer:** No to the first question. Yes to the second; an example is $f(x) = \sin(2\pi x)$.
- **Drill Question:** Can you give an example of a sequence $\{a_n\}$ that is monotonic and bounded above and below, but $\lim_{n\to\infty} a_n$ does not exist?

  **Answer:** No such sequence exists, by the Monotonic Sequence Theorem.

### MATERIALS FOR LECTURE

- Point out that if $a_n = f(n)$ for some function $f$, and if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. Thus, $a_n$ converges if $f$ has a horizontal asymptote as $x \to \infty$. Note that the converse is not true. For example, take $a_n = \sin n\pi$ and $f(x) = \sin(\pi x)$.
- Discuss monotonicity and the Monotonic Sequence Theorem. Perhaps apply the theorem to show that the sequence $\{0.1, 0.12, 0.123, 0.1234, \ldots, 0.123456789, 0.12345678910, 0.1234567891011, \ldots\}$ converges. (The limit of this sequence is called the Champernowne Constant.) One interesting fact about the Champernowne Constant is that its decimal expansion clearly contains every possible finite sequence of numbers. For example, the sequence 3483721589712 will appear somewhere, because of the counting nature of the constant. So if one were to take any book and convert it to a number using the code $A = 1$, $B = 2$, etc., that book appears somewhere in the Champernowne Constant. (This book could be already written, or even a book that has not been written yet, such as one that reveals any person's life story — including their future!)
- Consider $a_n = \dfrac{(\cos n)^n}{\ln(n+1)}$ and ask students how they might determine the convergence or divergence of this sequence. Then remind them that $-1 \le (\cos n)^n \le 1$ for all $n$ and hence the Squeeze Theorem can be used to show that the limit is 0.

## WORKSHOP/DISCUSSION

- Determine the convergence or divergence of the following sequences $\{a_n\} = f(n)$ by first looking at $f(x)$. Make sure to write out the first few terms of the sequences for each case, to emphasize their discrete nature.
  1. $a_n = \dfrac{n}{1+n^2}$
  2. $b_n = \dfrac{\ln(1+2e^n)}{n}$
  3. $c_n = (n+1)^{1/2} - n^{1/2}$
  4. $d_n = \dfrac{1+n\cos(2\pi n)}{n}$
- Do a non-obvious example that uses the Squeeze Theorem to establish convergence, such as $a_n = \dfrac{\sin n + \cos n}{n^{2/3}}$.
- Compute the limit of a recursive sequence such as $a_1 = 2$, $a_n = 4 - \dfrac{1}{a_{n-1}}$, after first either proving convergence (using induction and the Monotonic Sequence Theorem) or giving a numerical argument for convergence.

## GROUP WORK 1: Practice with Convergence

After the students have warmed up by doing one or two of the problems as a class, have them start working on the others, checking one another's work by plotting the sequences on a graph. If a group finishes early, give them Group Work 2, the Random Decimal, which makes a nice sequel.

**Answers:**

**1.** Converges to 0 **2.** Diverges **3.** Converges to 0 **4.** Converges to $-1$

**5.** Converges to $-1$ **6.** Converges to $\frac{4}{3}$ **7.** Diverges **8.** Converges to 0

## GROUP WORK 2: The Random Decimal

This works as an addition to Group Work 1. It can also stand alone. (Groups of four or five work best for this problem) The students may not be familiar with the idea of concatenation, so you may want to do an example for them if they seem to be having trouble understanding the idea.

**Answers:** (Answers to the first two problems will vary.)

**1.** $a_1 = 0.5358$

**2.** $a_1 = 0.5358$
$a_2 = 0.53589793$
$a_3 = 0.535897932384$
$a_4 = 0.5358979323846264$
$a_5 = 0.53589793238462643383$

**3.** The sequence is always increasing, and has an upper bound (1 will always be an upper bound, for example; 0.6 is a better upper bound in this case.) Therefore, by the Monotone Convergence Theorem, this sequence does converge. It can be proven that if the numbers generated are truly random, then this number will be irrational.

## GROUP WORK 3: Recursive Roots

This problem gives the tools to show that if $x > 0$, then $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}$ exists and equals $\dfrac{1+\sqrt{1+4x}}{2}$. The students may find Questions 2 and 3 difficult. One hint to give them is that for $a$ and $b > 0$, $a < b \Rightarrow a^2 < b^2$. In a less rigorous class, it may be acceptable for students to notice that there is a trend (the terms are increasing, and approaching $1.6 < 2$) but they should also realize that noticing a trend isn't the same thing as proving that the trend will continue forever. This exercise can be done with less rigor by having the students skip Questions 2 and 3 entirely.

**Answers:** (Answers to the first two problems will vary.)

**1.** $a_1 = 0$
$a_2 = \sqrt{1+0} = 1$
$a_3 = \sqrt{1+\sqrt{1+0}} = \sqrt{2} \approx 1.4142$
$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+0}}} \approx 1.5538$
$a_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+0}}}} \approx 1.5981$

**2.** The easiest way to prove this is by induction. We want to show that if $a_n < 2$, then $\sqrt{1+a_n} < 2$. The base case is trivial $(0 < 2)$. The induction step: If $a_n < 2$, then $\sqrt{1+a_n} < \sqrt{1+2} < 2$. If the students haven't learned mathematical induction, this argument can be put into less formal language.

**3.** We now want to show $a_n < \sqrt{1+a_n}$ It suffices to show that $(a_n)^2 < 1 + a_n$, or $(a_n)^2 - a_n - 1 < 0$. The quadratic formula, or a graph, can show that this is true if $0 < a_n < \frac{1+\sqrt{5}}{2} \approx 1.618$. (Actually, this is true for $\frac{1-\sqrt{5}}{2} < a_n < \frac{1+\sqrt{5}}{2}$.) We can use an induction argument like the one in the previous part to show that if $a_n < \frac{1+\sqrt{5}}{2}$, then $a_{n+1} < \frac{1+\sqrt{5}}{2}$.

**4.** $a = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$. Therefore $a = \sqrt{1+a}$.

**5.** Since $a = \sqrt{1+a}$, we have $a^2 = 1 + a$ and $a = \frac{1+\sqrt{5}}{2}$.

**6.** 2, 3

## GROUP WORK 4: Euler's Constant Revisited

This activity revisits Group Work 4 in Section 5.4 (which also appears as Group Work 2 in Section 7.2*).

**Answers:**

**1.** It is the sum of positive terms.

**2.** Each time $n$ is incremented, more positive area is added to the total.

**3.** The Monotone Convergence Theorem gives the result.

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 3, 8, 9, 19, 28, 36, 50, 55, 59, 62

**Sample Assignment:** 2, 3, 6, 8, 9, 13, 15, 19, 21, 28, 36, 43, 50, 51, 55, 59, 62, 73, 75

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | × | | | |
| 3 | | × | | |
| 6 | | × | | |
| 8 | | × | | |
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| 59 | × | | | |
| 62 | | × | | |
| 73 | | | | × |
| 75 | | × | | |

# GROUP WORK 1, SECTION 12.1

## Practice with Convergence

Do the following sequences $\{a_n\}$ converge or diverge? Justify your answers.

**1.** $a_n = \dfrac{e^n}{3^n}$

**2.** $a_n = (-1)^n \sqrt{n}$

**3.** $a_n = (-1)^n \dfrac{1}{\sqrt{n}}$

**4.** $a_n = (-1)^{2n+1}$

**5.** $a_n = \dfrac{(-1)^n + n}{(-1)^n - n}$

**6.** $a_n = \dfrac{\ln\left(\left(e^4\right)^n\right)}{3n}$

**7.** $a_n = (-1)^n \cos\left(\frac{\pi}{2}(n+1)\right)$

**8.** $a_n = (-1)^n \sin\left(\frac{\pi}{2}(2n+1)\right)$

## GROUP WORK 2, SECTION 12.1
### The Random Decimal

**1.** Have each person in your group think of a random integer from 0 through 9. Let $a_1$ be $0.wxyz$ where $w, x, y$ and $z$ are your numbers. For example, if you came up with 2, 4, 1, and 8, then you would write $a_1 = 0.2418$.

$$a_1 = \_\_\_\_\_\_\_\_$$

**2.** Have each person in your group think of a new integer, and add those integers to the end of $a_1$ to form $a_2$. For example, if you already had $a_1 = 0.2418$, you might come up with $a_2 = 0.24185299$. Continue the process to form $a_3$, $a_4$ and $a_5$.

$a_2 =$ ________________

$a_3 =$ ________________________

$a_4 =$ ________________________________

$a_5 =$ ________________________________________

**3.** If you continued this process infinitely many times, you would have an infinite sequence $\{a_n\}$. Does this sequence converge, diverge, or is it impossible to tell? Why?

# GROUP WORK 3, SECTION 12.1

## Recursive Roots

We want to find the value of

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

**1.** Consider the recursive sequence $a_0 = 0$, $a_{n+1} = \sqrt{1+a_n}$. Compute the next five terms $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$.

**2.** Show that $a_n < 2$ for all $n$.

**3.** Show that $a_{n+1} > a_n$.

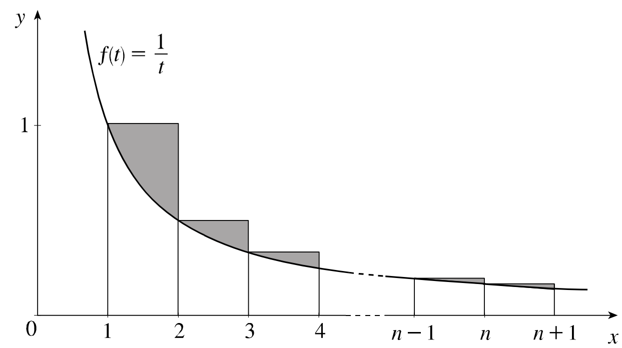
**4.** Since $\{a_n\}$ is increasing and bounded above by 2, the Monotone Sequence Theorem says that $\{a_n\}$ converges. If $\lim_{n\to\infty} a_n = a$, show that $a = \sqrt{1+a}$.

**5.** What is the value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$?

**6.** Using similar reasoning, try to compute $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$ and $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}}$.

## GROUP WORK 4, SECTION 12.1
### Euler's Constant Revisited

Recall the following picture:




and the sequence $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \int_1^{n+1} \frac{1}{t}\,dt$.

**1.** By a previous group work (Group Work 4 in Section 7.4), we know that $\gamma_n \leq 1$. Explain why $0 \leq \gamma_n$ for all $n$.

**2.** Using the picture above, explain why $\gamma_n$ is monotone increasing.

**3.** Why can we conclude that $\lim_{n\to\infty} \gamma_n = \gamma$ exists?

## LABORATORY PROJECT Logistic Sequences

This project is useful as an in-class extended exercise, if computers are available, or as an out-of-class project. Students are asked to numerically analyze the logistic difference equation, the discrete variant of the logistic differential equation, examining its long-term behavior. Problems 1 and 2 provide a basic analysis of the situation and can be covered in a shorter period. Problems 3 and 4 lead the students to discovering chaotic behavior.

In their report, students should include a paragraph about the similarities and differences between the behavior of the difference equation and the logistic growth differential equation from Section 10.4.

## 12.2 SERIES

### TRANSPARENCY AVAILABLE

#27 (Figures 2 and 3)

### SUGGESTED TIME AND EMPHASIS

2 classes  Essential material

### POINTS TO STRESS

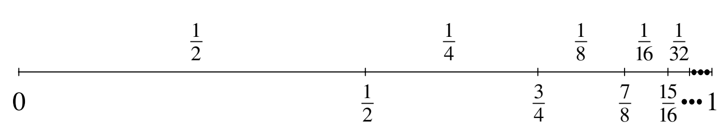
1. The basic concept of a series. The difference between the underlying sequence and the sequence of partial sums.
2. The relationship between $\lim_{n\to\infty} a_n$, and the convergence/divergence of $\sum_{n=1}^{\infty} a_n$.
3. The analysis and applications of geometric series.
4. The Test for Divergence.

### QUIZ QUESTIONS

- **Text Question:** Is the following always true, sometimes true, or always false? If the series $\sum_{n=1}^{\infty} a_n$ converges, and the series $\sum_{n=1}^{\infty} b_n$ converges, then their sum converges.
  **Answer:** Always true
- **Drill Question:** Does the series $0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + \cdots$ converge? If so, find its sum.
  **Answer:** $\frac{1}{9}$

### MATERIALS FOR LECTURE

- Present an intuitive approach to the definition of a series. Explain that the way we add an infinite number of terms is to keep adding more and more of them to a running total, in a systematic way, to create partial sums. If the limit of the partial sums exists, we say the series converges to that limit. Show the distinction between the $n$th term $a_n$, the $n$th partial sum $s_n$, and the connection between them ($s_{n-1} + a_n = s_n$).
- Discuss Theorems 6 and 7 and Note 2, explaining how the converse to Theorem 6 is not true in general. Use Theorem 7 to explain why $\sum_{n=1}^{\infty} \cos(1/n)$ diverges.
- Derive the formula for the sum of a geometric series. Illustrate why this type of series diverges for $|r| > 1$, and why it also diverges for $r = \pm 1$. Provide details about the geometric justification found in Figure 1.
- Represent a geometric series visually. For example, a geometric view of the equation $\sum_{n=1}^{\infty} 1/2^n = 1$ is given below.




An alternative geometric view is given in Group Work 3, Problem 2. If this group work is not assigned, the figure should be shown to the students at this time.

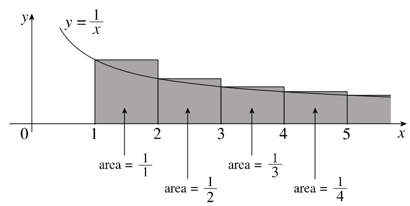
- Sometimes we can express a mathematical constant as the sum of a series. A classic example is $e = \sum_{k=1}^{\infty} \frac{1}{k!}$. Summations for $\pi$ are historically important, and a nice simple research project might be for students to find a few of the more unusual ones. In 1985, David and Gregory Chudnovsky used the series

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left[ \frac{\binom{2n}{n}}{16^n} \right]^3 \frac{42n+5}{16}$$

where $\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$, to compute $\pi$ to more than 4 billion decimal places. Each term of this series produces an additional 14 correct digits. (If the students have access to a CAS, this is a fun formula to play with.)

## WORKSHOP/DISCUSSION

- Foreshadow the Integral Test. Show that $\sum_{n=1}^{\infty} 1/n$ diverges by using an integral as a lower bound, as illustrated by the following figure:




- Ask the students to write down an example of a series $\sum_{n=1}^{\infty} a_n$ such that the terms of the series go to zero, yet the series diverges. Since the students have seen the harmonic series in both the lecture and in the text, this should be an easy question for them to answer. But there is something to be gained in their hearing an abstract question and thinking of a concrete example that they know.
- Introduce the idea that for any two real numbers $A$ and $B$, the statement $A = B$ is the same as saying that for any integer $N$, $|A - B| < 1/N$. Now use this idea to show that $0.9999\ldots = 0.\overline{9} = 1$, since $\left|1 - 0.\overline{9}\right| < \left|1 - 0.\underbrace{99999\ldots99}_{N \text{ nines}}\right| = 0.\underbrace{00000\ldots000}_{N-1 \text{ zeros}}1 = 10^{-N} = \frac{1}{10^N}$. Then use the usual approach to define $0.\overline{9}$ as $\sum_{n=1}^{\infty} 9/10^n$ and show directly that $0.\overline{9} = 1$. Generalize this result by pointing out that *any* repeating decimal ($0.\overline{3}$, $0.\overline{412}$, $0.24\overline{621}$) can be written as a geometric series, and can thus be written as a fraction using the formula for a geometric series. Demonstrate with $0.\overline{412} = \frac{412}{1000}\left(\frac{1}{1-1/1000}\right) = \frac{412}{999}$.
- Check the convergence/divergence of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{k^n},\ k > 1 \qquad \sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n} \qquad \sum_{n=1}^{\infty} (-1)^n$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n}{n+1}\right) \qquad \sum_{n=1}^{\infty} (-1)^{2n} \qquad \sum_{n=1}^{\infty} \left[\frac{5}{n\,(n+1)} - \left(-\frac{1}{2}\right)^n\right]$$

**Answers:**

$\sum_{n=1}^{\infty} \frac{1}{k^n}$, $k > 1$ is geometric and converges to $\frac{1}{k-1}$.

$\sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n}$ is a sum of two geometric series and converges to 10.

$\sum_{n=1}^{\infty} (-1)^n$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} \sin\left(\frac{n}{n+1}\right)$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} (-1)^{2n}$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - \left(-\frac{1}{2}\right)^n\right]$ is the sum of a geometric series and 5 times the series from Example 6, and converges to $\frac{16}{3}$.

- Using a diagram similar to Figure 2 in Section 12.3, show that $\ln n < \sum_{k=1}^{n} 1/k < 1 + \ln n$. Make sure the students know that $\sum_{k=1}^{\infty} 1/k$ goes to infinity. Now ask them this question: "We know that the harmonic series diverges. Assume that in the year 4000 B.C., you started adding up the terms of the harmonic series, at the rate of, say, one term per second. We know that the sum gets arbitrarily large, but approximately how big would your partial sum be as of right now?" (If you wish the students to discover some of these concepts for themselves, Have them explore Group Work 2: The Harmonic Series.)
- Define the "middle third" Cantor set for the students. (Let $C$ be the set of points obtained by taking the interval $[0, 1]$, throwing out the middle third to obtain $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, throwing out the middle third of each remaining interval to obtain $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and repeating this process *ad infinitum*). Point out that there are infinitely many points left after this process. (If a point winds up as the endpoint of an interval, it never gets removed, and new intervals are created with every step). Now calculate the total length of the sections that were thrown away: $\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \ldots = \sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} = 1$. Notice the apparent paradox: We've thrown away a total interval of length 1, but still infinitely many points remain. (See also Exercise 73.)

## GROUP WORK 1: The Leaning Tower

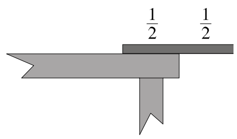
This exercise, an expansion of Problems Plus #12, should take about 45–50 minutes.

Group the students, and give them materials to stack. Packs of small notepads, CD jewel boxes, or wooden blocks all make good materials for stacking. Their goal is to make a stack with the top block one length away from the bottom, without having the stack fall over. (See the diagram below.)

Give them time to work. When a group achieves the goal, have them try to get two lengths out. After they have been working for a while, give them the hint that it is easiest to build onto the bottom, not the top. In other words, take a balanced stack, transfer it to a new bottom piece, and then slide it as far as possible.

When there are 25 minutes left, collect the blocks and model the situation on the board. The main thing to get across is the idea of center of mass. The center of mass will be the place where half of the mass is to the left, and half is to the right. (We don't care that much about the vertical component; it will be $n/2$ units up.) The stack balances if the center of mass is over the table, otherwise it falls.

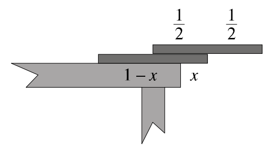
Have them try to solve the general problem: Given $n$ things to stack, what is the farthest that they can go? You may want to do $n = 1$ and $n = 2$ on the board to give them a start:




$n = 1$

Total weight on the right $= \frac{1}{2}n = \frac{1}{2}$

extension $= \frac{1}{2}$




$n = 2$

Total weight on the right $= \frac{1}{2}n = 1 = \frac{1}{2} + 2x \Leftrightarrow x = \frac{1}{4}$

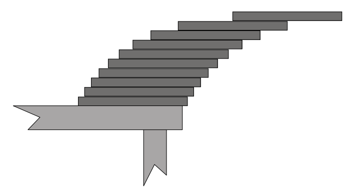
extension $= \frac{1}{2} + \frac{1}{4}$

After they have tried, if they did not succeed, write out the solution for $n = 3$ and $n = 4$ as follows:

$n = 3$: Total weight on the right $= \frac{3}{2} = 1 + 3x \quad \Leftrightarrow \quad x = \frac{1}{6}$; extension $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$.

$n = 4$: Total weight on the right $= 2 = \frac{3}{2} + 4x \quad \Leftrightarrow \quad x = \frac{1}{8}$; extension $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}$.

Therefore, if $n = k$, we have an extension of $\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}\right)$. *(See below.)



At this point they should recognize the harmonic series. So the answer to the question "What is the farthest that the stack can extend, given as many objects as desired?" is tied to the question "What is the sum of the harmonic series?" which they have already seen to be infinity. Emphasize how slowly it goes to infinity (perhaps by putting the figure at right on a transparency and noting how small the increments are at the bottom of the stack).

*Note that, since at some point the left edges of the blocks will begin to overhang, the expression $\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)$ is actually a lower bound on the possible extension of $k$ objects.

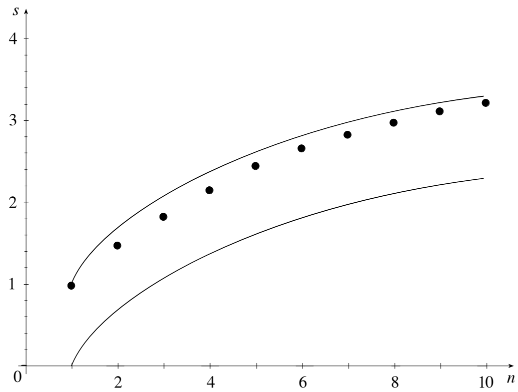
## GROUP WORK 2: The Harmonic Series

This activity was suggested by the Teachers Guide to AP Calculus published by the College Board. In addition to allowing the students to discover the divergence of the harmonic series for themselves, the last question will allow them to make an intuitive guess that will be confirmed or refuted by what they learn in the next section.

### Answers:

**1.** $s_1 = 1$, $s_2 = 1.5$, $s_3 \approx 1.8333$, $s_4 \approx 2.08333$, $s_5 \approx 2.28333$, $s_6 = 2.45$, $s_7 \approx 2.5929$, $s_8 \approx 2.7179$, $s_9 \approx 2.8290$, $s_{10} \approx 2.9290$

**2., 4.**




**3.** The partial sums appear to approach 3.

**5.** We know $\ln x$ goes to infinity, and the partial sum $s_n$ seems to always be larger than $\ln n$.

**6.** Neither of our answers is a proof, since we are just generalizing based on the first ten partial sums. There is a proof in the text that $\sum_{n=1}^{\infty} 1/n$ diverges, and one may have been given in class as well.

**7.** Defer revealing the answer to this question until the next section.

## GROUP WORK 3: Made in the Shade

Problem 1 attempts to help the students visualize geometric series. Problem 2 gives a geometric interpretation of the fact that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$.

**Answers:**

**1.** (a) $a = \frac{1}{2}\pi, r = \frac{1}{4}, A = \frac{2\pi}{3}$ (b) $a = \frac{1}{4}, r = \frac{1}{4}, A = \frac{1}{3}$

**2.** (a) $a_n = 1/2^n$

(b) 1. Note that the students may use the geometric series formula to get the answer, but it should be pointed out that the answer is immediate from the diagram.

## GROUP WORK 4: An Unusual Series and Its Sums

The initial student reaction may be "I have no idea where to start!" One option is to start the problem on the board for the students. Another approach may be to encourage the students to write down the length of the largest dotted line segment ($b$), then to figure out the length of the next one, which they can get using trigonometry, and keep going as long as they can. Many students still resist the idea of tackling a problem "one step at a time" if it seems difficult.

**Answers:**

**1.** $L = b + b\sin\theta + b\sin^2\theta + \cdots$ or $\sum_{n=1}^{\infty} b(\sin\theta)^n$. Because there are infinitely many terms, we need to write the answer as a series.

**2.** $L = \frac{b}{1-\sin\theta}$ **3.** $L$ approaches infinity.

**4.** Geometrically: As $\theta \to \frac{\pi}{2}$, the picture breaks down. The easiest way to see this is to have the students try to sketch what happens for $\theta$ close to $\frac{\pi}{2}$. The dotted lines become infinitely dense.

Using infinite sums: $\lim\limits_{\theta\to\pi/2} \dfrac{b}{1-\sin\theta}$ diverges.

## HOMEWORK PROBLEMS

**Core Exercises:** 1, 4, 14, 23, 29, 35, 42, 57, 58, 64

**Sample Assignment:** 1, 4, 5, 9, 14, 16, 23, 28, 29, 30, 35, 42, 47, 55, 57, 58, 59, 62, 64, 72

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 1 | × | | | |
| 4 | × | × | | |
| 5 | × | × | | |
| 9 | | × | | |
| 14 | | × | | |
| 16 | | × | | |
| 23 | | × | | |
| 28 | | × | | |
| 29 | | × | | |
| 30 | | × | | |

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 35 | | × | | |
| 42 | | × | | |
| 47 | | × | | |
| 55 | | × | | |
| 57 | × | × | | |
| 58 | | × | | |
| 59 | | × | | |
| 62 | | × | | × |
| 64 | | × | | |
| 72 | | × | | |

## GROUP WORK 2, SECTION 12.2

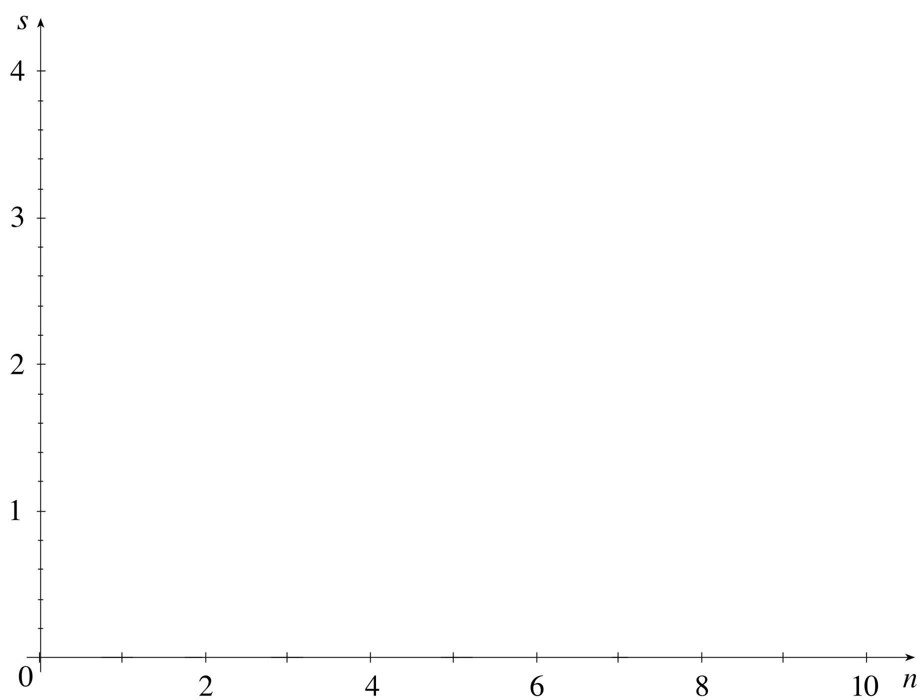
### The Harmonic Series

In this exercise, we look at $\sum_{n=1}^{\infty} \frac{1}{n}$.

**1.** What are the first ten partial sums $s_n$?

| | |
|---|---|
| $s_1 =$ | $s_6 =$ |
| $s_2 =$ | $s_7 =$ |
| $s_3 =$ | $s_8 =$ |
| $s_4 =$ | $s_9 =$ |
| $s_5 =$ | $s_{10} =$ |

**2.** The way we will compute $\sum_{n=1}^{\infty} \frac{1}{n}$ (or prove that it diverges) is to compute the limit of its partial sums. Plot the partial sums on the following graph, as accurately as you can.




**3.** The partial sums appear to be approaching a limit. What is that limit?

**4.** Now, on the same axes, graph $y = \ln x$ and $y = 1 + \ln x$ for $x \geq 1$. (Both of these graphs, as you know, go to infinity as $x$ gets arbitrarily large.)

**5.** Using your answer to Problem 4 and your graph, explain why it is reasonable to believe that $\sum_{n=1}^{\infty} \frac{1}{n}$ goes to infinity.

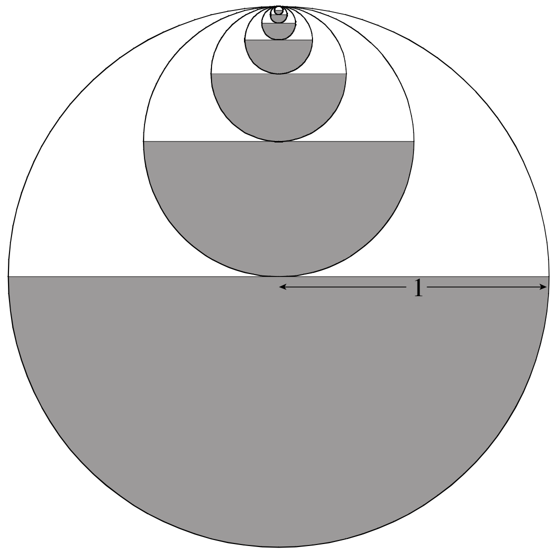
**6.** Did either of your answers to Problems 3 and 5 constitute a proof? Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

**7.** Assume that in the year 4000 B.C., you started adding up the terms of the harmonic series, at the rate of, say, one term per second. We know that the sum gets arbitrarily large, but approximately how big would your partial sum be as of now? Go ahead and make a guess, based on your best judgment and intuition.

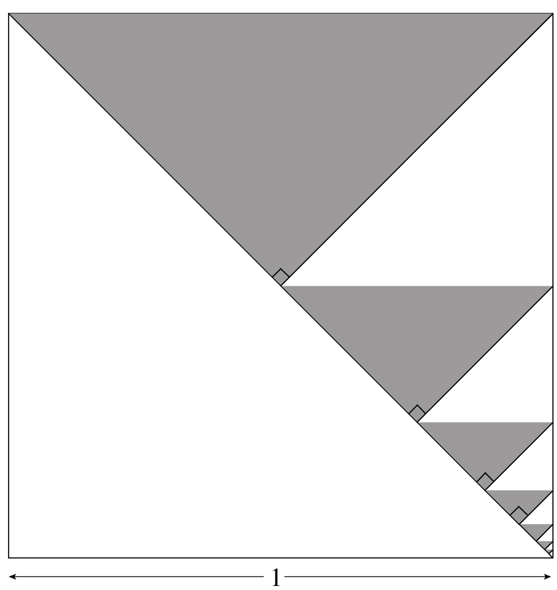
# GROUP WORK 3, SECTION 12.2
## Made in the Shade

**1.** Compute the sum of the shaded areas for each figure.

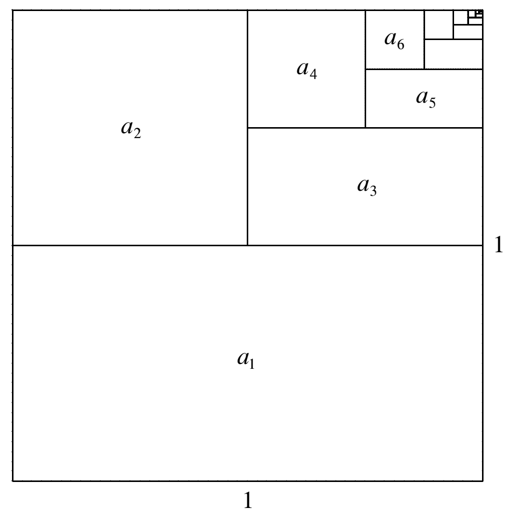
(a)




(b)



**2.** Consider the figure below.



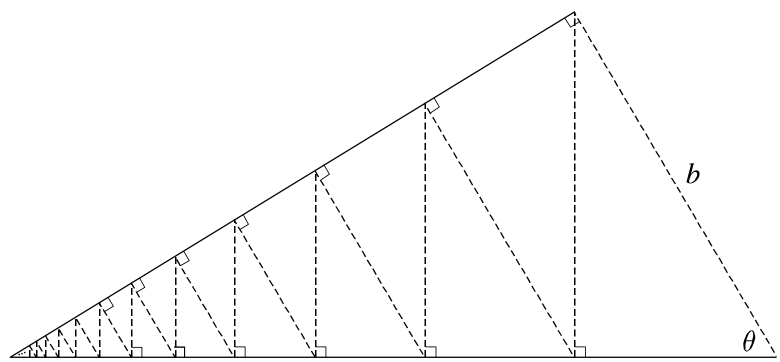

(a) Find an general expression for the area $a_n$.

(b) What is $\sum_{n=1}^{\infty} a_n$?

## GROUP WORK 4, SECTION 12.2

### An Unusual Series and Its Sums

Consider the following right triangle of side length $b$ and base angle $\theta$.




**1.** Express the total length of the dotted line in the triangle in terms of $b$ and $\theta$. Why should your answer be given in terms of a series?.

**2.** Compute the sum of this series.

**3.** What happens as $\theta \to \frac{\pi}{2}$?

**4.** Justify this answer geometrically and using infinite sums.

## 12.3 THE INTEGRAL TEST AND ESTIMATES OF SUMS

### TRANSPARENCY AVAILABLE

#28 (Figures 1 and 2)

### SUGGESTED TIME AND EMPHASIS

1 class Essential material

### POINT TO STRESS

1. The geometry and formal statement of the Integral Test, including the conditions on $f$ (continuous, positive, decreasing).
2. The Remainder Estimate for the Integral Test, and its use in bounding the error.
3. The use of the Remainder Estimate and partial sums to estimate the sum of a series (as in Example 6).

### QUIZ QUESTIONS

- **Text Question:** Why can the integral $\int_{100}^{\infty} f(x)\,dx$ be used to test the convergence of $\sum_{n=1}^{\infty} f(n)$?

  **Answer:** Convergence or divergence of a series depends only on the "tail".

- **Drill Question:** We know that $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. From this fact, we can conclude that

  (A) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (B) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1$ (C) Both (A) and (B) (D) Neither (A) nor (B)

  **Answer:** (A)

### MATERIALS FOR LECTURE

- State the Integral Test and give at least a geometric justification, as done prior to the formal statement in the text. Present examples of series that can be shown to be convergent by the Integral Test, such as $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.
- Discuss the basic $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, and determine the values of $p$ for which it converges and diverges. If time permits, similarly discuss $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$.
- State the remainder estimate $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$ (Formula 2) and the corresponding sum estimate (Formula 3), together with the geometric justification given in Figures 3 and 4.

### WORKSHOP/DISCUSSION

- Present examples of convergent and divergent series determined by the Integral Test, such as $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$.
- Using the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, find values of $n$ for which the remainder $R_n < 0.01$, and then values of $n$ for which $R_n < 0.001$. Then do the same for $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- Using the remainder estimate for the Integral Test, answer this question (posed at the end of Group Exercise 2 in Section 12.2): If you had started adding up the harmonic series at a rate of one term per second, starting in 4000 B.C., what would the partial sum be today? (Not all series that go to infinity do so quickly!) This question anticipates Exercise 35.

  **Answer:** As of the date and time of this writing, taking into account leap years, skipped leap years, leap seconds, and the change from the Gregorian calendar, the partial sum is approximately 25.967340.
- Check the following for convergence or divergence (or not enough information given):

  **1.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ **2.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$ **3.** $\displaystyle\sum_{n=1}^{\infty} e^{-x} \cos x$

  **Answers:** All converge

## GROUP WORK 1: The Integral Test

A straightforward series is presented, followed by one that seems to prove the Integral Test false, to make sure the students understand the importance of the conditions of the test.

**Answers:**

**1.** $\displaystyle\int_1^{\infty} \frac{x\,dx}{x^2+1} = \infty$

**2.** It is true that $\int_1^{\infty} x \sin(\pi x)\,dx$ is divergent, which may lead the students to think that the Integral Test says the series is divergent. This is not the case, as explained in Problem 4.

**3.** $0+0+0+0+0+\cdots$. The series converges to 0.

**4.** The fact that $\int_1^{\infty} x \sin(\pi x)\,dx$ is divergent is not relevant, because the Integral Test applies only to positive decreasing functions.

## GROUP WORK 2: Unusual Sums

Assign each group a different problem to work on first. Have the students do as many as they can. Leave time for each group to present the solution to their assigned problem.

**Answers:**

**1.** $\frac{3}{2}$ **2.** Diverges to $\infty$ **3.** 1 **4.** Diverges to $-\infty$

**5.** (a) They are the same. (b) $0 < p < 1$

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 5, 9, 11, 21, 27

**Sample Assignment:** 2, 4, 5, 8, 9, 11, 14, 20, 21, 25, 27, 34, 37, 39, 40

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | × | | | × |
| 4 | | × | | |
| 5 | | × | | |
| 8 | | × | | |
| 9 | | × | | |
| 11 | | × | | |
| 14 | | × | | |
| 20 | | × | | |

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 21 | | × | | |
| 25 | | × | | |
| 27 | | × | | |
| 34 | | × | | |
| 37 | | × | | |
| 39 | | × | | |
| 40 | × | × | | × |

# GROUP WORK 1, SECTION 12.3

## The Integral Test

**1.** Use the Integral Test to show that $\sum_{n=1}^{\infty} \frac{x}{x^2+1}$ diverges.

**2.** What does the Integral Test say about the series $\sum_{n=1}^{\infty} n \sin(\pi n)$?

**3.** Write out the first five terms of $\sum_{n=1}^{\infty} n \sin(\pi n)$. Does the series converge or diverge?

**4.** Do your answers to Problems 2 and 3 contradict each other? Explain.

# GROUP WORK 2, SECTION 12.3

## Unusual Sums

In each of the following problems, determine if the sum converges, diverges, or if there is not enough information to tell:

**1.** $\sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{1}{x^{5/3}}\,dx$

**2.** $\sum_{n=1}^{\infty} \int_{n}^{n+1} x^{2/3}\,dx$

**3.** $\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} x^{2/3}\,dx$

**4.** $\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} \frac{1}{x^{5/3}}\,dx$

**5.** (a) Suppose $p > 0$. What is the relationship between $\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$?

(b) Find the values of $p$ for which $\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} \frac{1}{x^p}\,dx$ converges.

## 12.4 THE COMPARISON TESTS

### SUGGESTED TIME AND EMPHASIS

1 class  Essential material

### POINT TO STRESS

**1.** The Comparison Test, stressing the fact that we need only consider the “tail” of the series to determine convergence or divergence.

**2.** The Limit Comparison Test

### QUIZ QUESTIONS

- **Text Question:** When using the Comparison Test, why do we need to check the conditions $a_n \leq b_n$ or $a_n \geq b_n$ only for $n \geq N$, where $N$ is some fixed integer (as stated in the text)?
  **Answer:** For any fixed integer $N$, the sum of the first $N$ terms of the series is finite. When checking for convergence, we are concerned only with the infinite “tail” of the series.

- **Drill Question:** If the improper integral $\int_5^\infty \frac{dx}{x^p}$ converges, then which of the following series *must* converge?
  (A) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$  (B) $\sum_{n=5}^{\infty} \frac{1}{n^{p+1}}$  (C) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$  (D) $\sum_{n=5}^{\infty} \frac{1}{n^{p-1}}$  (E) Both A and B  (F) Both C and D
  **Answer:** (E)

### MATERIALS FOR LECTURE

- State the Comparison Test, using a graph for informal justification. For example, illustrate how $\sum_{n=3}^{\infty} \frac{1}{n^2-5}$ can't be shown convergent by comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$, but *can* be shown convergent by comparison with $\sum_{n=2}^{\infty} \frac{2}{n^2}$, since for $n \geq 4$, we have $n^2 - 5 \geq n^2 - \frac{n^2}{2} = \frac{n^2}{2}$. Discuss the fact that we need consider only the “tail” of the series to determine convergence or divergence.

- Give some examples of using the Limit Comparison Test, such as $\sum_{n=1}^{\infty} \frac{n^2+2n+3}{3n^4+7n^3+11n^2+13n+17}$ [compare to $1/\left(3n^2\right)$] and $\sum_{n=1}^{\infty} \frac{2n+17}{n^2 \ln n + 5}$ [compare to $2/(n \ln n)$].

- Discuss why the series $\sum_{n=1}^{100} 2^n + \sum_{n=101}^{\infty} 1/2^n$ converges by comparison to $\sum_{n=1}^{\infty} 1/2^n$. Point out that $\sum_{n=1}^{100} 2^n$ is a very large number, namely 2,535,301,200,456,458,802,993,406,410,750.

## WORKSHOP/DISCUSSION

- Check the following series for convergence or divergence (or not enough information given):

  **1.** $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} - 1/2}$

  **2.** $\sum_{n=1}^{\infty} \frac{1}{\ln n + n}$

  **3.** $\sum_{n=1}^{\infty} a_n^2$, where $\sum_{n=1}^{\infty} a_n$ converges and $0 < a_n < 1$ for all $n$

  **Answers:**

  **1.** Converges. Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ or use the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

  **2.** Diverges. Compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n}$ or use the Limit Comparison Test with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.

  **3.** Converges. Compare $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{\infty} (a_n)^2$.

- Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ can be shown to converge using either the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, or the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ $\left(\frac{1}{2^n - 1} \leq \frac{1}{2^{n-1}}\right.$, since $2^n - 1 \geq 2^n - 2^{n-1} = 2^{n-1}$ for $\left.n \geq 1\right)$.

- Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$ for convergence in two ways: using the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and using a regular comparison with $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$ $\left(\text{That is, } \frac{1}{n^2 - n + 1} \leq \frac{1}{(n-1)^2} \text{ for } n \geq 2\right)$.

## GROUP WORK 1: Practicing with the Comparison Test

This activity consists of eight comparison test problems, ranging widely in difficulty. Feel free to have the students start with a problem other than the first one. Problems 4 and 5 were foreshadowed in Group Work 2 in Section 12.3.

**Answers:**

**1.** Converges. Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

**2.** Diverges. Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ or use the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

**3.** Converges. Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$.

**4.** Converges. Compare with $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{2}{3}\right)^n$.

**5.** Converges. The series is the same as $\displaystyle\int_1^{\infty} \frac{1}{x^{5/3}}\, dx = \frac{3}{2}$.

**6.** Converges. Compare with $\displaystyle\sum_{n=1}^{\infty} 2\left(2^{-n}\right)$.

**7.** Converges. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3 \sqrt[3]{2}}$, or use the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

**8.** Diverges. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$.

## GROUP WORK 2: How Do I Compare?

This group work should be attempted only after the students have had an opportunity to practice using the Comparison Test on some routine problems.

**Answers:**

**1.** Converges. Compare to $\sum_{n=1}^{\infty} \frac{1}{2n^2}$.

**2.** Diverges. Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

**3.** Diverges. The proof can be a bit tricky. Consider the odd numbered terms only: $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$.

One can compare this series to $\sum_{n=1}^{\infty} \frac{1}{2n}$, which diverges. The point of this problem is to get a little controversy going: encourage the students to discuss and debate this one with each other, instead of just putting down a guess.

## GROUP WORK 3: Sums of Squares

The purpose of this group work is to have students realize that comparisons need only take place after a finite number of terms, and that the Limit Comparison Test is frequently useful in the cases where the "obvious" convergent comparison series is not always larger. Be sure to first cover Exercise 40, which shows that the Limit Comparison Test holds for convergence when $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$. This group work is useful primarily for advanced students. Problem 1(d) can be done by showing that $\ln n \leq n^{1/6}$ for sufficiently large $n$.

**Answers:**

**1.** (a) Intuitive justifications will vary. Correct answers must include the fact that, for large $n$, $a_n < 1$.

(b) The Comparison Test works directly.

**2.** Compare $\sum_{n=1}^{\infty} a_n b_n$ to $\sum_{n=1}^{\infty} (c_n)^2$, where $c_n = \max(a_n, b_n)$.

## HOMEWORK PROBLEMS

**Core Exercises:** 1, 4, 9, 18, 24, 28, 34, 37

**Sample Assignment:** 1, 4, 7, 9, 14, 18, 19, 24, 28, 31, 34, 35, 37, 40, 43

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 1 | × | | | |
| 4 | | × | | |
| 7 | | × | | |
| 9 | | × | | |
| 14 | | × | | |
| 18 | | × | | |
| 19 | | × | | |
| 24 | | × | | |
| 28 | | × | | |
| 31 | | × | | |
| 34 | | × | | |
| 35 | | × | | |
| 37 | | × | | |
| 40 | | × | | |
| 43 | | × | | |

# GROUP WORK 1, SECTION 12.4
## Practicing with the Comparison Test

For each of the following problems, determine whether the series is convergent or divergent.

**1.** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

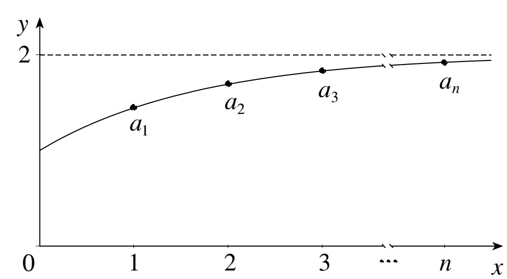
**2.** $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

**3.** $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$

**4.** $\sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^{5/3}}\,dx$

**5.** $\sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^{2/3}}\,dx$

**6.** $\sum_{n=1}^{\infty} 2^{-n} a_n$, where $a_n = f(n)$ as shown.




**7.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^9 - n^3 + 1}}$

**8.** $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

## GROUP WORK 2, SECTION 12.4

## How Do I Compare?

Determine if each of the following series converges or diverges.

**1.** $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2 \cdot 3^2} + \frac{1}{4^3} + \frac{1}{2 \cdot 5^2} + \frac{1}{6^3} + \cdots$

**2.** $\frac{1}{1} + \frac{1}{\ln 2} + \frac{1}{3} + \frac{1}{\ln 4} + \frac{1}{5} + \frac{1}{\ln 6} + \cdots$

**3.** $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3} + \frac{1}{4^2} + \frac{1}{5} + \frac{1}{6^2} + \cdots$

# GROUP WORK 3, SECTION 12.4

## Sums of Squares

**1.** Let $\sum_{n=1}^{\infty} a_n$ be a convergent positive series. Now consider the series $\sum_{n=1}^{\infty} a_n^2$.

(a) Why is it intuitively true that for sufficiently large $n$, $a_n^2 \le a_n$? Give a reason for your answer.

(b) Using part (a), it should be possible to compare $\sum_{n=1}^{\infty} a_n^2$ to $\sum_{n=1}^{\infty} a_n$ to check its convergence. Use the Comparison Test to show that $\sum_{n=1}^{\infty} a_n^2$ converges.

**2.** Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive series and both converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

*Hint:* What convergent series can you compare to $\sum_{n=1}^{\infty} a_n b_n$?

## 12.5 ALTERNATING SERIES

### TRANSPARENCY AVAILABLE

#29 (Figure 2 and Section 12.6 Figure 1)

### SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class    Recommended material

### POINTS TO STRESS

1. The Alternating Series Test.
2. The Alternating Series Estimation Theorem.

### QUIZ QUESTIONS

- **Text Question:** In Example 3, how did looking at the function $f(x) = \dfrac{x^2}{x^3+1}$ and its derivative help us determine that the series converges?

  **Answer:** We examined the derivative of $f(x)$ to show that $f$ was a decreasing function, and that therefore we were able to use the Alternating Series Test to show that $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+1}$ converges.

- **Drill Question:** Does $\sum_{n=1}^{\infty} \sin\left(\dfrac{\pi n}{2}\right) x^{-1/2}$ converge or diverge? Why?

  **Answers:** The series converges by the Alternating Series Test.

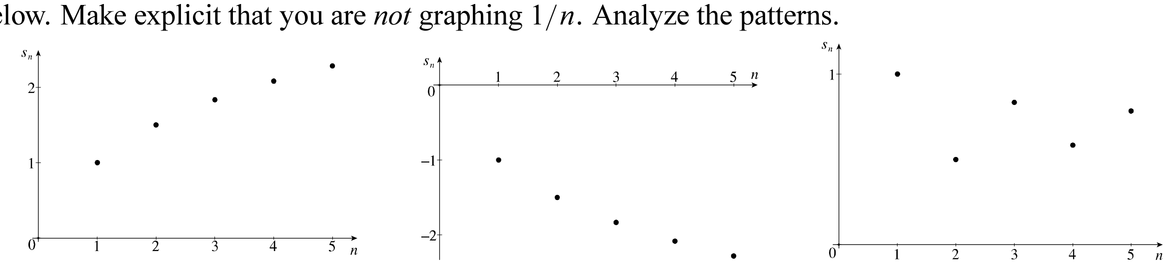
### MATERIALS FOR LECTURE

- Do Exercise 32. In this problem, students explore how changing the value of a parameter affects the convergence of a series, and are thus exposed to a type of reasoning used in analysis of power series.
- Some common terms that appear in alternating series are $(-1)^n$ and $\cos(\pi n)$. Ask the students if they can come up with other such alternating terms.

  **Answer:** $\sin\left(\frac{\pi}{2}n\right)$ alternates, as does $\sec(\pi n)$ and $4\left(\dfrac{n}{2} - \left[\!\left[\dfrac{n}{2}\right]\!\right]\right) - 1$.

### WORKSHOP/DISCUSSION

- Write out the first five partial sums of $\sum_{n=1}^{\infty} \dfrac{1}{n}$, $\sum_{n=1}^{\infty} -\dfrac{1}{n}$, and $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ and graph your answers as shown below. Make explicit that you are *not* graphing $1/n$. Analyze the patterns.




  Observe, by looking at the graphs, that all the partial sums $s_n$ of $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ appear to be between 0 and 1. This is a fact that can be proven by a more formal induction argument on the even-odd pairs of partial sums.

- Note that in the case of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the associated absolute value series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, while for $\sum_{n=0}^{\infty} (1.1)^{-n} \cos n\pi = \sum_{n=0}^{\infty} (-1)^n (1.1)^{-n}$, the associated series $\sum_{n=0}^{\infty} (1.1)^{-n}$ converges. Foreshadow the notion of absolute convergence.

## GROUP WORK 1: The Three Conditions

Set this up by making sure the students have read the test in the text. When students apply the Alternating Series Test, it is very tempting for them to neglect to check that the sizes of the terms are decreasing. This activity allows the students to explore that condition, and its importance in the Alternating Series Test. The students may not be able to complete this activity, but there is something to be gained by having the students work on it anyway. Wrestling with this problem will help them to really internalize both the general concept of the Alternating Series Test, and its subtleties. When wrapping up this activity, make sure the students see that if there were *no* divergent series satisfying conditions 1 and 3, but not condition 2, then we would not need to include condition 2 as part of the test.

**Answers:**

**1.** One possible answer (there are many) is:

$$1 - 2 + 1 \quad - \tfrac{1}{2} + 1 - \tfrac{1}{2} \quad + \tfrac{1}{4} - \tfrac{1}{2} + \tfrac{1}{4} \quad - \tfrac{1}{8} + \tfrac{1}{4} - \tfrac{1}{8} \quad + \cdots$$

The terms do go to zero, but they are not strictly decreasing. The partial sums of the series go to zero:

$$1, -1, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{-1}{4}, 0, -\tfrac{1}{8}, \tfrac{1}{8}, 0, \ldots$$

**2.** One possible answer is:

$$1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{8} + \tfrac{1}{4} - \tfrac{1}{16} + \tfrac{1}{5} - \tfrac{1}{32} + \cdots$$

The positive terms form the harmonic series, and thus tend to infinity. The sum of the negative terms is $-1$.

## GROUP WORK 2: Exploring Infinite Series (Part 1)

In this exercise the students explore a series that is conditionally convergent, and that converges slowly. They should explore the relationship between the partial sums $S_n$ and the actual sum $S$, noting that as $n \to \infty$, $S_n \to S$, and that for a given $n$ we can bound the error $|S - S_n|$.

**Answers:**

**1.** The Alternating Series Test

**2.** $R_3 = |S - S_3|$, $R_n = |S - S_n|$

**3.** $R_3 < \frac{1}{2}$, $R_6 < \frac{1}{\sqrt{7}} \approx 0.37796447$, $R_9 < \frac{1}{\sqrt{10}} \approx 0.31622777$

## HOMEWORK PROBLEMS

**Core Exercises:** 1, 3, 12, 21, 24, 32, 35

**Sample Assignment:** 1, 3, 6, 12, 15, 17, 21, 24, 29, 32, 35

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 1 | × | | | |
| 3 | | × | | |
| 6 | | × | | |
| 12 | | × | | |
| 15 | | × | | |
| 17 | | × | | |
| 21 | | × | | |
| 24 | | × | | |
| 29 | | × | | |
| 32 | | × | | |
| 35 | × | × | | |

# GROUP WORK 1, SECTION 12.5

## The Three Conditions

Notice that the Alternating Series Test has *three* conditions associated with it:

**1.** The series must alternate

**2.** The terms must decrease (in absolute value) for large $n$

**3.** The $n$th term must go to 0

The second condition seems redundant. Can you even have a series where the first two conditions hold, but the second doesn't?

**1.** Come up with a convergent series that satisfies conditions 1 and 3, but not 2.

**2.** Come up with a divergent series that satisfies conditions 1 and 3, but not 2.

# GROUP WORK 2, SECTION 12.5
## Exploring Infinite Series (Part 1)

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

**1.** How do we know that this series converges?

**2.** Call the sum of this series $S$, and the $n$th partial sum $S_n$. Write an equation for the remainder $R_3$. Write an equation for the remainder $R_n$.

**3.** Find upper limits on the remainders $R_3$, $R_6$, and $R_9$.

## 12.6 ABSOLUTE CONVERGENCE AND THE RATIO AND ROOT TESTS

### TRANSPARENCY AVAILABLE

#29 (Figure 1 and Section 12.5 Figure 2)

### SUGGESTED TIME AND EMPHASIS

1–2 classes    Essential material

### POINTS TO STRESS

1. Absolute convergence and conditional convergence.
2. The Ratio Test: When it gives useful information, and when it doesn't
3. The Root Test: When it gives useful information, and when it doesn't

### QUIZ QUESTIONS

- **Text Question:** Are the following statements true or false?

  1. If a series is absolutely convergent, then it is convergent.
  2. If a series is convergent, then it is absolutely convergent.

  **Answers: 1.** True    **2.** False

- **Drill Question:** What can we say about the convergence of $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$?

  **Answer:** Nothing

### MATERIALS FOR LECTURE

- Provide some intuition about the proof of the Ratio Test. For example, try to check the convergence of $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ by previous methods. First show the students that, by cleverly noting that $\frac{2^n}{n!} \le \left(\frac{2}{3}\right)^n$ for $n \ge 7$, we can use the Comparison Test to show that $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ converges. Alternately, if we note that $\frac{a_{n+1}}{a_n} = \frac{2}{n+1} \le \frac{2}{3}$ for $n \ge 2$, we can write $a_{n+1} \le \frac{2}{3}a_n \le \cdots \le \left(\frac{2}{3}\right)^{n-1} a_2 = 2\left(\frac{2}{3}\right)^{n-1}$ for $n \ge 2$, and hence the series converges by comparison with $\sum_{n=2}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$.

  Show that we can generalize these observations: if we know that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L < 1$, then $a_{n+1}$ "acts like" $cL^{n-N}$, for $n$ larger than some fixed $N$, and hence the series converges by comparison.

- Illustrate the power of the Ratio Test in summing recursively defined sequences. For example: let $a_1 = 1$ and $a_{n+1} = \dfrac{|\sin n|}{n} a_n$. The Ratio Test can show that $\sum_{n=1}^{\infty} a_n$ converges, even though the individual terms are hard to compute explicitly. Perhaps use the Ratio Test to check if $\sum_{n=1}^{\infty} b_n$ converges, where $\{b_n\}$ is the recursive sequence defined by $b_1 = 1$, $b_{n+1} = \left(1 + \dfrac{1}{n}\right)^n b_n$.

  **Answer:** $\lim_{n\to\infty} \dfrac{b_{n+1}}{b_n} = e > 1$, so the series diverges.
- The topic of rearrangements of conditionally convergent series can be both counterintuitive and beautiful. After going over the example in the text, perhaps have the students try to approximate numbers such as 2, $\sqrt{2}$ and $e$ only using distinct terms from the alternating harmonic series. Working on these problems "hands on" will help the students to understand this result.

## WORKSHOP/DISCUSSION

- When stating the Ratio Test, be sure to indicate that $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$ gives no information. Present examples of series for which the Ratio Test is not helpful, such as the $p$-series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ for various $p$.
- Apply the Ratio Test to $\sum_{n=1}^{\infty} \dfrac{(1.1)^n}{n^2}$, $\sum_{n=1}^{\infty} \dfrac{n+2}{n \cdot n!}$, and $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$. (The Ratio Test is inconclusive in the last case. The Integral Test would have been a better choice.)

  **Answer:** Diverges, converges, diverges (by the Integral Test)
- Test the convergence of $\sum_{n=1}^{\infty} \dfrac{(2n)!}{n^n}$, $\sum_{n=1}^{\infty} \dfrac{1}{n^n}$, $\sum_{n=1}^{\infty} \dfrac{e^n}{n!}$, and $\sum_{n=1}^{\infty} \dfrac{2^n}{n^2}$, using either the Ratio Test or the Root Test, as appropriate.

  **Answer:** Divergent, convergent, convergent, divergent

## GROUP WORK 1: Exploring Infinite Series (Part 2)

Several of the questions in this exercise have no answer. If the students are struggling, tell them that "No such series exists" is a possible answer.

**Answers:** (Answers to Problems 1, 4, 5, and 6 will vary.)

**1.** $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$

**2.** This is not possible, due to the Test for Divergence.

**3.** This is not possible, due to the Test for Divergence.

**4.** $\{-1, 2, -3, 4, -5, 6, \dots\}$. The Ratio Test requires that $\left| \dfrac{a_{n+1}}{a_n} \right| < 1$.

**5.** $a_n = \begin{cases} 10 & \text{if } n \le 100 \\ 0 & \text{if } n > 100 \end{cases}$

**6.** $f(x) = \sin(\pi x)$

## GROUP WORK 2: What's Your Ratio?

This group work foreshadows power series and the notion of the radius and interval of convergence.

**Answers:**

1. $(-1, 1)$, by the Ratio Test
2. $(-\infty, 1) \cup (1, \infty)$, by the Ratio Test
3. The Ratio Test is inconclusive for $x = \pm 1$. The Alternating Series Test and the Integral Test give us that the series is convergent for both of these values of $x$.

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 7, 11, 12, 17, 25, 29, 38

**Sample Assignment:** 2, 5, 7, 8, 11, 12, 17, 21, 24, 25, 29, 31, 34, 36, 38

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | | × | | |
| 5 | | × | | |
| 7 | | × | | |
| 8 | | × | | |
| 11 | | × | | |
| 12 | | × | | |
| 17 | | × | | |
| 21 | | × | | |
| 24 | | × | | |
| 25 | | × | | |
| 29 | | × | | |
| 31 | | × | | |
| 34 | | × | | |
| 36 | | × | | |
| 38 | × | × | | |

# GROUP WORK 1, SECTION 12.6
## Exploring Infinite Series (Part 2)

**1.** Can you find a sequence $\{a_k\}$ such that $\{a_k\}$ converges, and in fact tends to zero, and the series $\sum_{k=1}^{\infty} a_k$ diverges?

**2.** Can you find a sequence $\{a_k\}$ such that $\{a_k\}$ converges, tending to a number other than zero, and the series $\sum_{k=1}^{\infty} a_k$ converges?

**3.** Can you find a sequence $\{a_k\}$ such that $\{a_k\}$ diverges, and the series $\sum_{k=1}^{\infty} a_k$ converges?

**4.** Can you find a sequence $\{a_k\}$ such that $\frac{a_{n+1}}{a_n} < 1$ for all $n$, and the series $\sum_{k=1}^{\infty} a_k$ diverges?

**5.** Can you find a sequence $\{a_k\}$ such that $\frac{a_{n+1}}{a_n} > 9$ for all $n \leq 100$, and the series $\sum_{k=1}^{\infty} a_k$ converges?

**6.** Can you find a function $f(x)$ with the property that $\lim_{x\to\infty} f(x)$ does not exist, yet whose associated series $\sum_{k=1}^{\infty} a_k$ [with $a_k = f(k)$] converges?

# GROUP WORK 2, SECTION 12.6
## What's Your Ratio?

**1.** Use the Ratio Test to determine the values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges. Write your answer in the form of an interval.

**2.** Use the Ratio Test to determine the values of $x$ for which the series diverges.

**3.** For which value(s) of $x$ is the Ratio Test inconclusive? Use another test to determine if the series converges in these cases.

## 12.7 STRATEGY FOR TESTING SERIES

### SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$ class    Optional material

### POINTS TO STRESS

**1.** It is better to make an intelligent guess than a blind guess.

**2.** It is better to try something than to try nothing.

**Note:** The author cannot resist pointing out that the above wisdom applies as well to life as it does to determining the convergence of a series.

### QUIZ QUESTIONS

- **Text Question:** If the limit of the terms of a series is 0, what does that tell us about its convergence?
  **Answer:** All that we can conclude is that the series may possibly converge.
- **Drill Question:** What would a good test to use to determine the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n!}$? Why?

  **Answers:** The Ratio Test would be best, although an argument could be made for the Root Test, and perhaps the Comparison Test, although that would be pushing it.

### MATERIALS FOR LECTURE AND WORKSHOP/DISCUSSION

- The main idea here is to get the students to put together the material from the previous sections. After going over the heuristic, an example from the text can be put on the board. The students should be given a few minutes to figure out which test should be used. It is advisable for them to solve a few of the problems completely, just to verify that their ideas are correct.

**Possible problems:**

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n+3} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+6n}} \qquad \sum_{n=1}^{\infty} \left(\frac{1}{n!} - \frac{1}{2^n}\right) \qquad \sum_{n=1}^{\infty} \frac{n}{(n+e)(n+\pi)}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}} \qquad \sum_{n=1}^{\infty} \frac{n^n}{(n!)!} \qquad \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n} \qquad \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$$

### GROUP WORK

For this section, it is more important for the students to spend some time solving problems by themselves, as they would on an exam, than it is for them to be working in groups. I recommend having the students work independently on some of the unassigned homework exercises, perhaps checking their answers with a neighbor.

## HOMEWORK PROBLEMS

**Core Exercises:** 1, 2, 7, 9, 12, 20, 26

**Sample Assignment:** 1, 2, 6, 7, 9, 12, 13, 17, 20, 22, 23, 26, 36, 37

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 1 | | × | | |
| 2 | | × | | |
| 6 | | × | | |
| 7 | | × | | |
| 9 | | × | | |
| 12 | | × | | |
| 13 | | × | | |
| 17 | | × | | |
| 20 | | × | | |
| 22 | | × | | |
| 23 | | × | | |
| 26 | | × | | |
| 36 | | × | | |
| 37 | | × | | |

# 12.8 POWER SERIES

## TRANSPARENCY AVAILABLE

#30 (Figures 1 and 2 and Section 12.9 Figure 1)

## SUGGESTED TIME AND EMPHASIS

1 class Essential material. Endpoint discussions beyond Figure 3 are optional but recommended.

## POINT TO STRESS

**1.** The definition of a power series.

**2.** The radius and interval of convergence of a power series.

## QUIZ QUESTIONS

- **Text Question:** If the interval of convergence of a power series has length 2, what is the radius of convergence?
  **Answer:** The radius of convergence is 1.
- **Drill Question:** If the power series $\sum_{n=1}^{\infty} c_n x^n$ has radius of convergence 3, what do we know about $\sum_{n=1}^{\infty} c_n 3^n$, $\sum_{n=1}^{\infty} c_n (-2)^n$, and $\sum_{n=1}^{\infty} c_n 5^n$?
  **Answer:** We know nothing about the first series without further investigation. The second converges, and the third diverges.

## MATERIALS FOR LECTURE

- Mention that the photograph in the text of a drum cover juxtaposed with a mathematical model is an example of an application of Bessel functions.
- Give the definition of a power series, emphasizing that the value of the series depends on $x$. Point out how power series differ from the numerical series (such as $\sum_{n=1}^{\infty} 1/n^4$) that have been discussed so far. Go over Figure 3 in detail, to explain the radius and interval of convergence for a power series.
- Discuss the role of partial sums of power series and how these polynomials "approximate" the series for suitable values of $x$ near $a$. This foreshadows Taylor polynomials, which are covered in depth in Section 12.12.
- Use the Comparison Test to show that if $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = b > 0$, then it converges for $|x| < b$.
- Mention that power series are not the only kinds of infinite series that contain variables. For example, a Fourier series is an infinite series of the form $\sum_{n=0}^{\infty} [a_n \sin(nx) + b_n \cos(nx)]$ and is used in modeling periodic functions.

## WORKSHOP/DISCUSSION

- Using the Ratio Test, compute the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$ and $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

  **Answers:** Converges for all $x$, $R = \infty$, $I = (-\infty, \infty)$; $R = 1$, $I = [0, 2)$
- Show that the $p$-power series $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ for $p > 1$ have $[-1, 1]$ as their interval of convergence.
- Show that the $p$-power series $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ have interval of convergence $(-1, 1)$ if $0 < p < 1$.
- Note that all power series have a non-empty interval of convergence, but that interval may just be a single point. Demonstrate with $\sum_{n=1}^{\infty} n!\,x^n$.

## GROUP WORK 1: Recognition

As an introduction, ask the students to answer the questions, "What is a geometric series?" and "What is a power series?" Have each group resolve their individual answers to these questions before handing out the problem.

**Answers:**

**1.** Both **2.** Geometric **3.** Geometric

**4.** Power **5.** Neither **6.** Geometric

## GROUP WORK 2: From Power Series to Polynomials

This group work anticipates Taylor polynomials and their use in approximating the value of a series.

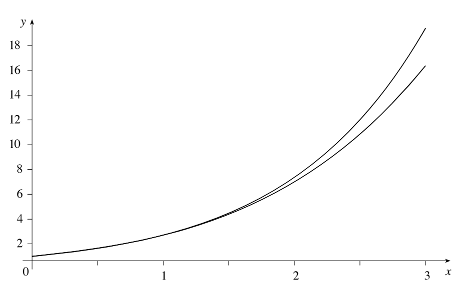
**Answers:**

**1.** $R = \infty$

**2.** $s_2 = 1 + x + \frac{x^2}{2}$, $s_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$, $s_6 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$

**3.** $x = 0$: $\sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 1$; $x = 1$: $\sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 2.7181$; $x = 2$: $\sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 7.3556$

**4.** $|s_6(0) - s_4(0)| = 0$,
$|s_6(1) - s_4(1)| = \frac{7}{120} \approx 0.009722$,
$|s_6(2) - s_4(2)| = \frac{16}{45} \approx 0.35555$. The differences are increasing because the graphs grow apart as shown at right.




**5.** $\lim_{n\to\infty} s_n(1) \approx 2.71828$ The limit is $e$.

## GROUP WORK 3: Intervals of Convergence

Problem 2 delves a little deeper into the concepts of radius and interval of convergence.

### Answers:

**1.** (a) 1 (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}, \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ (c) The first converges, the second diverges. (d) $[-1, 1)$

**2.** (a) $b_n = nx$

(b) For any fixed $x$, $b_n \to \infty$ and $(b_n)^n \to \infty$ as $n \to \infty$.

(c) The Root Test now tells us that the series diverges for $x > 0$.

(d) The interval of convergence is $[0, 0]$ and the radius of convergence is 0.

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 3, 10, 13, 24, 31, 34

**Sample Assignment:** 2, 3, 7, 10, 13, 15, 22, 24, 29, 31, 33, 34, 36, 41

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | × | | | |
| 3 | | × | | |
| 7 | | × | | |
| 10 | | × | | |
| 13 | | × | | |
| 15 | | × | | |
| 22 | | × | | |
| 24 | | × | | |
| 29 | | × | | |
| 31 | | × | | |
| 33 | × | | | |
| 34 | × | | | × |
| 36 | | × | | × |
| 41 | | × | | |

## GROUP WORK 1, SECTION 12.8
### Recognition

Which of the following are geometric series? Which are power series? (Circle one choice.) Be prepared to explain your answers.

**1.** $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

**2.** $1 + 1.1 + 1.21 + 1.331 + 1.4641 + 1.6105 + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

**3.** $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^8 + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

**4.** $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

**5.** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

**6.** $\frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + x^4 + \cdots$

**GEOMETRIC SERIES** **POWER SERIES** **BOTH** **NEITHER**

# GROUP WORK 2, SECTION 12.8

## From Power Series to Polynomials

**1.** Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

**2.** Compute the partial sums $s_2(x)$, $s_4(x)$, and $s_6(x)$ for values of $x$ where the series converges. Note that if we let $x$ vary, $s_2(x)$, $s_4(x)$, and $s_6(x)$ describe three polynomials in $x$.

**3.** Using the polynomials from Part 2, estimate the value of the series for $x = 0$, 1, and 2.

**4.** Compute the differences $|s_6(x) - s_4(x)|$ for $x = 0$, 1, and 2. Interpret your answer in terms of the graphs of the polynomials $s_4(x)$ and $s_6(x)$.

**5.** Estimate $\lim_{n\to\infty} s_n(1)$ to 5 decimal places. Does this number look familiar?

# GROUP WORK 3, SECTION 12.8
## Intervals of Convergence

**1.** Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

(a) Find the radius of convergence.

(b) Find expressions for the two series at the endpoints of the interval of convergence.

(c) Determine whether the series in part (b) converge.

(d) What is the interval of convergence?

**2.** Consider the power series $\sum_{n=1}^{\infty} n^n x^n$.

(a) Write each term of the series in the form $(b_n)^n$. What is $b_n$?

(b) Let $x$ be any fixed positive number. What can we say about $b_n$ and $(b_n)^n$ as $n \to \infty$?

(c) What does part (b) tell us about the convergence or divergence of $\sum_{n=1}^{\infty} n^n x^n$?

(d) Compute the radius and interval of convergence of $\sum_{n=1}^{\infty} n^n x^n$.

## 12.9 REPRESENTATION OF FUNCTIONS AS POWER SERIES

### TRANSPARENCY AVAILABLE

#30 (Figure 1 and Section 12.8 Figures 1 and 2)

### SUGGESTED TIME AND EMPHASIS

1 class    Essential material

### POINTS TO STRESS

1. Understanding the change of viewpoint: In the previous section we started with a power series and talked about its sum. Now we are starting with a function and finding its power series representation.
2. Starting with a given power series, operations such as differentiating and integrating yield new power series with the same radius of convergence as the original. Substituting an expression for $x$ will yield a new power series with possibly a different radius of convergence.
3. Starting with a given function represented by a power series, operations such as the ones listed above create power series representations for different functions.

### QUIZ QUESTIONS

- **Text Question:** In Example 5, how did we get from $\sum_{n=1}^{\infty} nx^{n-1}$ to $\sum_{n=0}^{\infty} (n+1)x^n$?

  **Answer:** We substituted $n+1$ for $n$. So $n=1$ became $n=0$, and so forth. (The idea here isn't to make the students create a perfectly worded paragraph on re-indexing series. The point is to make sure that the students are reading the text carefully, and asking themselves these sorts of questions.)
- **Drill Question:** If a power series representation for $f(x)$ can be written $f(x) = 1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$, write out a power series representation for $\frac{f(x)-1}{x}$.

  **Answer:** $\frac{f(x)-1}{x} = \frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\cdots$

### MATERIALS FOR LECTURE

- Many students miss the distinction between a polynomial approximating a function, and a power series replacing a function (on the appropriate interval). One way to help them is to write the following on the board:

  For $|x| < 1$,

$$\frac{1}{1-x} \approx 1+x+x^2+x^3+x^4$$
$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$$

  Point out that it would be missing the point to write " $\approx$ " in the second line.
- Point out that the representation of $f(x) = x^3+3x^2-5x+6$ as a power series is $f(x)$ itself, since $f(x) = 6-5x+3x^2+x^3+0x^4+0x^5+\cdots$. In fact, any polynomial is its own representation as a power series.

- Recall from the text that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Make sure that they understand that this is just a restatement of the formula for the sum of a geometric series, with $a = 1$ and $r = x$. (Perhaps plug $x = \frac{1}{3}$ into the equation to reinforce the point.)

  Next, plug $x^2$ in for $x$, to obtain the power series for $\dfrac{1}{1-x^2}$. (This particular result will be revisited in workshop.)

  Finally, derive power series for the following functions:

  **1.** $\dfrac{1}{1+x}$

  **2.** $\dfrac{1}{3+x}$ $\left[= \dfrac{1}{1+(2+x)} \text{ or } \dfrac{1}{3}\left(\dfrac{1}{1+(x/3)}\right)\right]$

  **3.** $-\dfrac{1}{(1+x)^2}$

  **4.** $\ln|x+1|$

  **5.** $\dfrac{x}{x+1}$ $\left[= x\dfrac{1}{1+x}\right]$

  **6.** $\dfrac{1}{3+x^2}$

  Several of these results can also be derived using polynomial division.

- Show how the material from this section can be used to do things like finding a closed form for the sum of the series $\sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2 + 6x + 12x^2 + 20x^3 + \cdots$. We first integrate the series twice:

$$\iint \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} x^n$$

  Then use the geometric series formula:

$$\sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x}$$

  Finally, we differentiate twice:

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{d^2}{dx^2}\frac{x^2}{1-x} = \frac{2}{(1-x)^3}$$

## WORKSHOP/DISCUSSION

- Using a power series for $\dfrac{1}{1+x^3}$, approximate $\displaystyle\int_0^{2/3} \frac{dx}{1+x^3}$ to two decimal places of accuracy.
- Derive the series $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right)$, $|x| < 1$ using $\ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$.
- Approach Exercise 35 from a different point of view. Begin with the differential equation $y' = y$, $y(0) = 1$. Now assume that $y = \sum_{n=0}^{\infty} a_n x^n$ and so $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$. Then the differential equation can be written

$$\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

or

$$\begin{array}{rllll} & 0+ & 1a_1+ & 2a_2x+ & 3a_3x^2+ & \cdots \\ = & & a_0+ & a_1x+ & a_2x^2+ & \cdots \end{array}$$

Now $y(0) = a_0 = 1$. Then by matching coefficients of the same powers of $x$, we get

$$\begin{array}{ll} a_1 = a_0 & a_1 = 1 \\ 2a_2 = a_1 & a_2 = \dfrac{1}{2 \cdot 1} \\ 3a_3 = a_2 & a_3 = \dfrac{1}{3 \cdot 2 \cdot 1} \\ & \cdots \\ na_n = a_{n-1} & a_n = \dfrac{1}{n!} \end{array}$$

So the series solution to this differential equation is $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, and since we know that the analytic solution is $y = e^x$, we get $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$. Show how $\dfrac{d}{dx} \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, as we would expect.

- Having derived the series for $f(x) = e^x$, show how this series can be used to obtain series representations of $e^{-3x}$, $xe^x$, $e^{x^2}$, and $e^x - \dfrac{1}{1-x}$.

## GROUP WORK: Find the Series

Note that in Problem 2, the function in part (c) is a polynomial and so is already in power series form. Part (f) can be done by factoring or long division, yielding the polynomial $x^2 - 3x$.

Hand out the pages separately. If the first page is sufficiently challenging, the second page can be saved for another time.

### Answers:

**1.** The equality is true when $|x| < 1$.

**2.** (a) $1 - x^4 + x^8 - x^{12} + \cdots = \sum\limits_{n=0}^{\infty} (-1)^n x^{4n}$ (b) $1 - 4x^2 + 16x^4 - 64x^6 + \cdots = \sum\limits_{n=0}^{\infty} (-4)^n x^{2n}$

(c) $5x^3 - 13x^2 + 7$ (d) $x^3 - \dfrac{x^6}{2} + \dfrac{x^9}{3} - \dfrac{x^{12}}{4} + \cdots + \sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{3n}}{n}$

(e) $2x^3 + x^5 + x^9 + x^{13} + \cdots = 2x^3 + \sum\limits_{n=0}^{\infty} x^{4n+5}$ f) $x^2 - 3x$

**3.** (a) $|x| < 1$ (b) $|x| < \frac{1}{2}$ (c) $\mathbb{R}$ (d) $|x| < 1$ (e) $|x| < 1$ (f) $\mathbb{R}$

**4.** $\int_0^{0.5} \ln\left|1 + x^3\right| dx \approx 0.01510$

## HOMEWORK PROBLEMS

**Core Exercises:** 1, 3, 11, 13, 20, 28, 35

**Sample Assignment:** 1, 3, 6, 8, 11, 13, 15, 20, 25, 28, 31, 32, 35

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 1 | × | | | |
| 3 | | × | | |
| 6 | | × | | |
| 8 | | × | | |
| 11 | | × | | |
| 13 | | × | | |
| 15 | | × | | |
| 20 | × | × | | × |
| 25 | | × | | |
| 28 | | × | | |
| 31 | | × | | |
| 32 | | × | | |
| 35 | | × | | |

# GROUP WORK, SECTION 12.9

## Find the Series

It has been said, both in class and in the text, that $\dfrac{1}{1-x} = \sum_{k=1}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$.

**1.** This statement seems to have some problems with it. For example, if $x = 2$, it is clearly false that $\dfrac{1}{1-2} = 1 + 2 + 4 + 8 + 16 + \cdots$. What do your instructor and textbook author mean when they say $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$?

**2.** Find series representations for the following functions:

(a) $f(x) = \dfrac{1}{1+x^4}$ (b) $f(x) = \dfrac{1}{1+4x^2}$ (c) $f(x) = 5x^3 - 13x^2 + 7$

(d) $f(x) = \ln\left|1 + x^3\right|$ (e) $f(x) = \dfrac{x}{1-x^4} + 2x^3 - x$ (f) $f(x) = \dfrac{x^3 - 5x^2 + 6x}{x-2}$

**3.** Find the radius of convergence for each of the series you found in Problem 2.

**4.** Find an approximate value for $\int_0^{0.5} \ln\left|1 + x^3\right| \, dx$.

## 12.10 TAYLOR AND MACLAURIN SERIES

### SUGGESTED TIME AND EMPHASIS

2 classes Essential material. Binomial series recommended; multiplication and division of power series optional.

### POINTS TO STRESS

**1.** Taylor polynomials and Taylor's theorem for $a = 0$.

**2.** Maclaurin series for important functions such as the ones in the text and $\ln(1 + x)$.

**3.** The binomial series formula for $(1 + x)^r$ ($r \in \mathbb{R}$) and its usefulness in finding certain Maclaurin series.

### QUIZ QUESTIONS

- **Text Question:** Which of the following statements is true?
  1. A Taylor series is a special type of Maclaurin series.
  2. A Maclaurin series is a special type of Taylor series.

  **Answer:** Statement 2
- **Drill Questions:**

  **1.** Let $g$ be a function that has derivatives of all orders for all real numbers. Assume that $g(0) = 3$, $g'(0) = 1$, $g''(0) = 4$, and $g'''(0) = -1$.

  (a) Write the third-degree Taylor polynomial for $g$ centered at $x = 0$ and use it to approximate $g(0.1)$.

  (b) Write the fifth-degree Taylor polynomial for $h'(0)$ centered at $x = 0$, where $h(x) = g(x^2)$.

  **Answer:** (a) $P_3(x) = 3 + x + 2x^2 - \frac{1}{6}x^3$, $g(0.1) \approx 3.1198$ (b) $2x + 8x^3 - x^5$

  **2.** (a) Compute $\binom{-2}{3}$. (b) Compute $\binom{-2}{n}$.

  **Answer:** (a) $-4$ (b) $(-1)^n(n + 1)$

### MATERIALS FOR LECTURE

- Start with the intuitive idea of Taylor polynomials. First remind students that the tangent line or linear approximation $L(x) = f'(0)x + f(0)$ is a linear model for $f$ near $x = 0$. What is a good way to derive a better model? One possible way is to start with a totally different kind of model, such as trying to model $y = \cos x$ near $x = 0$ by a function of the form $f(x) = ae^x + bx + c$. Note that one way to get a good fit would be to choose $a$, $b$, and $c$ so that $f(0) = y(0)$, $f'(0) = y'(0)$, and $f''(0) = y''(0)$. If desired, $a$, $b$, and $c$ can even be found for this model ($a = -1$, $b = 1$, and $c = 2$) and $f$ and $y$ can be graphed simultaneously. If we want a better model, what other functions could be used? Discuss how $f(x) = ax^2 + bx + c$ is a model that is preferred, because it can be more easily extended to cubics, quartics, and to polynomials of any degree, and because polynomials are relatively easy to analyze.

  Show that the general coefficient of $x^n$ for the approximating polynomial is $\dfrac{f^{(n)}(0)}{n!}$. [We want $f^{(n)}(0) = T^{(n)}(0)$. $T^{(n)}(0) = n!\,a_n$ where $a_n$ is the coefficient of $x^n$.] Then define Maclaurin series, followed by Taylor series.

- Starting with the Maclaurin series for $\sin x$, compute the series expression

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

which holds for all real $x$ except $x = 0$, and explain why it is *not* the Maclaurin series for $f(x) = \dfrac{\sin x}{x}$, $x \neq 0$, but for the extended function $F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

**Answer:** $\dfrac{\sin x}{x}$ is undefined at zero. The function $F(x)$ is defined at zero, and all its derivatives exist there.

- Note that if we know that $f$ has a Maclaurin series with infinite radius of convergence, we can't necessarily conclude that $f$ equals its Maclaurin series everywhere. Examine the Taylor series for $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ The first four derivatives of $f$ are given below. Point out that for all $n$, $f^{(n)}(0) = 0$ and that the radius of convergence is infinite, but that the Taylor polynomials are poor approximations of $f(x)$. Note that $f'(0)$ is $\lim_{x \to 0} \frac{1}{x} e^{-1/x^2}$, which requires l'Hospital's Rule to compute.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases} \qquad f(0) = 0$$

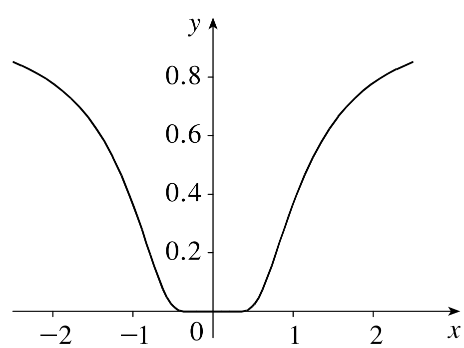
$$f'(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left(\dfrac{2}{x^3}\right) e^{-1/x^2} & \text{if } x \neq 0 \end{cases} \qquad f'(0) = 0$$

$$f''(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left(\dfrac{-6x^2 + 4}{x^6}\right) e^{-1/x^2} & \text{if } x \neq 0 \end{cases} \qquad f''(0) = 0$$

$$f'''(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left(\dfrac{24x^4 - 36x^2 + 8}{x^9}\right) e^{-1/x^2} & \text{if } x \neq 0 \end{cases} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left(\dfrac{-120x^6 + 300x^4 - 144x^2 + 16}{x^{12}}\right) e^{-1/x^2} & \text{if } x \neq 0 \end{cases} \qquad f^{(4)}(0) = 0$$

Notice that $e^{-1/x^2}$ is so flat at $x = 0$, it is hard to obtain a good graph, even with the use of technology.



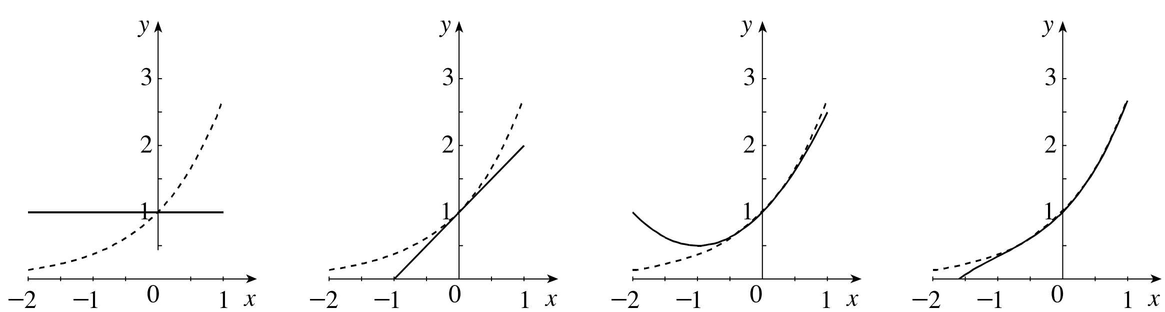
- Review that, when $k$ is a natural number, $\binom{n}{k}$ can be obtained by Pascal's triangle:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

- Check the binomial series for $\frac{1}{1+x}$ against the series obtained by using the geometric series and replacing $x$ by $-x$.

## WORKSHOP/DISCUSSION

- Estimate $\int_0^1 \sin x^3\, dx$ using a power series. Note that the fourth term of the expansion is very small.
- Show how power series can sometimes be used in place of l'Hospital's Rule. For example, compute $\lim_{x\to 0} \frac{\ln(1+2x^2)}{3x^2}$ and $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ using power series.
- Check the speed of convergence of $T_n(x)$ for $e^x$ near $x = 1$ by having the students graph $T_1$, $T_2$, $T_3$, and $T_4$ on their calculators.
**Answer:**




- TEC Have the students use technology to visually analyze the convergence of Taylor series for several different types of functions, such as $y = e^x$, $y = \cos x$, and $y = \frac{1}{6-x}$. For each function they should pay attention to how quickly the series converges, and on what interval. TEC Module 8.7/8.9 can be used for these comparisons.
- Show that the binomial series for $(1-x)^k$ is $\sum_{n=0}^{\infty} \binom{k}{n} (-1)^n x^n$.

• Compute $\dfrac{1}{(1-x)^3}$ by binomial series, and check the answer by differentiating $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ twice.

## GROUP WORK 1: Find the Error

**Answer:** Choose a level of tolerance, say 0.1. For each $k$, $|e^x - P_k(x)| < 0.1$ for a larger range of $x$. The remainder goes to zero for all $x$ only for the Taylor *series*, not for any of the Taylor polynomials.

## GROUP WORK 2: The Secret Function

If a group finishes early, ask them if they recognize the number, and if they do, ask them to show that the answer is in fact $e$, by using that $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$. Perhaps ask if they can compute the number that Oprah Winfrey purchased at auction.

Also note that the result is true only if we know that the function is equal to its Taylor series.

**Answer:** 2.71828 or $e$

## GROUP WORK 3: Taylor and Maclaurin Series

This exercise is too long to be assigned in its entirety. Pick and choose based on the desired emphasis of the course. Problem 2, in which students discover that the Maclaurin series of a polynomial is the polynomial itself, is particularly important. Problem 3 is an extension of Exercise 2 in the text.

**Answers:**

**1.** (a) $(2+x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

(b) $\ln\left(\dfrac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots$

(c) $x^3 \sin\left(x^2\right) = x^5 - \dfrac{1}{3!}x^9 + \dfrac{1}{5!}x^{13} - \dfrac{1}{7!}x^{17} + \cdots$

**2.** (a) $x^5 - 5x^4 + 27x^2 - 3x + 17$

(b) $(x-1)^5 - 10(x-1)^3 + 7(x-1)^2 + 36(x-1) + 37$

(c) $17x^{11} - 9x^8 + 27x^7 - 5x^4 + 13x^2 + 8x - 5$

**3.** $s(0) = -1$, eliminating $f$. $s'(0) > 0$, eliminating $g$. $s''(0) < 0$, eliminating $h$.

**4.** (a) $a_0 = b_0$

(b) $\dfrac{a_1}{b_1}$ by multiplying top and bottom by $\dfrac{1}{x}$ or by using l'Hospital's Rule.

(c) $\lim\limits_{x\to 0} \dfrac{\cos\left(x^2\right) - 1}{\ln(1+x) - x} = 0$

(d) Whichever method was not used in part (b)

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 5, 13, 27, 30, 39, 49, 56, 60

**Sample Assignment:** 2, 4, 5, 8, 10, 13, 18, 22, 27, 30, 35, 39, 42, 46, 49, 51, 56, 60, 64, 69

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | × | | | × |
| 4 | | × | | |
| 5 | | × | | |
| 8 | | × | | |
| 10 | | × | | |
| 13 | | × | | |
| 18 | | × | | |
| 22 | | × | | |
| 27 | | × | | |
| 30 | | × | | |

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 35 | | × | | |
| 39 | × | × | | × |
| 42 | × | × | | × |
| 46 | | × | | |
| 49 | | × | | |
| 51 | | × | | |
| 56 | | × | | |
| 60 | | × | | |
| 64 | | × | | |
| 69 | | × | | |

# GROUP WORK 1, SECTION 12.10
## Find the Error

It is a beautiful spring morning. You are waiting in line to get your picture taken with a man in an Easter Bunny costume, as an amusing gift for your friends. When you get to the head of the line, the bunny says, "My! You are a very big child."

"Oh!" you laugh, "I am not a child. I am just doing this as an amusing gift for my friends. I am actually a calculus student."

"What a sweet and precocious word to use, 'calculus' — did you learn that word from your older sister or brother?"

"No," you protest. "I really *am* a calculus student. Why, just yesterday I learned all about Taylor series."

"Taylor series? How cute! That is... if you think it is cute to learn about LIES!" The person in the Easter Bunny costume is none other than your wild-eyed, hungry-looking tormentor!

While you are frozen on his lap in horror, he asks you, "Please tell me, if you will, what is $\int_0^\infty 1\,dx$?"

Out of sheer reflex you answer his simple question. "The integral diverges. It doesn't go to any finite number."

"Bad person being mean to Easter Bunny!" says the little girl in line behind you.

The stranger ignores her. "How about $\displaystyle\int_0^\infty \left(1 - x + \frac{x^2}{2}\right) dx$? and $\displaystyle\int_0^\infty \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right) dx$?"

Leaving aside the question of how he can talk in math notation, you think for a moment, and then say "They both diverge. In fact, any polynomial like that is going to diverge, going off to infinity. It doesn't matter what the denominators of the coefficients are."

"So you are saying that no matter how large $k$ gets, $\displaystyle\int_0^\infty \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots \frac{x^k}{k!}\right) dx = \infty$?"

"Well... yes."

Several angry children start throwing chocolate eggs at you.

"But $e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{24} - \cdots \frac{x^k}{k!}$ as $k$ goes to infinity! And $\int_0^\infty e^{-x}\,dx$ is equal to one! Now get off my lap!" As hundreds of children boo you, you walk away, thinking about $\int_0^\infty e^{-x}\,dx$. The faux Easter Bunny has just proved that one is equal to infinity! Can this be true? Does it no longer "take one to know one"? Does this mean that all monotheists have suddenly become pantheistic? Or is there a chance, however small, that our hare-y friend has made a mistake?

Find the error.

# GROUP WORK 2, SECTION 12.10
## The Secret Function

This is an illustration of an application of Taylor's Theorem that we think is really amazing, and we hope you do, too. Although the problem statement in this exercise is contrived, the principle is one that is often actually used.

The U.S. Government derives all its mystical power from a secret infinitely-differentiable function whose origins are lost to history. This function is so secret that *no one person* seems to know all of its values. The most powerful person in the government, the President, knows all about the function at $x = 9$ and has the greatest mystical power. The second-to-most powerful person, the Senate Majority Leader, knows all about the function at $x = 8$. In a silent auction, Oprah Winfrey bought the information about $x = 7$, and so on.

Thus, if we could only find $f(10)$, we could run the whole show!

So we get the Vice President to talk. He is angry, because he only knows all about the function at $x = 0$. He knows that $f(0) = 1$, $f'(0) = 0.1$, $f''(0) = 0.01$, $f'''(0) = 0.001$, and so forth. "Wow," we say, "they really *do* trust you!"

"What are you talking about?" he complains. "Yeah, I know a lot about the function, but just at the measly point $x = 0$! I know nothing about $f(10)$, which is what I need to take over! Erm... not that I would *want* to, of course. Heh, heh, ahem."

We dismiss him, and he goes back to Washington, not knowing what he has just given us.

Find $f(10)$ to five decimal places, and *please*, do not compromise our national security with this knowledge!

# GROUP WORK 3, SECTION 12.10

## Taylor and Maclaurin Series

**1.** Compute Maclaurin series for the following functions:

(a) $f(x) = (2 + x)^5$

(b) $f(x) = \ln\left|\dfrac{1+x}{1-x}\right|$

(c) $f(x) = x^3 \sin x^2$

**2.** Let $f(x) = x^5 - 5x^4 + 27x^2 - 3x + 17$.

(a) Compute the Maclaurin series for $f$.

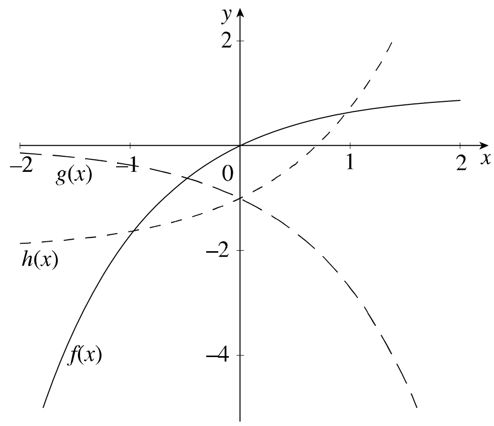
(b) Compute the Taylor series for $f$ at $a = 1$.

(c) Answer part (a) for $g(x) = 17x^{11} - 9x^8 + 27x^7 - 5x^4 + 13x^2 + 8x - 5$.

**3.** The graphs of $f$, $g$, and $h$ are shown. Explain why the series

$$s(x) = -1 + 0.3x - 0.1x^2 + 0.08x^3 + \cdots$$

cannot be the Maclaurin series for $f$, $g$, or $h$.




**4.** Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, with $f(0) = 0$ and $g(0) = 0$.

(a) What does the condition $f(0) = g(0) = 0$ mean in terms of the coefficients for these series?

(b) Compute $\lim_{x \to 0} \frac{f(x)}{g(x)}$.

(c) Compute $\lim_{x \to 0} \frac{\cos(x^2) - 1}{\ln(1+x) - x}$ using part (b).

(d) The limit in part (c) can also be computed by another technique. What is this other technique called? Use it to check your answer to part (c).

# LABORATORY PROJECT An Elusive Limit

Mathematicians have a variety of tools they can use to investigate mathematical phenomena. The bedrock of mathematics is *proof*. A proof distinguishes between what we believe to be true and what we *know* is true. In this activity, students use technology to make a conjecture, and then attempt to verify their conjecture with analytic tools. An unexpected twist occurs in this process.

## WRITING PROJECT How Newton Discovered the Binomial Series

This topic is useful for an extended out-of-class project that emphasizes fundamental material and the history of mathematics. The description provides an excellent summary of Newton's role, and clearly outlines the structure of a report, together with appropriate source materials. This project can be done by an individual, or by a group of students.

There are actually several possible projects here; the students can choose to emphasize the mathematics, the history, or some combination of the two. Make sure that they narrow their focus a little; it is better to have an in-depth exploration of a particular idea than a few pages of broad generalizations.

## 12.11 APPLICATIONS OF TAYLOR POLYNOMIALS

### TRANSPARENCY AVAILABLE

#31 (Figures 1 and 6)

### SUGGESTED TIME AND EMPHASIS

1 class Recommended material (Example 3 may be particularly interesting to students currently studying physics)

### POINTS TO STRESS

**1.** The definition of the $n$th degree Taylor polynomial $T_n(x)$, and $R_n(x) = f(x) - T_n(x)$. (This notation was first introduced in Section 12.10.)

**2.** Various ways to bound $R_n(x)$, both numerically and analytically.

### QUIZ QUESTIONS

- **Text Question:** Give an example of how Taylor polynomials are used in physics.
  **Answer:** A quick summary of the relativity or optics example in the text would suffice, as would a more general statement such as, "Taylor polynomials can be used to simplify a complicated function by considering only the first few terms in its Taylor series."
- **Drill Question:** Physicists use the approximation $\sin x \approx x$ for very small angles $x$. What similar approximation should they use if they were talking about angles that were very close to being right angles?
  **Answer:** $\sin x \approx 1$ or $\sin x \approx 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$

### MATERIALS FOR LECTURE

- Bring up the issue of how to choose an appropriate $a$ for computing the Taylor polynomial $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{n!}(x-a)^i$ for a given function. Perhaps let $f(x) = \frac{1}{\sqrt{x+99}}$ and compare the results with $a = 0$ and $a = 1$.
  **Answer:** $\frac{1}{\sqrt{x+99}} = \left(\frac{1}{33}\sqrt{11}\right) - \left(\frac{1}{6534}\sqrt{11}\right)x + \left(\frac{1}{862{,}488}\sqrt{11}\right)x^2 + \cdots$ or $\frac{1}{\sqrt{x+99}} = 0.1 - 0.0005(x-1) + \frac{3}{800{,}000}(x-1)^2 - \cdots$
- Point out that in general we do not find $R_n$, we merely bound it. If we were able to find our *exact* error, then we would be able to add it to our estimate, and get exact answers!
- Work through Exercise 32, particularly if your class contains physics and engineering students.

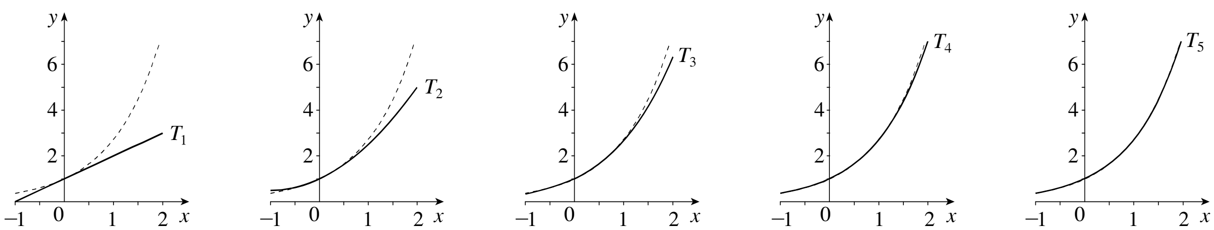
### WORKSHOP/DISCUSSION

- Present an example of bounding $R_n$ for a Maclaurin polynomial. For example, let $f(x) = x\cos x$, with $n = 3$ (or 4) on the interval $[-2, 2]$. Notice that as $n$ gets larger, the denominator of the remainder estimate gets larger as $n!$, so the remainder does go to zero.
- Work through Example 2, perhaps using a different function such as $e^{-x}$. That is, find the maximum error possible in using the approximation $e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ when $-0.2 \le x \le 0.2$. Use this approximation

to find good bounds on $e^{-0.1}$. Then find the range of $x$-values for which this estimate is accurate to within 0.00005.

- Compare the estimates of $e^{-0.1}$ given by the Taylor polynomials for $e^x$ with $n = 1, 2, 3, 4$, and 5, and present graphs of the relevant polynomials.

$$
\begin{aligned}
e^{-0.1} &\approx T_1(x) = 1 - 0.1 = 0.9 \\
e^{-0.1} &\approx T_2(x) = 1 - 0.1 + \tfrac{0.01}{2} = 0.905 \\
e^{-0.1} &\approx T_3(x) = 1 - 0.1 + \tfrac{0.01}{2} - \tfrac{0.001}{6} = 0.90483333 \\
e^{-0.1} &\approx T_4(x) = 1 - 0.1 + \tfrac{0.01}{2} - \tfrac{0.001}{6} + \tfrac{0.0001}{24} = 0.9048375 \\
e^{-0.1} &\approx T_5(x) = 1 - 0.1 + \tfrac{0.01}{2} - \tfrac{0.001}{6} + \tfrac{0.0001}{24} - \tfrac{0.00001}{120} = 0.90483742
\end{aligned}
$$




- Present several examples of finding $R_n$ for Maclaurin polynomials by numerical methods. Include illustrations of the speed at which $T_n$ converges. Three sample functions are given below.

$$
\begin{aligned}
f(x) &= \cos x,\ n = 4 \text{ on } [-1, 1] \\
g(x) &= \ln(1 + x),\ n = 4 \text{ on } [-0.5, 0.5] \\
h(x) &= \frac{x^2}{1 + x},\ n = 5 \text{ on } [-0.7, 0.7]
\end{aligned}
$$

## GROUP WORK: Ghosts of Departed Quantities

Start by putting the integral $\int_0^x te^{-t^3}dt$ on the board, and having the students try to evaluate it. After they have tried for some time, confess that it is impossible to get an exact answer, and then pass out the worksheet.

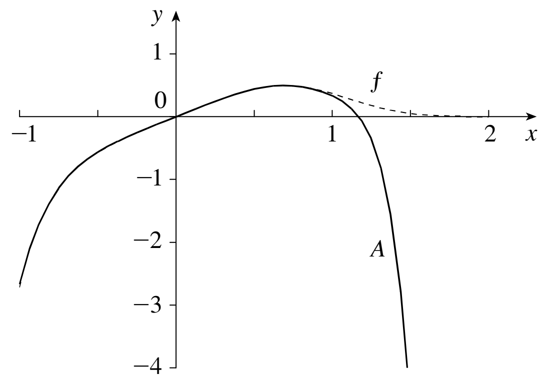
**Answers:**

**1.** This could be obtained using Taylor's Formula with $f(0) = 0$, $f'(x) = xe^{-x^3}$, but it is easier to start with the series for $e^x$, obtain the series for $xe^{-x^3}$, and integrate it.

**2.** The Alternating Series Estimation Theorem gives that $|R| < \dfrac{x^{14}}{336}$. $|R(0.3)| < 1.4235027 \times 10^{-10}$, $|R(0.5)| < 1.8165225 \times 10^{-7}$.

**3.** The error in our estimation is already smaller than the degree of precision of the calculator.

**4.**




The functions are similar, and they should be, because they are the graphs of $f'(x)$ and the derivative of our estimate.

## HOMEWORK PROBLEMS

**Core Exercises:** 2, 5, 23, 28

**Sample Assignment:** 2, 5, 9, 15, 18, 21, 23, 26, 28, 29, 37

| Exercise | D | A | N | G |
|---|---|---|---|---|
| 2 | × | × | | × |
| 5 | | × | | × |
| 9 | | × | | × |
| 15 | | × | | × |
| 18 | | × | | × |
| 21 | | × | | × |
| 23 | | × | | |
| 26 | | × | | |
| 28 | | × | | |
| 29 | × | × | | |
| 37 | × | × | | |

# GROUP WORK, SECTION 12.11

## Ghosts of Departed Quantities

Consider the function $f(x) = \int_0^x te^{-t^3}\,dt$.

Unfortunately, it is not possible to write a formula for $f$ any more explicitly than that. Doing so would involve computing a symbolic antiderivative of $te^{-t^3}$, which is impossible. However, we do not have to give up on working with such a function.

**1.** Show that near $x = 0$, $f(x) \approx A(x) = \dfrac{x^2}{2} - \dfrac{x^5}{5} + \dfrac{x^8}{16} - \dfrac{x^{11}}{66}$.

**2.** Estimate the error in using this formula to calculate $f(0.3)$ and $f(0.5)$.

**3.** Assume that our only other way of estimating values of $f(0.3)$ is to use the definite integral feature on a calculator with 8-digit accuracy. If that is the case, explain why computing a more exact polynomial estimate of $f$ would not be of any help at all. Justify your answer.

**4.** One way to compare $f(x)$ and its approximation is to graph them both. Graphing functions like $f(x)$ can be problematic, even with computer technology. (It would have to estimate a definite integral at every point plotted, which could take a long time, depending on the complexity of the integrand.) There is another way to see how accurate our estimate is. On the same axes, graph $xe^{-x^3}$ and the function $x - x^4 + \dfrac{x^7}{2} - \dfrac{x^{10}}{6}$. Are these two functions similar near $x = 0$? Should we expect them to be?

## APPLIED PROJECT Radiation from the Stars

This project analyzes Planck's Law for expressing the energy at different wavelengths emitted from the Sun, other stars, and more general blackbody systems. It is particularly useful as an extended in-class or out-of-class project for students interested in physics. The first two questions use l'Hospital's Rule and Taylor polynomials to study the asymptotic behavior of Planck's Law and its relation to its predecessor the Rayleigh-Jeans Law. These questions make good shorter projects. Questions 3–5 use graphical methods to provide more detail for advanced students.

Students will probably want to consult outside reference material before writing their final report so that they can discuss blackbody radiation qualitatively before answering the questions.

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

**1.** Consider the sequence defined by $a_n = \dfrac{\cos n}{n}$.

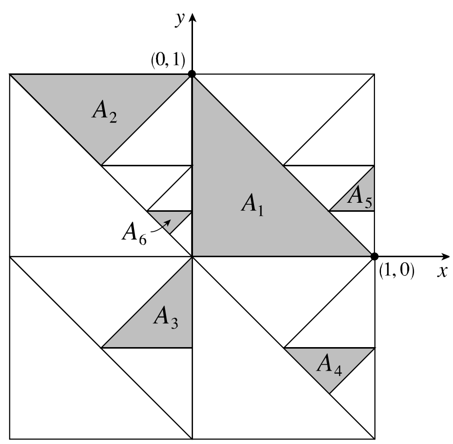
(a) Is $\{a_n\}$ increasing, decreasing, or neither?

(b) Determine whether $\{a_n\}$ converges or diverges. If it converges, find its limit.

**2.** Consider the recursively defined sequence $\{a_n\}$ given by $a_{n+1} = 1/a_n^{3/2}$.
For each of the following choices of $a_1$, find the limit of the sequence $\{a_n\}$ if it converges. Otherwise, explain why it diverges.

(a) $a_1 = \frac{1}{2}$ (b) $a_1 = 2$ (c) $a_1 = 1$

**3.** Find $\sum_{n=1}^{\infty} a_n$, where $a_n$ is the area of the triangle $A_n$ in the diagram.




**4.** Let $f(x) = 2 - 3x^2 + \frac{1}{2}x^3 - 171x^4 + 15x^6$.

(a) Find the Maclaurin polynomial $M_8(x)$ of $f(x)$.

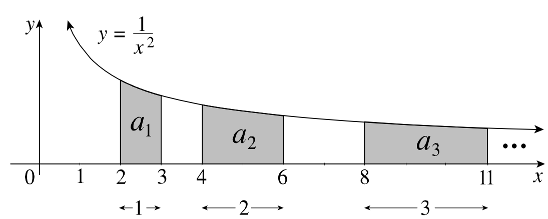
(b) Compute $f^{(5)}(0)$ and $f^{(6)}(0)$.

**5.** Let $f(x) = x^3 + x^2 + x + 1$.

(a) Find the Taylor series for $f(x)$ about $a = 1$.

(b) What expression will you get if you multiply out all the terms of the Taylor series?

**6.** Does the series $\sum_{n=1}^{\infty} a_n$ illustrated below converge or diverge?



**7.** Suppose $a_k > 0$ for all $k$, and $\sum_{k=1}^{\infty} a_k$ converges.

(a) Must $\sum_{n=1}^{\infty} 3a_n$ always converge?

(b) Must $\sum_{n=1}^{\infty} \frac{1}{a_n}$ always converge?

(c) Must $\sum_{n=1}^{\infty} (a_n)^2$ always converge?

(d) Must $\sum_{n=1}^{\infty} \sqrt{a_n}$ always converge?

**8.** Decide whether the following sequences converge or diverge.

(a) $\{1, 0, 1, 0, 1, ...\}$

(b) $\left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}, \ldots\right\}$

(c) $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...\}$

(d) $\{a_n\}$, where $a_n = 1 - \frac{(-1)^n}{n}$

**9.** Decide whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$

**10.** The Maclaurin series for $e^x$ is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

(a) Find the Maclaurin series for $f(x) = x^2 e^x$.

(b) Compute $f^{(100)}(0)$.

**11.** The Maclaurin series for $e^x$ is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

(a) Find the Maclaurin series for $e^{3x}$.

(b) Find the Maclaurin series for $3e^x$.

(c) Find the Maclaurin series for $e^{x+3}$.

(d) Find the Maclaurin series for $e^x + 3$.

**12.** The Maclaurin series expansions of $\sin x$ and $\cos x$ are as follows:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

(a) Determine the first four nonzero terms of $\cos\left(x^2\right)$.

(b) Determine the first four nonzero terms of $\cos^2 x$.

(c) What is $f^{(6)}(0)$ for $\cos\left(x^2\right)$?

**13.** Let $f(x)$ have a power series representation $T(x)$, and suppose that $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, and $f^{(3)}(0) = 6$.

(a) If the above is the only information we have, to what degree of accuracy can we estimate $f(1)$?

(b) If, in addition, we know that $T(x)$ converges on the interval $[-2, 2]$ and that $\left|f^{(4)}(x)\right| \le 15$ on that interval, then to what degree of accuracy can we estimate $f(1)$?

**14.** Consider the set of sequences $\{a_n\}$ that satisfy $1 \le \sum_{n=1}^{k} a_n \le 10$ for all $k$.

(a) Find an example of such a sequence that converges, or prove that none exists.

(b) Find an example of such a sequence that diverges, or prove that none exists.

**15.** Consider the three infinite series below.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)\left(n^2-1\right)}{4n^3-2n+1}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{5(-4)^{n+2}}{3^{2n+1}}$

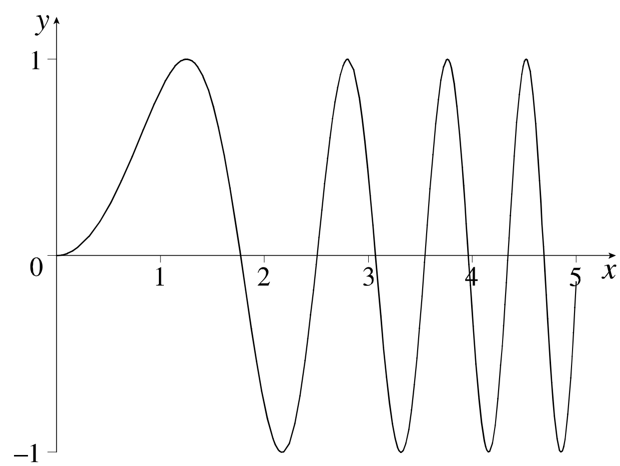
(a) Which of these series is (are) alternating?

(b) Which one of these series diverges, and why?

(c) One of these series converges absolutely. Which one? Compute its sum.

**16.** Using the power series representation of $\dfrac{1}{1+x}$, find a power series representation of $f(x) = x\ln(1+x)$ which holds for $|x| < 1$.

*** 17.** Consider $\int_0^{\infty} \sin\left(x^2\right) dx$. [The function $y = \sin\left(x^2\right)$ is graphed below.]




(a) Let $a_n = \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin\left(x^2\right) dx$, $n \ge 0$. Argue, using the graph above, that $|a_n| > |a_{n+1}|$ and that $\lim_{n\to\infty} a_n = 0$.

(b) Show that $\sum_{n=0}^{\infty} a_n$ is an alternating series.

(c) Use parts (a) and (b) to show that $\int_0^{\infty} \sin\left(x^2\right) dx$ converges.

**18.** Find the radius of convergence and the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} 5^n x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^5}$ (c) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

**19.** Consider the series $\sum_{n=1}^{\infty} \frac{n^2+2n+3}{n^4+n+2}$.

(a) Explain why each term "looks like" $\frac{n^2}{n^4}$ for large values of $n$.

(b) Determine if the series converges or diverges.

***20.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of positive terms.

(a) Why is it true that $a_n < 1$ for large values of $n$?

(b) Using part (a), show that $a_n b_n < b_n$ for large values of $n$.

(c) Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

(d) Using part (c), show that $\sum_{n=1}^{\infty} a_n^2$ converges.

## 12 SAMPLE EXAM SOLUTIONS

**1.** (a) $\{a_n\}$ is neither increasing nor decreasing. The cosine function oscillates between $-1$ and 1, and $\cos n$ is positive for some integers and negative for others.

(b) $\{a_n\}$ converges to 0, that is, $\lim_{n\to\infty} a_n = 0$, since $\left|\frac{\cos n}{n}\right| \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

**2.** (a) $a_1 = 2^{-1}, a_2 = 2^{3/2}, a_3 = 2^{-9/4}, a_4 = 2^{27/8}, \ldots, a_n = 2^{(-1)^n(3/2)^{n-1}}$

Since the even-indexed terms tend to infinity and the odd-indexed terms tend to 0, this sequence diverges.

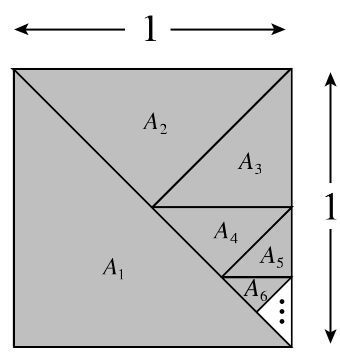
(b) $a_1 = 2^1, a_2 = 2^{-3/2}, a_3 = 2^{9/4}, a_4 = 2^{27/8}, \ldots, a_n = 2^{(-3/2)^{n-1}}$

Here the odd-indexed terms tend to infinity and the even-indexed terms tend to 0, so once again this sequence diverges.

(c) $a_1 = 1, a_2 = 1, \ldots, a_n = 1$

This sequence is constantly 1, so it converges and its limit is 1.

**3.** $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \ldots, a_n = 1/2^n$, so $S = \sum_{n=1}^{\infty} a_n$ is a geometric series and $S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. The answer can also be found geometrically, as in the figure.



**4.** $f(x) = 2 - 3x^2 + \frac{1}{2}x^3 - 171x^4 + 15x^6$.

(a) The Maclaurin polynomial of a given polynomial is always simply the given polynomial itself (up to the degree of the desired Maclaurin polynomial). In this case

$$M_8(x) = f(x) = 2 - 3x^2 + \tfrac{1}{2}x^3 - 171x^4 + 15x^6$$

(b) Since

$$\begin{aligned} M_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &\quad + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^5 + \frac{f^{(7)}(0)}{7!}x^5 + \frac{f^{(8)}(0)}{8!}x^5 \end{aligned}$$

we have $\dfrac{f^{(5)}(0)}{5!} = 0 \quad\Rightarrow\quad f^{(5)}(0) = 0$ and $\dfrac{f^{(6)}(0)}{6!} = 15 \quad\Rightarrow\quad f^{(6)}(0) = 15 \cdot 6! = 10{,}800$.

**5.** $f(x) = x^3 + x^2 + x + 1$.

(a) $f'(x) = 3x^2 + 2x + 1$, $f''(x) = 6x + 2$, $f^{(3)}(x) = 6$, and all other derivatives are 0, so

$$T(x) = 4 + 6(x-1) + \frac{8}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3$$

(b) $T(x) = f(x)$, which we check by expanding the powers of $(x-1)$:

$$T(x) = 4 + 6x - 6 + 4x^2 - 8x + 4 + x^3 - 3x^2 + 3x - 1 = x^3 + x^2 + x + 1$$

**6.** Each region with area $a_n$ is bounded above by a rectangle with area $\dfrac{n}{2^{2n}}$. The series $\sum\limits_{n=1}^{\infty} \dfrac{n}{2^{2n}}$ converges by the Ratio Test, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{n+1}{4n} = \frac{1}{4} < 1$. We can also show that $\sum\limits_{n=1}^{\infty} a_n$ converges by the Integral Test, since $\int_1^{\infty} (1/x^2)\,dx$ converges and

$$\sum_{i=1}^{n} a_i \le \int_1^{2^n+n} \left(1/x^2\right) dx$$

**7.** (a) $\sum\limits_{n=1}^{\infty} 3a_n = 3\sum\limits_{n=1}^{\infty} a_n$ must converge.

(b) $\sum\limits_{n=1}^{\infty} \dfrac{1}{a_n}$ need not converge. For example, if $a_n = 1/2^n$, then $\sum\limits_{n=1}^{\infty} \dfrac{1}{a_n}$ diverges.

(c) $\sum\limits_{n=1}^{\infty} (a_n)^2$ must converge. Because all $a_n > 0$, we have $\lim\limits_{n\to\infty} a_n = 0$, so for some $N$, $(a_n)^2 < a_n$ for all $n > N$. Thus the series converges by comparison.

(d) $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$ need not converge. For example, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, but $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ does not.

**8.** (a) $\{1, 0, 1, 0, 1, \ldots\}$ diverges.

(b) $\left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \ldots\right\}$ converges to 2, since the terms of the sequence are the partial sums of the geometric series $\sum\limits_{n=0}^{\infty} 1/2^n$.

(c) $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ...\}$ converges to some real number. (From the pattern, it appears to converge to $\pi$.)

(d) $\{a_n\}$, where $a_n = 1 - \frac{(-1)^n}{n}$, converges to 1 since $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = 0$.

**9.** (a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converges by comparison to the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ converges by comparison to the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$, since $2^n - n \geq 2^{n-1}$ for all $n$.

**10.** (a) The Maclaurin series for $f(x) = x^2 e^x$ is $x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots$.

(b) $M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}$. So, matching the coefficients of $x^{100}$, we see that

$$\frac{f^{(100)}(0)}{100!} = \frac{1}{98!} \quad \Rightarrow \quad f^{(100)}(0) = \frac{100!}{98!} = 9900.$$

**11.** (a) $e^{3x} = 1 + 3x + \frac{3^2}{2!} x^2 + \frac{3^3}{3!} x^3 + \cdots$

(b) $3e^x = 3 + 3x + \frac{3}{2!} x^2 + \frac{3}{3!} x^3 + \cdots$

(c) $e^{x+3} = e^3 + e^3 x + \frac{e^3}{2!} x^2 + \frac{e^3}{3!} x^3 + \cdots$

(d) $e^x + 3 = 4 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

**12.** (a) $\cos\left(x^2\right) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$

(b)
$$\begin{aligned}
\cos^2 x &= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2 \\
&= 1 - 2\frac{x^2}{2!} + \left[2\frac{x^4}{4!} + \left(\frac{x^2}{2!}\right)^2\right] + \left[-2\frac{x^6}{6!} - 2\frac{x^2}{2!} \cdot \frac{x^4}{4!}\right] + \cdots \\
&= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots \text{ (or use } \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x)
\end{aligned}$$

(c) Since there is no $x^6$ term in the Maclaurin series for $f(x) = \cos\left(x^2\right)$, $f^{(6)}(0) = 0$.

**13.** (a) We cannot estimate $f(1)$ with any accuracy because we have no information about $f(x)$ on the interval $(0, 1]$.

(b) By the Remainder Theorem, we know that the error is bounded by $\frac{M \cdot 1^4}{4!} \leq \frac{15}{4!} = \frac{5}{8}$ (where $M = 15$ is an upper bound on $\left|f^{(4)}(x)\right|$).

**14.** (a) One such sequence which converges has $a_n = \frac{1}{2^{n-1}}$. Then $1 \le \sum\limits_{n=1}^{k} a_n = \sum\limits_{n=1}^{k} \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{k-1}} \le 10$ for all $k$, and the sequence $\left\{\frac{1}{2^{n-1}}\right\}$ converges to 0.

(b) One such sequence which diverges has $a_1 = 5$ and then $a_n = (-1)^n$ for $n \ge 2$, so $1 \le \sum\limits_{n=1}^{k} a_n = \left\{\begin{array}{ll} 5 & \text{if } n \text{ is odd} \\ 6 & \text{if } n \text{ is even} \end{array}\right\} \le 10$ for all $k$, and the sequence is $\{5, 1, -1, 1, -1, 1, \ldots\}$, which diverges.

**15.** (a) Series (i) and (iii) are alternating.

(b) Series (ii) diverges, as $a_n = \frac{(n+1)\left(n^2-1\right)}{4n^3-2n+1} = \frac{n^3+n^2-n-1}{4n^3-2n+1} = \frac{1+\frac{1}{n}-\frac{1}{n^2}-\frac{1}{n^3}}{4-\frac{2}{n^2}+\frac{1}{n^3}} \to \frac{1}{4} \ne 0$ as $n \to \infty$.

(c) Series (iii) converges absolutely, and its sum is given by the geometric series

$$\sum_{n=1}^{\infty}\left(-\frac{320}{27}\right)\left(\frac{4}{9}\right)^{n-1} = -\frac{320}{27} \cdot \frac{1}{1+\frac{4}{9}} = -\frac{320}{27} \cdot \frac{9}{13} = -\frac{320}{39}$$

**16.** $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$, and $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$, so $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $|x| < 1$ and $x\ln(1+x) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots$ for $|x| < 1$.

**17.** (a) For $n \ge 0$, $a_n = \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin\left(x^2\right) dx$. We can show that $|a_{n+1}| < |a_n|$ by noting that for any $n$, $\sin\left(\left(\sqrt{n\pi}\right)^2\right) = \sin \pi n = 0$, so the integral is the (signed) area of the curve between successive roots of $y = \sin\left(x^2\right)$, and these roots are getting closer together. We also have that $\left|\sin\left(x^2\right)\right| \le 1$, and the same function values occur on each interval.

(b) The terms are alternating because the corresponding integrals are positive when the curve is above the $x$-axis and negative when the curve is below $x$-axis.

(c) $\int_0^\infty \sin\left(x^2\right) dx = \sum\limits_{n=0}^{\infty} \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin\left(x^2\right) dx$, and since this series is an alternating series with $|a_n| > |a_{n+1}|$, the series converges by the Alternating Series Test, and hence the integral converges.

**18.** (a) $\sum_{n=1}^\infty 5^n x^n = \sum_{n=1}^\infty (5x)^n$ converges for $|5x| < 1$, or $|x| < \frac{1}{5}$. At $x = \frac{1}{5}$, each term is 1 and the series diverges. At $x = -\frac{1}{5}$, each term is $\pm 1$ and the series diverges. The interval of convergence is $\left(-\frac{1}{5}, \frac{1}{5}\right)$.

(b) Using the Ratio Test, $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)^5} \cdot \frac{n^5}{x^n} = \frac{1}{\left(1+\frac{1}{n}\right)^5} x \to x$ as $n \to \infty$. Thus, the series converges for $|x| < 1$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^5}$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ both converge, the interval of convergence is $[-1, 1]$.

(c) By the Ratio Test, $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}\,x^{n+1}}{n^n x^n} = (n+1)\left(1+\dfrac{1}{n}\right)^n x$ diverges as $n \to \infty$ for any $x \neq 0$. The radius of convergence is 0 and the interval of convergence is the single point 0.

(d) By the Ratio Test, $\dfrac{a_{n+1}}{a_n} = \dfrac{x^{n+1}}{(n+1)^{n+1}} \cdot \dfrac{n^n}{x^n} = \dfrac{x}{n+1} \cdot \dfrac{1}{\left(1+\frac{1}{n}\right)^n} \to 0$ as $n \to \infty$ for any $x$. The radius of convergence is $\infty$ and the interval of convergence is $\mathbb{R}$.

**19.** (a) We can write $\dfrac{n^2+2n+3}{n^4+n+2} = \dfrac{n^2}{n^4}\left(\dfrac{1+\dfrac{2}{n}+\dfrac{3}{n^2}}{1+\dfrac{1}{n^3}+\dfrac{2}{n^4}}\right)$, and $\dfrac{1+\dfrac{2}{n}+\dfrac{3}{n^2}}{1+\dfrac{1}{n^3}+\dfrac{2}{n^4}}$ is very close to 1 for large $n$.

(b) Use the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. As indicated above,

$\dfrac{\dfrac{n^2+2n+3}{n^4+n+2}}{\dfrac{1}{n^2}} = \dfrac{1+\dfrac{2}{n}+\dfrac{3}{n^2}}{1+\dfrac{1}{n^3}+\dfrac{2}{n^4}} \to 1$ as $n \to \infty$. So the original series converges.

**20.** (a) Since $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ as $n \to \infty$, so $a_n < 1$ for large values of $n$.

(b) Since the terms are all positive, if $a_n < 1$, then $a_n b_n < b_n$.

(c) Since $\sum_{n=1}^{\infty} b_n$ converges and $0 < a_n b_n < b_n$, $\sum_{n=1}^{\infty} a_n b_n$ converges.

(d) Use the series $\sum_{n=1}^{\infty} a_n$ in place of $\sum_{n=1}^{\infty} b_n$ in part (c), since $\sum_{n=1}^{\infty} a_n$ converges.