2022 NTU MATH TALENTS SELECTION

(1) By the fundamental theorem of algebra, a degree n complex polynomial has n complex roots. Let $\alpha_n = \cos(2\pi/n) + i\sin(2\pi/n)$, where $i^2 = -1$. Then we have $z^n - 1 = (z-1)(z-\alpha_n^1)\cdots(z-\alpha_n^{n-1})$. Using the product notation Π, we have the expression

$$z^n - 1 = \prod_{j=0}^{n-1}(z - \alpha_n^j).$$

If n, k are coprime, α_n^k is called a primitive n-th root of unity. Given a positive integer d, we define the cyclotomic polynomial $\Phi_d(z)$ by

$$\Phi_d(z) = \prod_{\alpha:\text{ primitive } d\text{-th root of unity}} (z - \alpha).$$

Prove:
(a) $z^{12} - 1 = \Phi_1(z)\Phi_2(z)\Phi_3(z)\Phi_4(z)\Phi_6(z)\Phi_{12}(z)$.
(b)

$$z^n - 1 = \prod_{d|n, d\geq 1} \Phi_d(z).$$

(c) $\Phi_d(z)$ is a polynomial with integral coefficients, for any d.

(2) Choose three vectors $v_1, v_2, v_3 \in \mathbb{R}^n$. Define the parallel-tope spanned by v_1 and v_2 to be $P(v_1, v_2) := \{s_1 v_1 + s_2 v_2 \,|\, 0 \leq s_1, s_2 \leq 1\}$ in $\mathbb{R}^n$ and the parallel-piped spanned by v_1, v_2 and v_3 to be $P(v_1, v_2, v_3) := \{s_1 v_1 + s_2 v_2 + s_3 v_3 \,|\, 0 \leq s_1, s_2, s_3 \leq 1\}$ in $\mathbb{R}^n$. Here, let $\mathbb{R}^n$ be equipped with its standard Euclidean inner product.
  (a) Let $v_1 = (3, 1, -1)$, $v_2 = (-2, 0, 1)$ and $v_3 = (2, 2, 1)$ in $\mathbb{R}^3$. Transform v_1, v_2 and v_3 into three orthogonal vectors v_1, v_2', v_3' and then compute the area of the parallel-tope $P(v_1, v_2)$ and the volume of the parallel-piped $P(v_1, v_2, v_3)$ in $\mathbb{R}^3$.
  (Justify your answers.)
  (b) Let $v_1 = (1, -1, 1, -1)$, $v_2 = (2, -1, 1, 0)$ and $v_3 = (3, -1, 3, -1)$. Try to transform v_1, v_2 and v_3 into three orthogonal vectors v_1, v_2', v_3' such that the subspaces generated by v_1, v_2, v_3 and v_1, v_2', v_3' are the same. Define the notion of "area" of the parallel-tope $P(v_1, v_2)$ in $\mathbb{R}^4$ and compute its value. Also define the notion of "volume" of the parallel-piped $P(v_1, v_2, v_3)$ in $\mathbb{R}^4$ and compute its value.

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(3) Let $P(x) = a_n x^n + \cdots + a_0$ be a real coefficients polynomial of degree n where $n \geq 0$. A real number b is called a balance point of $P(x)$ if whenever $b = \frac{a+c}{2}$ for some real numbers a, c, then $P(b) = \frac{P(a)+P(c)}{2}$. Prove the following.

(a) If b is a balance point of $P(x)$, then 0 is a balance point of $\widetilde{P}(x)$ where $\widetilde{P}(x) = P(x+b)$.

(b) If $P(x)$ has two distinct balance points, then the degree of $P(x)$ is at most 1.

(4) (a) Suppose P_1, P_2, P_3 are three distinct points on a line l and P_0 is a point not on the line l. Let l_i be the line determined by P_0 and P_i for $i = 1, 2, 3$. Let d be the distance between P_0 and l, d_{ij} be the distance between P_i and l_j for $1 \leq i \neq j \leq 3$. Prove that $\min\{d, d_{12}, d_{13}, d_{21}, d_{23}, d_{31}, d_{32}\} < d$.

(b) Let $n \geq 2$. Let S be a set of n distinct points in the plane, not all on a line and L be the set of lines through at least two points in S. Prove that the set L contains a line which contains exactly two of the points in S.

(c) Let $n \geq 3$. Let S be a set of n distinct points in the plane, not all on a line and L be the set of lines through at least two points in S. Prove that the set L contains at least n lines.