# 2020 台大數學系特殊選才
# 面談筆試試題

(1) 對任兩個正整數 $m$ 和 $n$, 考慮以下多項式

$$f_{m,n}(x) = x^4 - 2(m+n)x^2 + (m-n)^2 .$$

刻劃所有 $m, n$ 使得 $f_{m,n}(x)$ 可以分解成兩個 (非常數) 整係數多項式的乘積.

(2) 給定平面 $E$ 上一定點 $O$與一直線 $L$. 令 $\Gamma \subset E$ 爲所有滿足

$$\frac{|\overline{OP}|}{d(P,L)} = e$$

爲一個定值 $e \geq 0$ 的點 $P \in E$ 所形成的的軌跡.

(a) 根據 $e$ 值分類 $\Gamma$.

(b) 證明 $\Gamma$ 等價於非圓的 “圓錐曲線”, 即平面與圓錐的截痕.

(3) 假設 $f : \mathbb{R} \to \mathbb{R}$ 是一個連續函數. 考慮集合

$$E = \{x \in \mathbb{R};\ f(x+h) > f(x) \text{ for some } h = h_x > 0\}.$$

(a) 如果 $E \neq \emptyset$, 證明 $E$ 是個開集合 (open set).

(b) 假定已知 $E = \coprod_{j=1}^{\infty}(a_j, b_j)$ 並且 $|a_k|, |b_k| < \infty$, 證明

$$f(a_k) = f(b_k).$$

(4) 給一個 $m \times n$ 的實係數矩陣 $A$, 我們透過將 $\mathbb{R}^n$ 中的 (行) 向量 $v$ 送到 $Av \in \mathbb{R}^m$ 來把 $A$ 看作一個從 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的線性變換.

(a) 令

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -2 & -2 \\ 1 & -1 & -2 & 0 & -2 \\ 2 & -2 & -1 & 3 & 2 \end{pmatrix}.$$

計算 $\text{image}(A)$ 的維度.

(b) 找一個矩陣 $B$ 使得 $\text{kernel}(B) = \text{image}(A)$.

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**日期: 2020 年 12 月 13 日, 9:00-11:20. 請嚴謹作答. 缺乏實質內容或直接背誦公式的作答並不會獲得任何分數. 入圍下午口試名單於 12:50 前公布.**

For those people who are not familiar with Chinese, we provide also the English translation below.

Do explain your arguments in details. No partial credits will be assigned to non-substantial solutions.

(1) For any two positive integers $m$ and $n$, consider the polynomial

$$f_{m,n}(x) = x^4 - 2(m+n)x^2 + (m-n)^2 .$$

Characterize the values of $m$ and $n$ so that $f_{m,n}(x)$ is the product of two non-constant polynomials with integer coefficients.

(2) Let $E$ be a plane, $O \in E$ be a point and $L \subset E$ be a line. Let $\Gamma \subset E$ be the set of all points $P \in E$ such that

$$\frac{|\overline{OP}|}{d(P,L)} = e$$

is a fixed value $e \geq 0$.

(a) Classify $\Gamma$ according to the value of $e$.

(b) Show that $\Gamma$ is equivalent to a conic section, that is the intersection of a plane in space with a circular cone, which is not a circle.

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider

$$E = \{x \in \mathbb{R};\ f(x+h) > f(x) \text{ for some } h = h_x > 0\}.$$

(a) If $E \neq \emptyset$, show that $E$ is an open set.

(b) Suppose we now have $E = \coprod_{j=1}^{\infty}(a_j, b_j)$. Show that for those finite intervals $(a_k, b_k)$, we must have

$$f(a_k) = f(b_k).$$

(4) If $A$ is an $m \times n$ matrix over $\mathbb{R}$, then $A$ can be regarded as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by sending a column vector $v$ to $Av$.

(a) Let

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -2 & -2 \\ 1 & -1 & -2 & 0 & -2 \\ 2 & -2 & -1 & 3 & 2 \end{pmatrix}.$$

Compute the dimension of image$(A)$.

(b) Find a matrix $B$ such that kernel$(B) =$ image$(A)$.