# 臺灣大學數學系113學年度第1學期博士班一般資格考試

## 科目：統計

2024.09.06

1. (15 points) Let $X_c = \min\{X, c\}$, where $X$ is a random variable and $c$ is a constant. Assume $E[X^2] < \infty$. Show that $Var(X_c) \leq Var(X)$.

2. (7 points) (8 points) Let $X|Y = y \sim$ Binomial$(y, p)$, $Y|\Lambda = \lambda \sim$ Poisson$(\lambda)$, and $\Lambda \sim$ Exponential$(\beta)$. Compute the expectation and variance of $X$.

3. (15 pts) Let $X = (X_1^\top, X_2^\top)^\top$ be a random vector that follows a multivariate normal distribution with mean vector $\mu$ and variance $\Sigma$, where $\Sigma$ is at least positive semi-definite. Assume that $X_1$ and $X_2$ are uncorrelated. Show that $X_1$ and $X_2$ are independent.

4. (10 points) Let $X \sim$ Negative Binomial$(r, p)$. Approximate the probability $P(X \leq x)$ by an appropriate Chi-square distribution for each $x \in \{0, 1, \cdots\}$.

5. (5 points) (10 points) State the conditions under which the maximum likelihood estimator is asymptotically normal and show this theoretical property.

6. (7 points) (8 points) Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be two random samples from Bernoulli$(p_1)$ and Bernoulli$(p_2)$, respectively. Consider the hypothesis test of $H_0 : p_1 = p_2$ versus $H_A : p_1 \neq p_2$. Derive the Wald and score test statistics and their asymptotic distributions under $H_0$.

7. (15 points) Let $X_1, \ldots, X_n$ be a random sample from a probability density function $f(x|\theta)$, where $\theta \in \Theta$ and $dim(\Theta) = k$. Let $\lambda_n$ be the likelihood ratio test statistic for testing the hypotheses $H_0 : \theta \in \Theta_0$ versus $H_A : \theta \in \Theta - \Theta_0$, where $\Theta_0 = \{\theta : \theta = g(\eta)\} \subset \Theta$ with $\eta$ being a $(k - r) \times 1$ unknown parameter vector and $g(\cdot)$ being a continuously differential function from $R^{k-r}$ to $R^k$. Show that under $H_0$ and the regularity conditions, $-2\ln(\lambda_n) \xrightarrow{d} \chi_r^2$.