臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：統計

2022.09.12

1. Let X be a random variable with a geometric distribution, i.e. $\mathbb{P}(X = x) = \theta(1-\theta)^{x-1}$, $x = 1, 2, \ldots$. Consider the estimation of θ under squared error loss.

   (a) (8%) Suppose θ has a prior distribution $P(\theta = 1/4) = 2/3, P(\theta = 1) = 1/3$. Find the Bayes estimator for θ.

   (b) (7%) Show that the Bayes estimator obtained in (a) is minimax.

2. Suppose that $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Poisson(λ).

   (a) (5%) Find the UMVUE of $\exp(-\lambda)$, the probability that $X = 0$.

   (b) (10%) Find the asymptotic distribution of the UMVUE in (a).

3. Suppose $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} U(\mu - \sigma, \mu + \sigma)$, and we want to test $H_0 : \mu = 0$ versus $H_A : \mu \neq 0$ based on $X_1, X_2, \ldots, X_n$.

   (a) (10%) Derive the likelihood ratio test and express it as a function of $W = X_{(n)} - X_{(1)}$ and $U = \max\left(X_{(n)}, -X_{(1)}\right)$ where $X_{(n)} = \max(X_1, \ldots, X_n)$ and $X_{(1)} = \min(X_1, \ldots, X_n)$.

   (b) (10%) Show that, under H_0, U is complete and sufficient and U and W/U are independent.

4. Starting with a density f with mean 0 and covariance matrix $\sigma^2 I_p$, create the location family $\{f(x-\theta) : |\theta| < \infty\}$. Let $\boldsymbol{X}_{p\times 1}|\theta \sim f(x-\theta)$ and consider a prior on θ to be $\theta \sim f^{*n}$, the n-fold convolution of f with itself. (The convolution of f with itself is $f^{*2}(x) = \int f(x-y)f(y)dy$. The n-fold convolution is $f^{*n}(x) = \int f^{*(n-1)}(x-y)f(y)dy$.)

   (a) (10%) Show that the Bayes estimator under squared error loss is $\frac{n}{n+1}\boldsymbol{X}$. Note that n is a prior parameter.

   (b) (10%) Calculate the mean squared error of $\frac{n}{n+1}\boldsymbol{X}$, and compare it to the mean squared error of $\boldsymbol{X}$. Under what circumstances would you prefer $\frac{n}{n+1}\boldsymbol{X}$?

(c) (10%) Show that, marginally, $|\boldsymbol{X}|^2/\left(p\sigma^2\right)$ is an unbiased estimator of $n+1$. Use this fact to construct an empirical Bayes estimator of θ that resembles a Stein estimator.

5. Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} N\left(\mu, \sigma^2\right)$, where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are both unknown.

   (a) (10%) Define $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S_n^2 = n^{-1}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2$. Find the asymptotic distribution of $\sqrt{n}\left(\bar{X}_n - \mu, S_n^2 - \sigma^2\right)^T$.

   (b) (10%) Suppose now the normality assumption is dropped, but the X_i's are assumed to have a finite fourth moment. Let $\mu_3 = \mathbb{E}\left(X_1 - \mu\right)^3$ and $\mu_4 = \mathbb{E}\left(X_1 - \mu\right)^4$. Find the asymptotic distribution of $\sqrt{n}\left(\bar{X}_n - \mu, S_n^2 - \sigma^2\right)^T$.