臺灣大學數學系
101 學年度第 2 學期博士班資格考
科目：統計

1. (15%) Let $X_1, \ldots, X_n$ be a random sample from a population with probability density function $f(x)$ and differentiable cumulative distribution function $F(x)$. Show that $\sqrt{n}(M_n - \mu)$ is asymptotically normal with mean zero and variance $1/(2f(\mu))^2$, where M_n and μ are separately the sample median and population median.

2. (8%) (7%) Let $X_1, \ldots, X_n$ be a random sample from a population with probability density function

$$f_X(x|\theta) = \theta x^{\theta-1}, \ 0 < x < 1, \ 0 < \theta < \infty.$$

Find the uniformly minimum variance unbiased estimator (UMVUE) of θ and show that the variance of the UMVUE cannot attain the Cramér-Rao lower bound.

3. (8%) (7%) Let $p_j(X_1, \ldots, X_n)$ denote a valid p-value for $H_{0j} : \theta \in \Theta_j$ versus $H_{Aj} : \theta \notin \Theta_j$, $j = 1, \ldots, k$. Define a valid p-value $p(X_1, \ldots, X_n)$ for the hypotheses $H_0 : \theta \in \bigcup_{j=1}^k \Theta_j$ versus $H_A : \theta \in \bigcap_{j=1}^k \Theta_j^c$ and find a level α test defined by $p(X_1, \ldots, X_n)$.

4. Let $\widehat{\theta}_n$ be an estimator of θ_0 based on a random sample $X_1, \ldots, X_n$.

(4a) (7%) (8%) Define the jackknife estimator of θ_0 and construct an approximated $100(1-\alpha)\%$, $0 < \alpha < 1$, confidence interval for θ_0 based on the jackknife estimator and its asymptotic variance.

(4b) (10%) Construct a $100(1-\alpha)\%$ bootstrap quantile confidence interval for θ_0.

5. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, a\theta)$, where a is a known positive constant and $\theta > 0$.

(5a) (7%) Find a minimal sufficient statistic for θ.

(5b) (3%) (5%) Show that, for any constant $0 \le c \le 1$, $E[c\bar{X}_n + (1-c)dS_n|\theta] = \theta$ with $d = [(n-1)\Gamma((n-1)/2)]/[2a\Gamma((n+1)/2)]$, and find the minimizer, say $\widehat{c}$, of $Var(c\bar{X}_n + (1-c)dS_n|\theta)$.

6. (8%) (7%) Let $X_1, \ldots, X_n$ be a random sample from $Uniform(\theta, \theta+1)$. Find the uniformly most powerful size α test of the hypotheses $H_0 : \theta = 0$ versus $H_A : \theta > 0$, and calculate the power of the test at θ_1, where $\theta_1 > 0$.