臺灣大學數學系113學年度第2學期博士班一般資格考試

科目：實分析

2025.02.20

1. (15%) Given $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$. Given $t \in \mathbb{R}$, define $f_t(x) := f(x-t)$. Show that $\lim_{t\to 0} ||f_t - f||_p = 0$.

2. (15%) Assume that $\{f_k\}$ is a sequence of measurable functions in $L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$ and f_k converges in L^p norm to some f. Now given a sequence of measurable function $\{g_k\}$ so that $||g_k||_\infty \leq M < \infty$ and $g_k \to g$ almost everywhere. Show that $f_k g_k$ also converges in L^p to fg.

3. (15%) Suppose $f(x)$ is real-valued and bounded on $\mathbb{R}$, and is of $C^2(\mathbb{R})$. Show that there must exist a point x_0 such that $f''(x_0) = 0$.

4. (10%) Suppose $f(x)$ is of bounded variation on $[0,1]$. Show that $g(x) := f(x^2)$ is of bounded variation on $[-1.1]$.

5. (15%) Let E denote the set of real numbers in $[0,1]$ without digit 9 in their decimal expansion, that is, $x \in E$ then $x = \sum_{k\geq 0} \frac{a_k}{10^k}$ and none of a_k is 9. Show that E has Lebesgue measure zero.

6. (15%) If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k\in Z} 2^{kp}\omega(2^k) < \infty$, where ω is the distribution function.

7. (15%) Suppose $f(x) \in L^1(\mathbb{R})$. Does the limit $\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)|^{\frac{1}{n}} dx$ exist ? Justify your result.