臺灣大學數學系113學年度第1學期博士班一般資格考試

科目：實分析

2024.09.04

1. (30%) Let E be a measurable set in R^n with its Lebesgue measure $|E|>0$.

   (a) Let $H=\{|x|x \mid x\in E\}$. Show that $|H|>0$.

   (b) Let $K=\{x-y \mid x\in E, y\in E\}$. Show that there exists an $r>0$ such that the ball $B_r=\{x\in R^n \mid |x|<r\}\subset K$.

   (c) Let f be a nonnegative real-valued measurable function defined on E. Show that the graph of f, $G=\{(x,f(x))\in R^{n+1} \mid x\in E\}$, has measure zero in R^{n+1}.

2. (30%) Let f, $\{f_k\}\in L^p(E)$, $1\le p\le\infty$.

   (a) Assume $\|f-f_k\|_p\to 0$. Show that there exists a subsequence of $\{f_k\}$ which converges to f a.e. in E.

   (b) Assume $f_k\to f$ a.e. and $\|f_k\|_p\to\|f\|_p$, $1\le p<\infty$. Show that $\|f-f_k\|_p\to 0$.

   (c) Find g, $\{g_k\}\in L^3([0,1])$ such that $g_k\to g$ a.e. in $[0, 1]$, $\lim_{k\to\infty}\|g_k\|_2=\|g\|_2$, and $\limsup_{k\to\infty}\|g_k\|_3>\|g\|_3$.

3. (20%) Let f be nonnegative, bounded, and Riemann integrable on $[a, b]$. Show that f is Lebesgue integrable on $[a, b]$.

4. (20%) Prove the following result concerning *changes of variable*. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha,\beta]$, and let f be bounded and measurable on $[a, b]$, $a=g(\alpha)$, $b=g(\beta)$. Then $f(g(t))$ is measurable on $[\alpha,\beta]$ and

$$\int_a^b f(x)dx=\int_\alpha^\beta f(g(t))g'(t)dt.$$

(Hint: Consider the case when f is the characteristic function of an interval, an open set, etc.)