臺灣大學數學系112學年度第2學期博士班一般資格考試

科目：實分析

2024.02.23

1. Show that an open set in $\mathbb{R}$ can be written down as a disjoint union of countable open intervals. (14%)

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Show that for any $\epsilon > 0$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $m(\{x|f(x) - g(x) \neq 0\}) < \epsilon$. Here, $m(E)$ denotes the Lebesgue measure of set E. (14%)

3 Show that $E \subset \mathbb{R}^n$ is measurable if and only if $E = H \setminus Z$ for some H of G_δ type and Z of measure zero. (14%)

4. Let E be a measurable subset of $\mathbb{R}^n$. Suppose a sequence of measurable functions $\{f_k\}$ converges in measure to f on E and $|f_k| \leq \phi_k$, where ϕ_k are measurable functions on E such that $\phi_k \to \phi \in L(E)$ a.e. Show that $f \in L(E)$ and $\int_E f_k \to \int_E f$. (15%)

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Suppose

$$\int_E f = 0$$

for every measurable subset E of $\mathbb{R}^n$. Show that $f = 0$ almost everywhere. (14%)

6. Let f be measurable and periodic with period l. Suppose that

$$\int_0^l |f(\alpha - t) + f(\beta + t)|dt \leq 1$$

for all α and β. Show that $f \in L[0, l]$. (14%)

7. Let $K(x, y) \geq 0$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define $T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$. Suppose there exist positive measurable functions $p,\ q$ and positive numbers $a,\ b$ such that

$$\int_{\mathbb{R}^n} K(x, y)p(y)dy \leq aq(x),$$
$$\int_{\mathbb{R}^n} K(x, y)q(x)dx \leq ap(y).$$

Show that

$$\int_{\mathbb{R}^n} |T(f)(x)|^2 dx \leq ab \int_{\mathbb{R}^n} |f(y)|^2 dy.$$

(15%)