臺灣大學數學系112學年度第1學期博士班一般資格考試

科目：實分析

2023.09.08

1. (20%) Let $E_k, k = 1, 2, 3, ...,$ be measurable subsets of $\mathbb{R}^n$ and let $m(E_k)$ denote the Lebesgue measure of E_k.
   (a) Show that $m(\cup_{k=1}^{\infty} E_k) = \lim_{n\to\infty} m(\cup_{k=1}^{n} E_k)$.
   (b) Assume $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let $E = \{x \in \mathbb{R}^d \,|\, x \in E_k \text{ for infinitely many k}\}$. Show that E is measurable and $m(E) = 0$.

2. (15%) Assume that E_1 and E_2 are measurable in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Show that $E = E_1 \times E_2$ is measurable in $\mathbb{R}^{m+n}$ and
$$m(E) = m(E_1)\, m(E_2).$$

3. (30%) Suppose f is integrable on $\mathbb{R}^n$.
   (a) Show that if $\int_E f(x)\,dx \geq 0$ for every open set E, then $f(x) \geq 0$ for a.e. x.
   (b) Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that
$$\int_E |f| < \epsilon \text{ whenever } E \text{ is measurable and } m(E) < \delta.$$
   (c) Show that $f(\dfrac{|x|x}{\alpha + |x|})$ converges to $f(x)$ in the L^1-norm as $\alpha \to 0^+$.

4. (20%) Show that if f is of bounded variation on $[a, b]$, then

   (a)
$$\int_a^b |f'(x)|\,dx \leq T_f(a,b), \text{ where } T_f(a,b) \text{ denotes the total variation of } f \text{ on } [a,b];$$
   (b)
$$\int_a^b |f'(x)|\,dx = T_f(a,b) \text{ if and only if } f \text{ is absolutely continuous.}$$

5. (15%) Show that if X is a normed vector space and Y is a Banach space, then the space $L(X, Y)$ of all bounded linear maps is a Banach space.