臺灣大學數學系
108 學年度下學期博士班資格考試題
科目：實分析

2020.03.05

1. (15 points)

(i) Show that $\exp(x)$ is a convex function.

(ii) State and prove the Young's inequality (you may use the result from (i)).

(iii) Prove the Hölder's inequality. (for vectors in $\mathbb{R}^N$)

2. (20 points) Prove the following integral version of Minkowski's inequality for $1 \le p < \infty$:

$$[\int |\int f(x,y)dx|^p dy]^{1/p} \le \int [\int |f(x,y)|^p dy]^{1/p} dx.$$

3. (30 points)

(i) Show that any continuous function defined on a closed interval $[a,b]$ $(a<b)$ is uniformly continuous.

(ii) For a function f defined on the closed interval $[a,b]$ $(a<b)$, write down the definition of Riemann integrable for function f.

(iii) Suppose f is a continuous function defined on the closed interval $[a,b]$ $(a<b)$, show that f is Riemann integrable over $[a,b]$.

4. (20 points)

(i) State Fatou's Lemma for nonnegative measurable functions.

(ii) State Lebesque's Dominated Convergence Theorem for nonnegative measurable functions.

(iii) Use Fatou's Lemma to prove Lebesgue's Dominated Convergence Theorem.

5. (15 points) Suppose $1 < p < N$ and $u \in C_0^1(\mathbb{R}^N)$. Show that

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \le (\frac{p}{N-p})^p \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Hint : scaling in space variable, fundamental theorem of calculus and Hölder's inequaity.