# 臺灣大學數學系
# 107 學年度上學期博士班資格考試題
## 科目：實分析　　　　2018.09.13

**PH.D QUALIFYING EXAM : REAL ANALYSIS 2018 FALL**

1.(15%) Assume $\{E_j\}_{j=1}^m$ is a collection of measurable sets in $[0,1]$ with the property that every $x \in [0,1]$ belongs to at least $n$ of $E_j$ $(n \le m)$. Show that $|E_j| \ge \frac{n}{m}$ for some $j$.

2.(10%) Suppose $A$ is a measure zero set in $\mathbb{R}$. Show that there exists a sequence of open sets $U_n$ such that $A \subset \cap_{n=1}^{\infty} U_n$ and $\lim |U_n| = 0$.

3.(15%) Is it possible to construct a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for all $p \ge 1$ but $f \notin L^\infty(\mathbb{R})$ ? Show your result.

4.(15%) Let $g(x)$ be a bounded measurable function with the property that

$$\lim_{n\to\infty} \int_E g(nx)dx = 0,$$

for any measurable set $E$ with finite measure. Given $f \in L^1(\mathbb{R})$, does

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)g(nx)dx = 0?$$

Prove or disprove your result.

5.(15%) Let $\{\phi_k\}$ be a complete orthogonal system in $L^2$. Given a bounded sequence of numbers $m = \{m_k\}$ and for every $f \in L^2$, we define $T(f)(x) = \sum_k m_k c_k \phi_k(x)$, where $c_k = \int f(x)\bar{\phi_k}(x)$. Show that $||T(f)||_2 \le C||f||_2$ for some constant $C$ that is independent of $f$.

6.(15%) Let $f(x_1, x_2, x_3) = (x_1^2 + x_2^4 + x_3^6)^{-p}$, where $p > 0$. Find the range of $p$ so that $f$ is integrable on a bounded set that contains the origin.

7.(15%) Given a measurable set $A \subset [0,1]$ with $|A| > 0$. Let $B = \cos(A) = \{cos(x), x \in A\}$. Show that the measure of $B$ is strictly less than the measure of $A$, i.e $|B| < |A|$.